1 Continuous physics

1.1 Einstein-Hilbert action :

$$S_{EH}[g] = \frac{1}{16\pi}\int d^4x \sqrt{g}\, R \tag{1}$$

☆ Equations of motion: $\frac{\delta S_{EH}}{\delta g_{\mu\nu}} = 0 \Leftrightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$

Note: possible (necessary!) to have $\Lambda \neq 0$, not discussed here.

☆ S_{EH} is symmetric under diffeomorphism:
Under the infinitesimal transformation $x^\mu \mapsto x^\mu + \xi^\mu(x)$, the metric varies as
$\delta_\xi g_{\mu\nu} = \partial_\mu \xi^\alpha g_{\alpha\nu} + \partial_\nu \xi^\rho g_{\mu\rho} + \xi^\alpha \partial_\alpha g_{\mu\nu}$. The action then only varies as
$\delta_\xi S_{EH} = \int d^4x\, \partial_\mu (\sqrt{-g}\, \xi^\mu R)$, a boundary term.

☆ The gauge group of the theory is $\mathcal{G} = \mathrm{Diff}(\mathcal{M})$

1.2 The Einstein-Cartan theory

☆ To couple to fermions, one needs a flat $SL(2,\mathbb{C})$ index $I = 0, \dots, 3$ to contract with the γ-matrices ($\{\gamma^I, \gamma^J\} = 2\eta^{IJ}$). The metric is written in term of vielbeins:

$$g_{\mu\nu}(x) = e_\mu{}^I(x)\, e_\nu{}^J(x)\, \eta_{IJ} \tag{2}$$

Note: while fermions are square-root of vector ($\bar{\psi}\gamma^\mu\psi$ is a vector), vielbeins are the square-root of the geometry.

☆ This gives an extra Local Lorentz gauge symmetry

$$e^I{}_\mu(x) \mapsto \Lambda^I{}_J(x)\, e^J{}_\mu(x) \tag{3}$$

$$\left(e^I{}_\mu e^J{}_\nu \eta_{IJ} \mapsto \Lambda^I{}_K \eta_{IJ} \Lambda^J{}_L e^K{}_\mu e^L{}_\nu = e^K{}_\mu e^L{}_\nu \eta_{KL} \right)$$

↳ The symmetry group is now $\mathcal{G} = \mathrm{Diff}(\mathcal{M}) \ltimes SO(1,3)_{loc}$

☆ We introduce a connection ω such that

$$0 = De^I = de^I + \omega^I{}_J \wedge e^J = \partial_{[\mu} e^I{}_{\nu]} + \eta_{KJ}\, \omega^{IK}{}_{[\mu}\, e^J{}_{\nu]}$$

Note : $De^I \equiv T^I$ is the torsion. A torsion-free connection requirement is called the 1st Cartan equation. A torsion-free connection transforms as : $\omega'^{IJ}{}_\mu \mapsto \Lambda \omega \Lambda^{-1} + \Lambda d \Lambda^{-1}$

The curvature 2-form is $F^I{}_J \equiv D\omega^I{}_J = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J$

It is then possible to write $R = F^{IJ}{}_{\mu\nu}\, e_I{}^\mu e_J{}^\nu$ to obtain the Einstein-Cartan theory :

$$S_P[e,\omega] = \frac{1}{16\pi}\int e^I \wedge e^J \wedge * F_{IJ} \qquad (4)$$

1.3 The Holst term

Up to a boundary term, the theory doesn't change if one adds the Holst term: $\int e\wedge e \wedge F$. Picking its coupling constant (analogous to θ_{QCD}) $(16\pi\gamma)^{-1}$, one gets the classical theory considered in LQG:

$$S[e,\omega] = \frac{1}{16\pi}\int e\wedge e\wedge * F + \frac{1}{16\pi\gamma}\int e\wedge e\wedge F$$

$$= \frac{1}{16\pi}\int B^{IJ}[e]\wedge F_{IJ}[\omega] \text{ with } B^{IJ} \equiv (* + 1/\gamma)\, e^I \wedge e^J \qquad (5)$$

↳ simplicity constraint

1.4 Boundaries

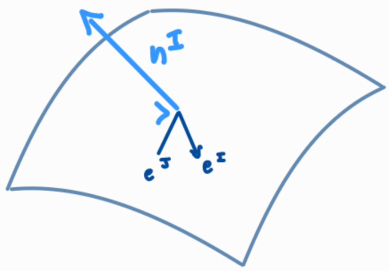
☆ The generators of the Lorentz group J^{IJ} respect the following algebra

$$[J^{IJ}, J^{KL}] = i\left(\eta^{JK} J^{IL} - \eta^{IL} J^{JK} - \eta^{JK} J^{IL} + \eta^{JL} J^{IK}\right)$$

On the boundary defined by $n_I e^I = 0$, J^{IJ} split into $\begin{cases} \text{boosts } n\cdot J \\ \text{rotations } n\cdot * J \end{cases}$

$B \equiv * e\wedge e + 1/\gamma\, e\wedge e$

Now, $\begin{cases} n_I B^{IJ} = n_I * (e^I \wedge e^J) \\ n_I * B^{IJ} = (1/\gamma)\, n_I * (e^I\wedge e^J) \end{cases}$



$\Rightarrow n_I B^{IJ} = \gamma\, n_I * B^{IJ} \Leftrightarrow K^J = \gamma L^J$ on the $\partial\mathcal{M}$ (6)

2 Discrete physics

2.1 Discrete variables 2.2 geometrical interpretation 2.3 Boundary states

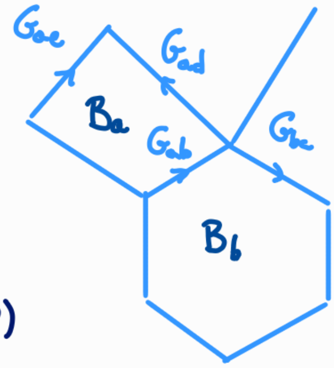
2.1 : Discrete variables

☆ The discrete physics use the BF theory, agnosticly of the simplicity constraint. The B variable is integrated on surface f and the connection ω on lines e :

$$B_f \equiv \int_f B^{IJ}{}_{\mu\nu} J_{IJ} \, dx^\mu dx^\nu \quad \text{and} \quad G_e \equiv \mathcal{P}\exp\left\{ \int_e \omega^{IJ}{}_\mu J_{IJ} \, dx^\mu \right\} \tag{7}$$

Flux Holonomy

↳ Diffeomorphism invariant!



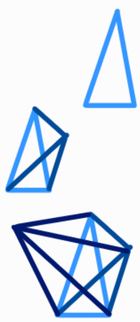
☆ The discrete action reads :

$$S_{BF} = \sum_f \mathrm{Tr}\left\{ B_f \prod_{e \in f} G_e \right\} + \sum_e \mathrm{Tr}\left\{ \lambda_e \sum_{f \in e} B_f \right\} \tag{8}$$

curvature = holonomy around a face Gauss constraint

2.2 : Geometrical interpretation

The 4 manifold is triangulated, and the smeared variables live on its dual.



	Δ		Δ^*		$\partial\Delta^*$
triangle	—		face	—	link
tetrahedra	—		edge	—	node
4-simplex	—		vertex		

2.3 Boundary states

☆ A spin network state $\phi_\Gamma(\{g_\ell\})$ is such that $\phi(\{g_\ell\}) = \phi(\{h_{s(\ell)}\, g_\ell\, h_{t(\ell)}\})$

One implements it using the Plancherel decomposition:

$$L_2[SU(2)] \ni \phi_\Gamma = \prod_\ell \sum_{j_\ell}^{N/2} \sum_{m,n=-j_\ell}^{j_\ell} \hat{\phi}^{(j_\ell)}_{mn}\, D^{(j_\ell)}_{mn}(g_\ell) \tag{9}$$

Then select the invariant part under SU(2) rotation at nodes:

$$\phi(\{g_\ell; j_\ell, \iota_n\}) = \bigotimes_n \iota_n \cdot_\Gamma \bigotimes D^{(j_\ell)}(g_\ell) \tag{10}$$

$$= \sum_{\{m_{t(\ell)}, m_{s(\ell)}\}} \left(\prod_n \left\langle \bigotimes_{\ell | s(\ell)=n} j_\ell\, m_{s(\ell)} \middle| \iota_n \middle| \bigotimes_{\ell | t(\ell)=n} j_\ell\, m_{t(\ell)} \right\rangle \right) \left(\prod_\ell D^{(j_\ell)}_{m_{t(\ell)} m_{s(\ell)}}(g_\ell) \right)$$

---

3 Spinfoam partition function and representation theory

3.1 Path integral — 3.2 BF theory — 3.3 Quantum gravity

3.1 Path integral

☆ Given two geometries $\partial\Delta$ and $\partial\Delta'$, the path integral reads:

$$Z_\Delta = \int_{\partial\Delta}^{\partial\Delta'} \mathcal{D}[B]\, \mathcal{D}[\omega]\, e^{iS}$$

The discretized action (10) gives flatness as it should for a topological theory.

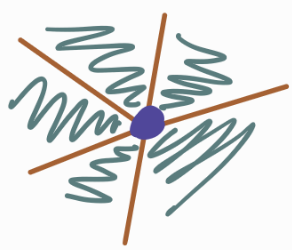
$$Z^{BF}_\Delta = \int \mathcal{D}[B]\mathcal{D}[\omega] \exp\left\{ i \,\mathrm{Tr}\, B_f \prod_{e\in f} G_e \right\}$$

$$= \int \mathcal{D}[\omega]\, \delta\left(\prod_{e\in f} G_e \right)$$

↳ What about the simplicity constraint $B = (* + 1/\gamma)\, e \wedge e$?

3.2 BF theory

☆ Let us start from an ansatz from BF theory:



$$Z^{BF}_\Delta = \prod_f \mathcal{A}_f(j_f) \prod_e \mathcal{A}_e(j_f, \iota_n) \prod_v \mathcal{A}_v(j_f, \iota_e)$$

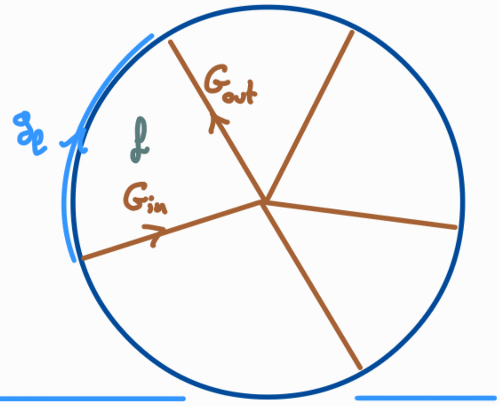
$$= \prod_f d_{j_f} \prod_e d_{\iota_e} \prod_v \{15j\}_v$$

☆ The γ-simplicity of B only affect vertices, so that

$$Z_\Delta = \prod_f d_{j_f} \prod_e d_{i_e} \prod_v \mathcal{A}_v (j_f, i_e) \qquad (11)$$

☆ How to compute $\mathcal{A}_v$? It has to be Lorentz invariant and match the SN around it:

no curvature around a face: $G_{in}^{-1} \circ G_{out}^{-1} \circ g_\ell = \mathbb{1}$ ($G_{in}^{-1} \circ G_{out}^{-1}$: $SL(2,\mathbb{C})$; g_ℓ: $SU(2)$)



How to embed $g_\ell \mapsto G_{in} \circ G_{out}$ while imposing $K = \gamma L$?

3.3 Quantum gravity

☆ The solution is the Y_γ map. It is explicitly build using unitary rep. of $SL(2,\mathbb{C})$. As for the boundary state, we use Plancherel decomposition:

$$L_2[SL(2,\mathbb{C})] \ni \Psi(G) = \int d\mu(\rho) \sum_k^{N/2} \sum_{j,\ell=k}^{\infty} \sum_{m=-j}^{j} \sum_{n=-\ell}^{\ell} \hat{\Psi}^{(\rho,k)}_{jm\ell n} D^{(\rho,k)}_{jm\ell n}(G) \qquad (12)$$

☆ The Y_γ map embeds $\phi \in L_2[SU(2)] \longrightarrow \Psi \in SL(2,\mathbb{C})$ in the following non trivial way:

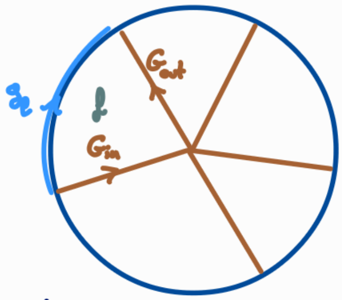
$$|j,m\rangle \mapsto |(\rho,k); j,m\rangle = |(\gamma j, j); j,m\rangle \qquad (13)$$

Note: in canonical LQG, the introduction of the tetrads shifts the constraints from 1[st] to second order. Picking specific variables (Ashtekar ones) allows to recast them as 1[st] class. In an analogous way, $K=\gamma L$ cannot be imposed strongly since $[K,L]\neq 0$. The Y_γ map is a way to weakly implement the simplicity constraint.

☆ The EPRL vertex amplitude reads:

$$\mathcal{A}_v^{EPRL}(g_e) = \int_{SL(2,\mathbb{C})} d\mu(G) \prod_f \sum_j \sum_{m,n} d_j D^{(j)}_{mn}(g_e) D^{(\gamma j, j)}_{jm,jn}(G)$$

Or a bit more explicitly: $\mathcal{A}_v^{EPRL}(\{g_\ell; j_\ell, L_n\}^{n=1,\dots,5}_{\ell=1,\dots,10})$



$$= \int_{SL(2,\mathbb{C})} \prod_{w=1}^{5} d\mu(G_w) \prod_{\ell=1}^{10} \sum_{j_\ell}^{N/2} d_{j_\ell} D^{(j_\ell)}_{m m'}(g_\ell)\, D^{(\gamma j,j)}_{jm\, jm'}(G_w) \cdot_r (L_n)_{m m'}$$

---

4 Cartan decomposition

☆ The key element of the EPRL model is the $D^{(\gamma j,j)}_{jm\,jm'}$. Using Cartan decomposition, $G = g\exp\{-i\beta K_3\}h = g\exp\{\beta\sigma_3/2\}h$, we can further decomposed $D^{(\rho,k)}$:

$$D^{(\rho,k)}_{jm\,\ell n}(G) = \sum_{p=-\min(j,\ell)}^{\min(j,\ell)} D^{(j)}_{jp}(g)\, d^{(\rho,k)}_{j\ell p}(\beta)\, D^{(\ell)}_{pn}(h)$$

Now, all the boost part of Z_Δ is factorized in these reduced $d^{(\gamma j,j)}(\beta)$ matrices.