Problem Set 4: Quasi-Normal modes, Part II PHYS-F484

March 20, 2025

1 Isospectrality of the Perturbations

As mentioned in the previous exercise session, the Regge-Wheeler and Zerilli potentials contain the same information. We saw that both potentials can be expressed as follows:

$$V^{\pm} = W(r)^2 \mp \frac{dW(r)}{dr_*} - \beta^2, \tag{1}$$

with

$$W(r) = \frac{6M(2M-r)}{r^2(6M+\mu^2 r)} - \beta, \quad \beta = \frac{\mu^2(\mu^2+2)}{12M} \tag{2}$$

and $\mu^2 = (l-1)(l+2)$.

Problem 1.1. *Show that spectra arising from odd and even perturbations are actually the same.*

This isospectrality allows us to consider our favourite potential to go on with our analysis. Because the analytical form of the Regge-Wheeler potential is simpler, we will focus on axial perturbations.

2 QNMs of Schwarzschild Black Hole

The master equations derived in Problem Set 3 describe the response of a Schwarzschild black hole to external perturbations. They tell us about the *vibration modes* of such a spacetime. A black hole can be perturbed in different way, either by gravitational waves incidence, an object entering the black hole, matter accretion or self-force.

We will now make the assumption that the perturbations have a temporal dependence such that we are looking for solutions of the form

$$Q_{\omega_n}(t, r_*) = e^{i\omega_n t}\Psi(r_*), \tag{3}$$

with ω_n a discrete spectrum of oscillation frequency that we can decompose as

$$\omega_n = \omega_n^R + i\omega_n^I, \quad n = 0, 1, 2, \ldots \tag{4}$$

The real part is associated with the oscillation frequency as $f = \omega^R/2\pi$, while the imaginary part describes the damping rate of each mode, as a response to the emitted radiation, through $\tau = 1/\omega^I$.

Problem 2.1. *Inserting this ansatz in the Regge-Wheeler equation, show that*

$$\frac{d^2\Psi_{lm}^{\pm}}{dr_*^2} + \left(\omega^2 - V_{lm}^{\pm}\right)\Psi_{lm}^{\pm} = 0, \tag{5}$$

where l, m refer to the harmonic decomposition and $\pm$ to the parity.

The solutions to this equation are called the *quasi-normal modes* of the black hole, and the associated frequencies are referred to as *quasi-normal frequencies*. Furthermore, they satisfy a *pure outgoing wave* boundary condition at null infinity and an *pure ingoing wave* boundary condition at the event horizon

$$\Psi \sim e^{-i\omega r_*} \qquad \text{for} \quad r_* \to -\infty, \tag{6}$$
$$\Psi \sim e^{i\omega r_*} \qquad \text{for} \quad r_* \to \infty. \tag{7}$$

These boundary conditions are chosen to reflect the physical behavior of perturbations: pure outgoing waves at infinity correspond to the radiation that escapes the black hole, while pure ingoing waves at the event horizon reflect the fact that no information can escape from within the event horizon. For each value of l and polarisation $\pm$, we have an infinite set of frequencies, $\omega_{l,n}^{\pm}$, labelled by n. The mode corresponding to $n = 0$, called the *fundamental mode*, is the one with the smallest damping, thus the longest-lasting. Having $\mathrm{Im}(\omega) > 0$ implies an instability. Hence, these QNMs give insight into the stability of the black hole and the timescale for it to settle after a perturbation.

Eikonal Limit and QNMs

One can also introduce the *eikonal limit*, which corresponds to the regime where the angular momentum quantum number is large: $l \gg 1$. In this limit, the QNM frequencies can be interpreted in terms of the properties of the *photon sphere*. The QNMs are closely related to the null geodesics trapped at this orbit.

While talking of the eikonal limit, one can make further physical interpretations of the real and imaginary parts of the QNM. The real part is the angular frequency Ω_c of photons at the photon sphere

$$\mathrm{Re}(\omega) \approx \Omega_c l. \tag{8}$$

The imaginary part is governed by the Lyapunov exponent λ_c, which quantifies the instability timescale of the photon orbit:

$$\mathrm{Im}(\omega) \approx -\lambda_c(n + 1/2). \tag{9}$$

Problem 2.2. *Consider a massless scalar field perturbation in the Schwarzschild spacetime. The effective potential governing the perturbation is given by*

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right). \tag{10}$$

Show that the location of the photon sphere corresponds to the maximum of the effective potential in the eikonal limit $l \gg 1$. That is, find the value of r where $dV/dr = 0$ in this limit.

The associated frequency and Lyapunov exponent are

$$\Omega_c = \frac{1}{3\sqrt{3}M}, \quad \lambda_c = \frac{1}{3\sqrt{3}M}. \tag{11}$$

In this particular limit, the QNM spectrum takes the following form:

$$\omega_{l,n}^{(s)} = \frac{l+1/2}{3\sqrt{3}M} - i\frac{n+1/2}{3\sqrt{3}M} + \mathcal{O}(1/l), \quad l \to \infty. \tag{12}$$

Some Numerical Values

Several analytical methods are used to compute the QNM spectrum of Schwarzschild black holes. The *WKB approximation* is one of the most well-known, particularly useful in the eikonal limit. It provides a semi-classical method to approximate solutions to wave equations with potential barriers. Other approaches include Poschl-Teller potential approximation, monodromy methods, matched asymptotic expansions, and continued fraction techniques, each suited for different regimes of QNMs.

Because of the limited amount of time, we will not have time to develop them. Instead, we will use the *Black Hole Perturbation Toolkit*[1]. For the Schwarzschild black hole, we obtained some QNM frequencies, listed in Table 1.

	Scalaire $s=0$	Vecteur $s=1$	Tenseur $s=2$
$l=2, n=0$	0.483644 - 0.0967588 i	0.457596 - 0.0950044 i	0.373672 - 0.0889623 i
$l=2, n=1$	0.463851 - 0.295604 i	0.436542 - 0.29071 i	0.346711 - 0.273915 i
$l=2, n=2$	0.430544 - 0.508558 i	0.401187 - 0.501587 i	0.301053 - 0.478277 i
$l=3, n=0$	0.675366 - 0.0964996 i	0.656899 - 0.0956162 i	0.599443 - 0.092703 i
$l=3, n=1$	0.660671 - 0.292285 i	0.641737 - 0.289728 i	0.582644 - 0.281298 i
$l=3, n=2$	0.633626 - 0.496008 i	0.613832 - 0.492066 i	0.551685 - 0.479093 i

Table 1: QNMs of the Schwarzschild black hole.

Problem 2.3. *Give an interpretation of those values with respect to the different parameters.*

3 QNMs of Kerr Black Hole

The Kerr black hole, discovered by Roy Kerr in 1963, generalizes the Schwarzschild solution to include the effects of angular momentum. While the Schwarzschild black hole is static and spherically symmetric, the Kerr black hole is dynamically rotating and axially symmetric.

The *no-hair theorem* conjectures that all stationary black holes solutions to Einstein equations can be entirely characterised by three parameters: their mass M, their charge Q and their angular momentum J. The most general black hole should then be the Kerr-Newman black hole, charged and rotating. But, astrophysical black holes are surrounded by an accretion disk essentially composed of plasma. Thus, any charged black hole will neutralise itself

[1] `https://bhptoolkit.org/QuasiNormalModes/`

with the surrounding plasma. This boils down to consider $Q = 0$. However, one cannot take $J = 0$ to describe the most general astrophysical black hole, as the angular conservation of a dying contracting star would not allow it.

Hence, the metric of a Kerr black hole can be written in Boyer-Lindquist coordinates (t, r, θ, ϕ) and is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\varphi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\ &+ \frac{\sin^2\theta}{\rho^2}\left[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2 \end{aligned} \tag{13}$$

with $\rho^2(r, \theta) \equiv r^2 + a^2\cos^2\theta$ and $\Delta(r) = r^2 - 2Mr + a^2$, where a is the specific angular momentum related to J via $a = J/M$. It is a dimensionless parameter measuring the angular momentum. The presence of a introduces several important effects, including:

- **Ergosphere:** The region outside the event horizon where spacetime is dragged around by the black hole's rotation. In this region, it is impossible for any object to remain at rest relative to infinity due to the frame-dragging effect.
- **Event Horizon and Inner Horizon:** The Kerr black hole has two horizons. The outer event horizon and the inner Cauchy horizon. The event horizon, like in the Schwarzschild black hole, marks the point beyond which nothing, not even light, can escape the black hole.
- **Singularity:** Unlike the Schwarzschild black hole, the singularity of a Kerr black hole is not a point but a ring singularity, located at the center of the black hole. This ring-like singularity is associated with the rotation of the black hole.

One can study perturbations of such a black hole, but losing spherical symmetry has a cost: equations become more complicated. The spherical harmonic basis on which we develop our fields is now spheroidal: the rotation lifts degeneracy of the m index.

Problem 3.1 (Mathematica)**.** *Using the associated Mathematica notebook, plot the evolution of the QNM frequency taking $M = 1$, $l = 0$ and $n = 0$, letting a vary. What can you observe?*