The Post Newtonian formalism and its regularizations

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Abstract

This work reviews both the Hadamard and the dimensional regularization of the post-newtonian formalism in the case of a compact binary coalescence. A short introduction on the validity of the Post-Newtonian expansion is given, followed by a review of the Landau-Lifshitz action in the harmonic gauge, with a discussion on the derivation of the gauge fixing term with QFT techniques. The Einstein Field Equations are then solved in the near zone at lowest order, allowing us to introduce the Hadamard regularization. Finally, we review the dimensional regularization of Poisson integrals, needed at higher post-newtonian orders.

This work is based on [Compère, 2024] for the introduction and the general theory of the MPN/PM formalism and on [Poisson and Will, 2014; Blanchet, 2014; Bernard et al., 2016; Blanchet, 2018, 2019] for the Hadamard and dimensional regularization.

Contents

1 Introduction

We consider here the Multipolar Post-Newtonian/Post Minkowskian formalism. Close to the Compact Binary Coalescence (**CBC**) (the *source zone*), relativistic effects are negligible and a *Post-Newtonian* (**PN**) expansion is valid. Far away from the CBC, (the *exterior zone*), the absence of source allows us to use the *Post-Minkowskian* (**PM**) formalism. To grasp the former property, let us consider two length scales of the system: the distance d between the two bodies and the typical wavelength of a Gravitational Wave (**GW**) $\lambda_{\rm GW} = 2\pi c/\omega_{\rm GW}$. It is easy to see that

$$\lambda_{\rm GW} \sim \frac{c}{\omega_{\rm GW}} \sim \frac{c}{\omega_s} \sim \frac{c}{v} d \sim \frac{1}{\beta} d \tag{1.1}$$

where $\omega_s = \omega_{\rm GW}/2$ is the angular frequency of the sources and $\beta = v/c$ is the relativistic beta.
For non-relativistic systems, $\beta \to 0$ and $\lambda_{\rm GW} \gg d$: the source is much smaller than the typical GW wavelength. On the contrary, when $\beta \to 1$, $\lambda_{\rm GW} \sim d$. In the first case, d and $\lambda_{\rm GW}$ provide us with two distinct length scales. We can consider two asymptotic regions: the source $r \sim d$ and the exterior $r \sim \lambda_{\rm GW}$. In the source region (or the interior), one solves a Poisson-type equations and expands in β: this is the PN expansion. In the wave-zone (or the exterior), one solves a wave equation and expands in G_N: this is the PM expansion. In the overlap region where $d \ll r \ll \lambda_{\rm GW}$, both solutions must correspond. To enforce the matching condition, the field is decomposed in terms of spherical harmonics.

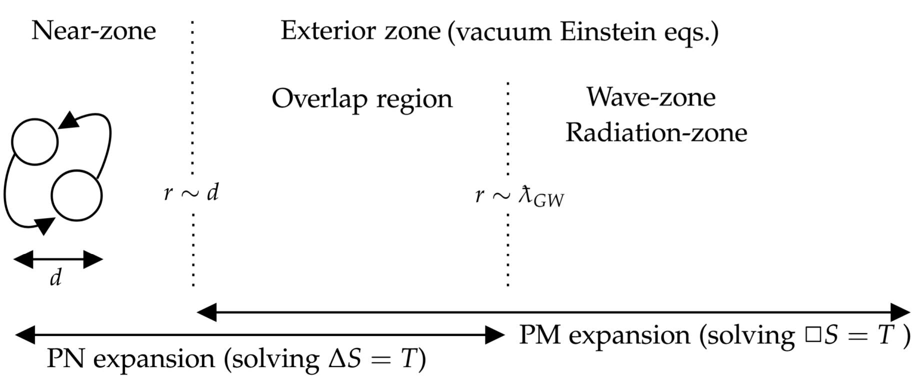

Figure 1: Illustration of the different regimes in the PN/PM formalism. From [Compère, 2024].

In this work, we will focus ourselves on the treatment of the source zone and the regularizations associated.

2 Near-zone PN expansion

2.1 Einstein Field Equations

General Relativity's (**GR**) dynamics can be derived from a variational principle, where the action is given by the Einstein-Hilbert action, traditionally expressed in terms of the metric $g_{\mu\nu}$ and its first and second derivatives: $S_{\rm EH}[g, \partial g, \partial^2 g]$. It is possible to rewrite it in terms of the metric and its first derivatives only, giving the *Landau-Lifshitz* form of the gravitational action $S_{\rm LL}[g, \partial g]$. One introduces the *densitized metric* (or gothic metric) $G^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ to write the Landau-Lifshitz action $S_{\rm LL}[G, \partial G]$ solely in terms of the (densitized) metric (expressing the Christoffel symbols in terms of the metric and its derivatives: $\Gamma = \Gamma[G, \partial G]$). To continue further, one can pick a gauge choice, giving a gauge-fixed action. The gauge choice considered here is the *harmonic gauge* defined by the condition:

$$\partial_\mu G^{\mu\nu} = 0 \tag{2.1}$$

With the condition eq. (2.1), we can add a new vanishing term to the action. The simplest way to do so is to square the harmonic gauge condition to obtain Lorentz-invariant term:

$$S_{\rm g.f.} = -\frac{c^3}{16\pi G_{\mathcal{N}}} \int {\rm d}^4 x \frac{1}{2} G_{\mu\nu} \partial_\rho G^{\mu\rho} \partial_\sigma G^{\nu\sigma} \tag{2.2}$$

Note that another way to obtain eq. (2.2) is to use similar techniques as in Quantum Field Theory (**QFT**). We must emphasize that the following derivation is purely formal, and serves as a pedagogical example, since classical GR is, as its name suggests, a *classical* field theory.

Let us consider a hypothetical path integral representation of GR, where the metric $G^{\mu\nu}$ is a dynamical field:

$$\mathcal{Z} = \int \mathcal{D}[G] \, \exp\left(i \frac{c^3}{16\pi G_{\mathcal{N}}} S[G] \right) \tag{2.3}$$

We want to impose the harmonic constraint eq. (2.1) through a *gauge-fixing term* in the action. Traditionally, one introduces a functional delta function that enforces the constraint, making a Lagrange multiplier appear, then completing the square and integrating out the multiplier field.

$$\delta[\partial_\mu G^{\mu\nu}] = \mathcal{N} \int \mathcal{D}[B] \, \exp\left(i \frac{c^3}{16\pi G_{\mathcal{N}}} \int {\rm d}^4 x \, B_\nu \partial_\mu G^{\mu\nu} \right) \tag{2.4}$$

where $B_\nu(x)$ is a Lagrange-multiplier field and $\mathcal{N}$ is the normalization. Following the standard QFT trick, we can always choose to insert a Gaussian integral over the Lagrange multiplier:

$$\int \mathcal{D}[B] \, \exp\left(i \frac{c^3}{16\pi G_{\mathcal{N}}} \int {\rm d}^4 x \, \frac{-1}{2} B_\nu B^\nu \right) = \tilde{\mathcal{N}} \tag{2.5}$$

Eq. (2.4) becomes:

$$\delta[\partial_\mu G^{\mu\nu}] = \mathcal{N}' \int \mathcal{D}[B] \, \exp\left(i \frac{c^3}{16\pi G_{\mathcal{N}}} \int {\rm d}^4 x \, \left\{ B_\nu \partial_\mu G^{\mu\nu} - \frac{1}{2} B_\nu B^\nu \right\} \right) \tag{2.6}$$

Completing the square, we get:

$$B_\nu \partial_\mu G^{\mu\nu} - \frac{1}{2} B_\nu B^\nu = -\frac{1}{2} \left(B_\nu - \partial^\mu G_{\mu\nu} \right)^2 + \frac{1}{2} (\partial^\mu G_{\mu\nu})^2 \tag{2.7}$$

We can now shift the auxiliary field $B_\nu \to \tilde{B}_\nu \equiv B_\nu - \partial^\mu G_{\mu\nu}$ and integrate over $\tilde{B}_\nu$. We are left with:

$$\delta[\partial_\mu G^{\mu\nu}] = \mathcal{N}'' \int \mathcal{D}[\tilde{B}] \, \exp\left(i \frac{c^3}{16\pi G_{\mathcal{N}}} \int {\rm d}^4 x \, \frac{1}{2} \partial_\alpha G^{\alpha\nu} \partial_\beta G^{\beta}{}_{\nu} \right) \tag{2.8}$$

We recover the harmonic gauge-fixing term eq. (2.2).

The gauge-fixed action in the Landau-Lifshitz form expressed in terms of the densitized metric $G^{\mu\nu}$ reads:

$$S = S_{\rm LL} + S_{\rm g.f.} + S_{\rm matter} \tag{2.9}$$

$$= \frac{c^3}{32\pi G_{\mathcal{N}}} \int \mathrm{d}^4x \left\{ -\frac{1}{2}\left(G_{\mu\rho}G_{\nu\sigma} - \frac{1}{2}G_{\mu\nu}G_{\rho\sigma}\right) G^{\lambda\tau}\partial_\lambda G^{\mu\nu}\partial_\tau G^{\rho\sigma} \right.$$
$$\left. + G_{\mu\nu}\left(\partial_\rho G^{\mu\sigma}\partial_\sigma G^{\nu\rho} - \partial_\rho G^{\mu\rho}\partial_\sigma G^{\nu\sigma}\right) \right\} + S_{\rm matter} \tag{2.10}$$

Varying the action with respect to G gives the Equations of Motion (**EOM**):

$$\begin{cases} \Box G^{\mu\nu} \equiv G^{\rho\sigma}\partial_\rho\partial_\sigma G^{\mu\nu} = \dfrac{16\pi G_{\mathcal{N}}}{c^4}|g|T^{\mu\nu} + \Sigma^{\mu\nu}[G, \partial G] & \text{EOM} \quad (2.11\text{a}) \\ \nabla_\mu T^{\mu\nu} = 0 & \text{Bianchi} \quad (2.11\text{b}) \end{cases}$$

where $T^{\mu\nu} = 2\delta S_{\rm matter}/(\sqrt{-g}\delta g_{\mu\nu})$ is the stress-energy tensor, $\Sigma^{\mu\nu}$ is a source term that accounts for the non-linearities of the EFE. We can now expand the metric around a flat background: $G^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu *}$, so that the EFE become:

$$\partial^2 h^{\mu\nu} = \frac{16\pi G_{\mathcal{N}}}{c^4} T^{\mu\nu} + \Lambda^{\mu\nu} \tag{2.12}$$

where $\partial^2 \equiv \eta\cdot\partial\cdot\partial$ and $\Lambda^{\mu\nu} = \Sigma^{\mu\nu} - h^{\rho\sigma}\partial_\rho\partial_\sigma h^{\mu\nu} \sim \mathcal{O}(h^2)^\dagger$. Thanks to the separation of scale argued in sec. 1, solving the EFE eq. (2.12) can be done using perturbative expansions: in powers of $G_{\mathcal{N}}$ in the PM formalism, or in powers of β in the PN formalism.

2.2 Post-Newtonian expansion

Perturbative expansion of the metric and stress-energy tensor

In the source zone where the PN expansion is valid, the perturbations of the metric $h^{\mu\nu}$ and of the stress-energy tensor are written as:

$$\overline{h}^{\mu\nu} \equiv h^{\mu\nu}|_{\rm interior} = \sum_{n=2}^{\infty} \frac{1}{\beta^{-n}} \overline{h}^{\mu\nu}_{(n)} \tag{2.13}$$

$$\overline{\tau}^{\mu\nu} \equiv \left(\frac{16\pi G_{\mathcal{N}}}{c^4} T^{\mu\nu} + \Lambda^{\mu\nu} \right)\Bigg|_{\rm interior} = \sum_{n=-2}^{\infty} \frac{1}{\beta^{-n}} \overline{\tau}^{\mu\nu}_{(n)} \tag{2.14}$$

We denote the level of the expansion as

$$n\text{PN} = \mathcal{O}\left(\beta^{2n}\right) \tag{2.15}$$

Splitting the flat wave operator ∂^2 into its temporal and spatial components, we can rewrite the EFE eq. (2.12) at a given order n as:

$$\Delta \overline{h}^{\mu\nu}_{(n)} = 16\pi G_{\mathcal{N}} \overline{\tau}^{\mu\nu}_{(n-4)} + \partial_t^2 \overline{h}^{\mu\nu}_{(n-2)} \tag{2.16}$$

The EFE are then solved iteratively:

$$\begin{cases} \Delta \overline{h}^{\mu\nu}_{(n)} = 16\pi G_{\mathcal{N}} \overline{\tau}^{\mu\nu}_{(n-4)}\left[\overline{h}_{(1)}, \ldots, \overline{h}_{(n-5)}\right] + \partial_t^2 \overline{h}^{\mu\nu}_{(n-2)}\left[\overline{h}_{(1)}, \ldots, \overline{h}_{(n-3)}\right] & (2.17\text{a}) \\ \partial_\mu \overline{h}^{\mu\nu}_{(n)} = 0 & (2.17\text{b}) \end{cases}$$

*Note that $h^{\mu\nu}$ is not the usual $g^{\mu\nu} - \eta^{\mu\nu}$ but $G^{\mu\nu} - \eta^{\mu\nu}$.

†The explicit form of Λ can be found in sec. 2 of [Blanchet, 2014]

0PN order: the newtonian limit

The stress-energy tensor $T^{\mu\nu}$, in the newtonian limit (i.e. $\beta \ll 1$), is of order:

$$\begin{cases} T^{00} \sim \rho\, c^2 \sim \mathcal{O}(c^2) & \text{(2.18a)} \\ T^{0i} \sim \rho\, c v^i \sim \mathcal{O}(c) & \text{(2.18b)} \\ T^{ij} \sim \rho\, v^i v^j \sim \mathcal{O}(1) & \text{(2.18c)} \end{cases}$$

Thus we can neglect any term of order $\mathcal{O}(c)$ or lower. Additionally, the flat wave operator ∂^2 can be approximated by the Laplace operator Δ:

$$\partial^2 = -\frac{1}{c^2}\partial_t^2 + \Delta \approx \Delta \tag{2.19}$$

At 0PN, the EFE reduce to the Poisson equation:

$$\Delta \overline{h}^{00} = \frac{16\pi G_{\mathcal{N}}}{c^2}\rho \tag{2.20}$$

We can solve eq. (2.20) in terms of the newtonian potential $U(t,\vec{x})$, defined as the solution of the Poisson equation $\Delta U = -4\pi G_{\mathcal{N}}\rho$:

$$\overline{h}^{00}(t,\vec{x}) = -\frac{4}{c^2}U(t,\vec{x}) \tag{2.21}$$

2.3 Compact Binary Coalescence

Let us consider two bodies labeled with $A = 1, 2$ with positions $\vec{z}_1(t)$ and velocities $\vec{v}_A(t)$. For point-like bodies, the energy density is a Dirac distribution:

$$\rho(t,\vec{x})/c^2 = m_1\delta^{(3)}(\vec{x} - \vec{z}_1(t)) + m_2\delta^{(3)}(\vec{x} - \vec{z}_2(t)) \tag{2.22}$$

Now recall that the Green function of the Laplace operator in $d = 3$ dimension (such that $\Delta G = \delta$) is given by

$$G(r) = -\frac{1}{4\pi}\frac{1}{r} \tag{2.23}$$

Solving eq. (2.20) with the source eq. (2.22) gives

$$U(t,\vec{x}) = \frac{G_{\mathcal{N}}m_1}{r_1} + \frac{G_{\mathcal{N}}m_2}{r_2} \tag{2.24}$$

To obtain the force on the body A, we simply compute

$$-\vec{F}(\vec{x}) = \partial_i U \tag{2.25a}$$

$$= \left(\frac{-G_{\mathcal{N}}m_1}{r_1^2}\vec{n}_1 - \frac{G_{\mathcal{N}}m_2}{r_2^2}\vec{n}_2\right) \tag{2.25b}$$

We want to know the force acting on the body A at position $\vec{z}_A(t)$, but clearly eq. (2.25) diverges at $\vec{x} = \vec{z}_A(t)$: we need to *regularize* the potential.

2.4 Hadamard regularization

Let $F(\vec{x})$ be a function singular in $\vec{z}_1$ and $\vec{z}_2$ and smooth otherwise that admits an expansion in powers of the distance to the singular points (say r_1 and r_2, with $r_A \equiv |\vec{x} - \vec{z}_A|$):

$$F(\vec{x}) \underset{\lim \vec{x}\to\vec{z}_A}{=} \sum_{p_0 \le p \le P} {r_A}^p {f^A}_p(\vec{n}_A) + \mathcal{O}\left(r_A^P\right) \qquad \forall P \in \mathbb{N}, \forall A \in \{1,2\} \tag{2.26}$$

where $\vec{n}_A = (\vec{x} - \vec{z}_A)/r_A$ is the (unit) direction of approach to the singularity $\vec{z}_A$. Two notions of *Hadamard partie finie* can be defined; the first one associated with the function $F(\vec{x})$ itself, the second one associated with its integral.

Definition 2.1. The Hadamard partie finie of $F(\vec{x})$ at z_A is defined as

$$\langle F \rangle_1 \equiv \int \frac{\mathrm{d}\Omega_A}{4\pi} {f^A}_0(\vec{n}_A) \tag{2.27}$$

It is the average of the finite part coefficient ${f^A}_0(\vec{n}_A)$ over the solid angle $\mathrm{d}\Omega_A = \sin(\theta_A)\mathrm{d}\theta_A\mathrm{d}\phi_A$ around the direction of approach $\vec{n}_A$. The Hadamard partie finie is illustrated in fig. 2.

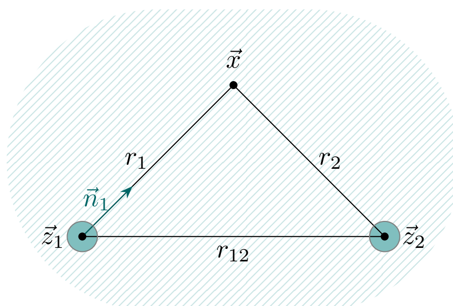

Figure 2: Illustration of the Hadamard regularization in the case $A = 1$. The two singular points are $\vec{z}_1$ and $\vec{z}_2$. The field point $\vec{x}$ is at distance r_A from $\vec{z}_A$.

Let us consider a simple example with two functions:

$$F(\vec{x}) = \frac{1}{r_1} + \vec{a}\cdot\vec{n}_1 \text{ and } G(\vec{x}) = \frac{1}{r_1} + \vec{b}\cdot\vec{n}_1 \tag{2.28a}$$

where $\vec{a}$ and $\vec{b}$ are constant vectors. The first term of F and G diverges at $\vec{z}_1$, but the second term is smooth. Expanding

$$F \sim \sum {r_1}^p {f^1}_p \text{ and } G \sim \sum {r_1}^p {g^1}_p \tag{2.28b}$$

we identify ${f^1}_{-1} = {g^1}_{-1} = 1$, ${f^1}_0 = \vec{a}\cdot\vec{n}_1$ and ${g^1}_0 = \vec{b}\cdot\vec{n}_1$. Since $\langle \vec{n}_1 \rangle_1 = 0$, we have $\langle F \rangle_1 = \langle G \rangle_1 = 0$, but

$$\langle FG \rangle_1 = a_i b_j \langle {n_1}^i {n_1}^j \rangle_1 = \frac{1}{3}\vec{a}\cdot\vec{b} \neq 0 = \langle F \rangle_1 \langle G \rangle_1 \tag{2.28c}$$

Note that for

$$\langle G(\vec{x}) \rangle_1 = \begin{cases} \langle F(\vec{x}) \rangle_1 & = \langle {f^1}_{(0)} \rangle & (2.29a) \\ \left\langle \dfrac{F(\vec{x})}{r_1^p} \right\rangle_1 & = \langle {f^1}_{(p)} \rangle & (2.29b) \\ \langle r_1^p F(\vec{x}) \rangle_1 & = \langle {f^1}_{(-p)} \rangle & (2.29c) \end{cases}$$

In the example above, we see that the Hadamard partie finie is in general not distributive over the product of functions:

$$\langle FG \rangle_A \neq \langle F \rangle_A \langle G \rangle_A \tag{2.30}$$

This is a serious problem: the variation of the field will not commute with the Hadamard regularization:

$$\langle \delta(FG) \rangle_A - \delta\left(\langle FG \rangle_A\right) \neq 0 \tag{2.31}$$

This is not a problem at order 2.5PN or below, but at 3PN and above, the metric includes terms representing the GW scattering off themselves, which require a stronger regularization method.
At 0PN, the potential is given by eq. (2.24) so that around $\vec{z}_1$, $f_0(\vec{n}_1)$ is a constant:

$$\langle U\rangle_1 = \frac{G_N m_2}{r_{12}} \text{ and } \langle h^{00}\rangle_1 = -\frac{4G_N m_2}{c^2\, r_{12}} \tag{2.32}$$

When computing higher order terms, we will need to consider the Hadamard partie finie of the integral of $F(\vec{x})$ over the whole space $\mathbb{R}^3$.

Definition 2.2. The Hadamard *partie finie* of the integral of $F(\vec{x})$ over the whole space $\mathbb{R}^3$ is defined as

$$\underset{s_1,s_2}{\mathrm{Pf}} \int_{\mathbb{R}^3} \mathrm{d}^3x\, F(\vec{x}) \equiv \lim_{s\to 0}\left(\int_{\mathcal{D}(s)} \mathrm{d}^3x\, F + 4\pi \ln\left(\frac{s}{s_1}\right)\langle {r_1}^3 F\rangle_1 + 4\pi \sum_{p+3<0} \frac{s^{p+3}}{p+3}\left\langle \frac{F}{{r_1}^p}\right\rangle_1 + \{1 \leftrightarrow 2\}\right) \tag{2.33}$$

where $\mathcal{D}(s) = \mathbb{R}^3 \backslash \mathcal{B}(z_1, s) \cup \mathcal{B}(z_2, s)$ is the domain of integration, s_1 and s_2 are two cut-off parameters, and $\mathcal{B}(z_A, s)$ is a ball of radius s centered at $\vec{z}_A$.

Def. 2.2 can be understood as follows:

- When integrating, we subtract from the domain of integration balls containing the singularities (hence the domain $\mathcal{D}(s)$). This gives the first term of eq. (2.33).
- Let us now consider the pathological part for a fixed p; inside $\mathcal{B}(z_A, s)$:

$$\int_{\mathcal{B}} \mathrm{d}^3x\, r^p f_p = \int_0^s r^2 \mathrm{d}r \int \mathrm{d}\Omega\, r^p f_p = 4\pi\langle f_p\rangle \int_0^s r^{p+2}\mathrm{d}r = 4\pi\langle f_p\rangle \cdot \begin{cases} \dfrac{s^{p+3}}{p+3} & \text{if } p+2 \neq -1 \quad (2.34\text{a}) \\ \ln(r) & \text{if } p+2 = -1 \quad (2.34\text{b}) \end{cases}$$

  - If $p+2 = -1$, the integral diverges logarithmically, and we need to subtract it (hence the second term of eq. (2.33)).
  - If $p+2 > -1$, the term in the integral $\sim s^{p+3}$ vanishes in the limit $s \to 0$.
  - If $p+2 < -1$, the integral diverges polynomially, and we need to subtract it (hence the third term of eq. (2.33)).

Note that $\underset{s_1,s_2}{\mathrm{Pf}}$ depends on the cut-off parameters $s_A \in \mathbb{R}^+$ which eventually will lead to problematic behavior of the fields.
Let us now use the Hadamard regularization to solve a Poisson equation $\Delta P = F$ where F admits the same expansion as in eq. (2.26):

$$P(\vec{x}') = -\frac{1}{4\pi} \underset{s_1,s_2}{\mathrm{Pf}} \int_{\mathbb{R}^3} \mathrm{d}^3x \frac{F(\vec{x})}{|\vec{x}-\vec{x}'|} \tag{2.35}$$

where $\vec{x}'$ is the field point distinct from the singularities of F as illustrated in fig. 2. To evaluate eq. (2.35) when $\vec{x}' \to \vec{z}_1$ (when $r_1' \equiv |\vec{x}' - \vec{z}_1| \to 0$), it is not enough to use the Hadamard partie finie of the integral (def. 2.2), we need to dress it with another counterterm that will cancel the logarithmic divergence of the integral.

$$\langle P\rangle_{\substack{1 \\ r_1',s_2}} = -\frac{1}{4\pi} \underset{s_1,s_2}{\mathrm{Pf}} \int \mathrm{d}^3x \frac{F(\vec{x})}{r_1} + \left\{\ln\left(\frac{r_1'}{s_1}\right) - 1\right\}\left\langle r_1^2 F\right\rangle_1 \tag{2.36}$$

The demonstration of the above result eq. (2.36) is a bit technical and can be found explicitly in the proof of theorem 3 of [Blanchet and Faye, 2000]. Note that the regularized Poisson integral is independent of s_1 since

$$\underset{s_1,s_2}{\mathrm{Pf}} \int \mathrm{d}^3x \frac{F(\vec{x})}{r_1} \ni 4\pi \ln\left(\frac{s}{s_1}\right)\left\langle r_1^2 F\right\rangle_1 \tag{2.37}$$

2.5 Dimensional regularization

As stated before, the ambiguity of the Hadamard regularization becomes pathological at 3PN and above. A priori, the partie finie of the Poisson integral eq. (2.36) contains 4 parameters: s_1, s_2, r'_1 and r'_2. However, it can be shown that s_1 actually cancels out from the two terms on the RHS of eq. (2.36) and that r'_1 and r'_2 can be removed by a coordinate transformation. This leaves us with only one parameter: s_2 if we compute $\langle P \rangle_1$ (or s_1 if we compute $\langle P \rangle_2$). Luckily, Dimensional Regularization (**dimreg**) provides an elegant way to remove this ambiguity, in addition to keeping the diffeomorphism invariance while solving the EFE.

In dimreg, one works in spacetime with $D = d + 1$ dimensions, where d is arbitrary and *complex*: $d \in \mathbb{C}$. One can thus use all the tools of complex analysis, and in particular analytic continuation of functions. This will define our expressions for any values of d, except at poles. We thus obtain a well defined quantity in the vicinity of the physical dimension, as illustrated in fig. 3.

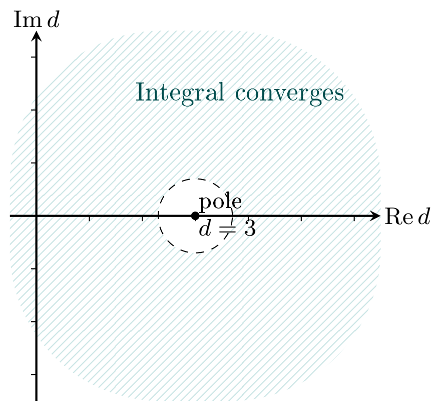

Figure 3: Illustration of the convergence of the complex analytical continuation of a diverging integral in dimension $d = 3$.

The Einstein Field Equations keep the same structure:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G_{\mathcal{N}}^{(d)}}{c^4} T^{\mu\nu} \tag{2.38}$$

where we introduce $G_{\mathcal{N}}^{(d)} = G_{\mathcal{N}} l_0^{d-3}$ and l_0^{d-3} is a constant length associated with the dimreg. One can rewrite the EFE as

$$R^{\mu\nu} = \frac{8\pi G_{\mathcal{N}}^{(d)}}{c^4} \left(T^{\mu\nu} - \frac{1}{d-1} g^{\mu\nu} T \right) \tag{2.39}$$

Keeping the same harmonic gauge condition eq. (2.1), the EFE become:

$$\partial^2 h^{\mu\nu} = \frac{16\pi G_{\mathcal{N}}^{(d)}}{c^4} |g| T^{\mu\nu} + \Lambda^{\mu\nu} \tag{2.40}$$

where $\partial^2 \equiv \eta^{(d)} \cdot \partial \cdot \partial$, and $\Lambda^{\mu\nu}$ depends explicitly on d. Since we focus on the physical dimension $d = 3$, we will denote $\epsilon \equiv d - 3$ and expand around it. As before, obtaining the EOM requires solving Poisson integrals iteratively. In d-dimension, the Green function $G^{(d)}$ of the Laplace operator Δ (such that $\Delta G^{(d)} = -4\pi\delta^{(d)}$) is given by:

$$G^{(d)}(r) = k^{(d)} r^{2-d} \qquad \text{with } k^{(d)} = \frac{4\pi}{d-2} \frac{1}{\Omega^{(d-1)}} \tag{2.41}$$

where $\Omega^{(d-1)} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the unit sphere in $d - 1$ dimensions.

Indeed, for a purely radial function $R(r)$, the Laplace operator in d-dimension reads:

$$\Delta R(r) = R'' + \frac{d-1}{r} R' \tag{2.42a}$$

For $R(r) = R_0 r^{2-d}$, its Laplace operator vanishes identically for $r \neq 0$. To determine R_0, we reduce the domain of integration to a ball of radius s centered at the origin:

$$\begin{aligned}
-4\pi\delta_d(r) &= \int_{\mathcal{R}^d} \mathrm{d}^d x\ \Delta R(r) && (2.42\text{b})\\
&= \int_{\mathcal{B}(s)} \mathrm{d}^d x\ \Delta R(r) && (2.42\text{c})\\
&= \oint_{\partial\mathcal{B}(s)} \vec{\nabla} R \cdot \mathrm{d}\vec{S} && (2.42\text{d})\\
&= \int R_0 (2-d) r^{1-d} \cdot r^{d-1} \mathrm{d}\Omega^{(d-1)} && (2.42\text{e})\\
&= R_0 (2-d) \Omega^{(d-1)} && (2.42\text{f})
\end{aligned}$$

We thus find $R_0 = -4\pi/((2-d)\Omega^{(d-1)})$.

For point-like sources, the density is singular at the position of the source, say $\vec{z}_A$, and one must perform a double Laurent expansion, when $\vec{x}$ approaches $\vec{z}_A$, and when the dimension d approaches 3:

$$F^{(d)}(\vec{x}) \underset{\substack{\lim \vec{x}\to\vec{z}_A \\ \lim d\to 3}}{=} \sum_{\substack{p_0 \le p \le P \\ q_0 \le q \le Q}} r_1^{p+q\epsilon} f_{p,q}^{1(\epsilon)}(\vec{n}_1) + \mathcal{O}\left(r_1^P\right) + \mathcal{O}\left(r_1^{Q\epsilon}\right) \tag{2.43}$$

The Green function in d-dimension eq. (2.41) is precisely the above expression eq. (2.43) with $(p,q) = (-1,-1)$ and $f_{-1,-1}^{1(\epsilon)}(\vec{n}_1) = k^{(d)}$.

The Hadamard partie finie of d-dimensional function $F^{(d)}(\vec{x})$ at $\vec{z}_A$ is a straightforward generalization of def. 2.2:

$$\left\langle f_{p,q}^{1(\epsilon)} \right\rangle_1 = \int \mathrm{d}\Omega_1^{(d-1)} f_{p,q}^{1(\epsilon)}(\vec{n}_1) \tag{2.44}$$

We change a bit the convention in eq. (2.44) since the Hadamard partie finie is not normalized anymore:

$$\langle \mathbf{1} \rangle_1 = \Omega_1^{(d-1)}$$

Note that until 4PN, there is no singular behavior when $\epsilon \to 0$, so that

$$\sum_{q_0}^{q_1} f_{p,q}^{1(\epsilon=0)}(\vec{n}_1) = f_p^1(\vec{n}_1) \tag{2.45}$$

Solving $\Delta P^{(d)} = F^{(d)}$ gives:

$$P^{(d)}(\vec{x}') = \Delta^{-1} F^{(d)}(\vec{x}') = -\frac{k^{(d)}}{4\pi} \int \mathrm{d}^d x \frac{F^{(d)}(\vec{x})}{|\vec{x}-\vec{x}'|^{d-2}} \tag{2.46}$$

The evaluation of $P^{(d)}$ in the vicinity of the singularity $\vec{z}_A$ is straightforward:

$$P^{(d)}(\vec{z}_1) = -\frac{k^{(d)}}{4\pi} \int \mathrm{d}^d x \frac{F^{(d)}(\vec{x})}{r_1^{d-2}} \tag{2.47}$$

Let us analyze the behavior of eq. (2.47). The volume element $\mathrm{d}^d x$ in spherical coordinates reads $\mathrm{d}^d x = r^{d-1} \mathrm{d}r \mathrm{d}\Omega^{(d-1)}$. As in def. 2.2, we split the integral into $\mathbb{R}^d \equiv \mathcal{D}(z_A, s_A) \cup \mathcal{B}(z_1, s) \cup \mathcal{B}(z_2, s)$. In

$\mathcal{D}(z_A, s_A)$, the integral is well-defined and converges. The self-interaction term in $\mathcal{B}(z_1, s)$ gives:

$$P^{(d)}(z_1)\Big|_{\mathcal{B}(z_1,s)} = \frac{k^{(d)}}{4\pi}\int_{\mathcal{B}(z_1,s)} r^{d-1}\mathrm{d}r\mathrm{d}\Omega^{(d-1)}\sum_{p,q} r^{p+q\epsilon} f^{1(\epsilon)}_{p,q}(\vec{n}_1) r^{2-d} \tag{2.48a}$$

$$= \frac{k^{(d)}}{4\pi}\sum_{p,q}\left\langle f^{1(\epsilon)}_{p,q}\right\rangle_1 \int_0^s r^{p+q\epsilon+1}\mathrm{d}r \tag{2.48b}$$

$$= \frac{k^{(d)}}{4\pi}\sum_{p,q}\left\langle f^{1(\epsilon)}_{p,q}\right\rangle_1 \frac{s^{p+q\epsilon+2}}{p+q\epsilon+2} \tag{2.48c}$$

$$= \frac{k^{(d)}}{4\pi}\sum_{q}\left\langle f^{1(\epsilon)}_{-2,q}\right\rangle_1 \left(\frac{1}{q\epsilon}+\ln\left(\frac{s}{s_1}\right)\right) + \text{ terms regular in } \epsilon \tag{2.48d}$$

where we used $s^{q\epsilon} = e^{q\epsilon\ln(s)} = 1 + q\epsilon\ln(s) + \mathcal{O}(\epsilon^2)$ to go from eq. (2.48c) to eq. (2.48d). The cross term in $\mathcal{B}(z_2, s)$ is better handled by first extending $r_1^{-(d-2)}$ in terms of the distance r_{12} between the two singularities $\vec{z}_1$ and $\vec{z}_2$:

$$r_1^{-(d-2)} = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left(\frac{1}{r_{12}^{d-2}}\right) r_2^\ell n_2^L \tag{2.49}$$

Recall the definition of the Taylor expansion of a smooth function $f(\vec{x})$ around a point $\vec{z}_2$ that you have probably seen in kindergarten:

$$f(\vec{x}) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}(\vec{x}-\vec{z}_2)^L\ \partial_L^{(x)} f(\vec{x})\Big|_{\vec{x}=\vec{z}_2} \tag{2.50}$$

where $L = i_1, \ldots, i_l$ is a multi-index of size ℓ, with each $i = 1, \ldots, d$. Note that

$$\partial_i^{(x)} r_1^{-(d-2)} \equiv \frac{\partial}{\partial x^i}\frac{1}{|\vec{x}-\vec{z}_1|^{d-2}} = -\frac{\partial}{\partial z_1^i}\frac{1}{|\vec{x}-\vec{z}_1|^{d-2}} \equiv -\partial_i^{(z_1)} r_1^{-(d-2)} \tag{2.51}$$

so that evaluated at $\vec{z}_2$, we have $\partial_L^{(z_2)} r_{12}^{-(d-2)} = (-1)^\ell \partial_L^{(z_1)} r_{12}^{-(d-2)}$. Rewriting $(\vec{x}-\vec{z}_2)^L = r_2^\ell n_2^L$, we recover eq. (2.49).

The cross term in $\mathcal{B}(z_2, s)$ then gives:

$$P^{(d)}(z_1)\Big|_{\mathcal{B}(z_2,s)} = \frac{k^{(d)}}{4\pi}\int_{\mathcal{B}(z_2,s)} r^{d-1}\mathrm{d}r\mathrm{d}\Omega^{(d-1)}\sum_{p,q} r^{p+q\epsilon} f^{2(\epsilon)}_{p,q}(\vec{n}_2)\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left(\frac{1}{r_{12}^{d-2}}\right) r^\ell n_2^L \tag{2.52a}$$

$$= \frac{k^{(d)}}{4\pi}\sum_{p,q}\sum_{\ell=0}^{\infty}\left\langle n_2^L f^{2(\epsilon)}_{p,q}\right\rangle_2 \frac{(-1)^\ell}{\ell!}\partial_L\left(\frac{1}{r_{12}^{d-2}}\right)\int_0^s r^{d-1+p+q\epsilon+\ell}\mathrm{d}r \tag{2.52b}$$

$$= \frac{k^{(d)}}{4\pi}\sum_{p,q}\sum_{\ell=0}^{\infty}\left\langle n_2^L f^{2(\epsilon)}_{p,q}\right\rangle_2 \frac{(-1)^\ell}{\ell!}\partial_L\left(\frac{1}{r_{12}^{d-2}}\right)\frac{s^{\epsilon(q+1)+p+\ell+3}}{\epsilon(q+1)+p+\ell+3} \tag{2.52c}$$

$$= \frac{k^{(d)}}{4\pi}\sum_{q}\sum_{\ell=0}^{\infty}\left\langle n_2^L f^{2(\epsilon)}_{-\ell-3,q}\right\rangle_2 \frac{(-1)^\ell}{\ell!}\partial_L\left(\frac{1}{r_{12}^{d-2}}\right)\left(\frac{1}{\epsilon(q+1)}+\ln\left(\frac{s}{s_2}\right)\right) + \text{ terms regular in } \epsilon \tag{2.52d}$$

Collecting terms from equations (2.48) and (2.52), we find that the dimreg of the Poisson integral eq. (2.47) reads:

$$\begin{aligned}\left\langle P^{(d)}\right\rangle_1^{\mathrm{DR}} = -\frac{k^{(d)}}{4\pi}\Bigg\{ &\int \mathrm{d}^d x\frac{F^{(d)}(\vec{x})}{r_1^{d-2}} \\ &- \sum_q \left\langle f^{1(\epsilon)}_{-2,q}\right\rangle_1 \left(\frac{1}{q\epsilon}+\ln\left(\frac{s}{s_1}\right)\right) \\ &- \sum_q\sum_{\ell=0}^{\infty}\left\langle n_2^L f^{2(\epsilon)}_{-\ell-3,q}\right\rangle_2 \frac{(-1)^\ell}{\ell!}\partial_L\left(\frac{1}{r_{12}^{d-2}}\right)\left(\frac{1}{\epsilon(q+1)}+\ln\left(\frac{s}{s_2}\right)\right)\Bigg\} \\ &+ \text{ terms regular in } \epsilon\end{aligned} \tag{2.53}$$

We are now interested in the correction to the Hadamard regularization (given in eq. (2.36)), which can be obtained by subtracting the (d=3) Hadamard regularization from the dimreg in arbitrary d-dimension.

$$\mathcal{D}P_A \equiv \left\langle P^{(d)} \right\rangle_A^{\mathrm{DR}} - \langle P \rangle_A \tag{2.54}$$

The regularization correction of the Poisson integral $\mathcal{D}P_1$ is given by:

$$\begin{aligned}\mathcal{D}P_1 = -\frac{k^{(d)}}{4\pi} \Bigg(& \sum_{q_0 \leq q \leq q_1} \left(\frac{1}{q\epsilon} + \ln(r_1') - 1 \right) \left\langle f^{1(\epsilon)}_{-2,q} \right\rangle_1 \\ & + \sum_{q_0 \leq q \leq q_1} \left(\frac{1}{(q+1)\epsilon} + \ln(s_2) \right) \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left(\frac{1}{r_{12}^{1+\epsilon}} \right) \left\langle n_2^L f^{2(\epsilon)}_{-\ell-3,q} \right\rangle_2 + \mathcal{O}(\epsilon) \Bigg)\end{aligned} \tag{2.55}$$

Crucially, eq. (2.55) depends on the parameters s_2 and r_1' in the such a way to perfectly cancel out the corresponding terms in eq. (2.36). The last subtlety is the dependence of the expansion coefficients $f^{A(\epsilon)}_{p,q}$ in the dimreg length l_0 appearing in the d-dimensional EFE eq. (2.38). It can be shown that taking the l_0-dependence explicitly leads to the modification of eq. (2.55) only in the logarithmic terms, adimensionalizing the terms $\ln(s_2/l_0)$ and $\ln(r_1'/l_0)$. However, when taking the physical limit $d \to 3$, $l_0 \to 1$ and the dimensionally regularized result is independent of arbitrary parameters.

3 Conclusion

We have seen that the Hadamard regularization offers a practical but not fully satisfactory way to deal with the singularities of the gravitational potential in the vicinity of point-like sources. The Hadamard partie finie is not distributive over the product of functions, which leads to a non-commutativity of the variation and the regularization. This is not a problem at 2.5PN and below, but at 3PN and above, we need a stronger regularization method. Dimensional Regularization offers a parameter-free way to deal with the singularities of the gravitational potential, while keeping the diffeomorphism invariance of the EFE.

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