Problem Set III: Quantum strings **PHYS-F483**

February 18, 2025

1 Light-cone quantisation

There are a few ways to quantise bosonic string theory. We will choose the more pedestrian approach of *light-cone quantisation*. By working on the light-cone, we fix a gauge, which effectively destroys Lorentz invariance. Starting from a Lorentz invariant theory *does not guarantee* Lorentz invariance through the quantisation procedure, since *anomalies*[1] can occur. Let us nevertheless carry on with this approach: when we are done, we will take a look at what Lorentz has to say.

First, let us discuss a *residual* symmetry that has not been fixed yet by going to the conformal gauge. Remember that on the worldsheet, in light-cone coordinates,

$$ds^2 = -d\sigma^+ d\sigma^-. \tag{1}$$

Then, any transformation of the light-cone coordinates

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+), \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-) \tag{2}$$

will leave the metric unchanged up to a factor that can be reabsorbed by a Weyl transformation. Our goal will be now to exploit this residual gauge freedom. To this effect, introduce the light-cone coordinates *in target space*

$$X^\pm = \sqrt{\frac{1}{2}}\left(X^0 \pm X^{D-1}\right). \tag{3}$$

At this point, we break Lorentz invariance, as previewed. In these coordinates, the target space metric is

$$ds^2 = -2dX^+ dX^- + \sum_{i=2}^{D-2} dX^i dX^i. \tag{4}$$

For two target space vectors A and B

$$A_+ = -A^-, \;\; A_- = -A^+, \;\; A \cdot B = -A^+B^- - A^-B^+ + A^iB^i. \tag{5}$$

Using the residual reparametrisation invariance (2), one can set

$$X^+(\tau, \sigma) = x^+ + \alpha' p^+ \tau, \tag{6}$$

[1] Anomalies refer to the breaking of a classical symmetry of the theory when it is quantised. See for example Schwartz, *Quantum Field Theory and the Standard Model*, Ch. 30, for a beautiful introduction.

or

$$X_L^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^+, \quad X_L^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^-. \tag{7}$$

The magic of the light-cone gauge is that it determines X^- by the constraints only. Remember that the constraints read, in the worldsheet light-cone coordinates,

$$(\partial_+ X^\mu)^2 = 0 = (\partial_- X^\mu)^2. \tag{8}$$

The first constraint is

$$2\partial_+ X^+ \partial_+ X^- = \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i, \tag{9}$$

that is

$$\partial_+ X_L^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i. \tag{10}$$

Similarly,

$$\partial_+ X_R^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i. \tag{11}$$

Consider the ansatz we have used in the previous problem set:

$$\begin{aligned} X_L^-(\sigma^+) &= \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^- e^{-in\sigma^+}, \\ X_R^-(\sigma^-) &= \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^- e^{-in\sigma^-}. \end{aligned}$$

In that case, x^- is the integration constant arising from the constraints, while the momentum and the mode coefficients are completely fixed.

Problem 1.1. *a) Find the expression for the modes α_n^- and $\tilde{\alpha}_n^-$ from the constraints.*

b) Derive the classical level matching conditions in the light-cone coordinates and gauge.

This problem shows that the most general classical solution is described by $2(D-2)$ transverse oscillators. Let us now discuss the quantisation of the closed bosonic string. Remember that

$$[\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{n+m,0}. \tag{12}$$

By defining

$$a_n^\dagger = \frac{\alpha_{-n}}{\sqrt{n}}, \quad a_n = \frac{\alpha_n}{\sqrt{n}}, \qquad \text{for } n > 0, \tag{13}$$

the commutation relations become the usual ladder operator relations

$$[a_n^i, a_m^{j\dagger}] = \delta^{ij}\delta_{mn}, \quad m, n > 0. \tag{14}$$

We define a vacuum state $|0; p\rangle$ such that

$$\hat{p}^\mu |0;\, p\rangle = p^\mu |0;\, p\rangle, \quad a_n^i |0;\, p\rangle = \tilde{a}_n^i |0;\, p\rangle = 0. \tag{15}$$

Then, we generate the Fock space by acting on the vacuum with the creation operators $a_m^{i\dagger}$ and $\tilde{a}_m^{i\dagger}$. The crucial difference with the "old covariant" way of quantising the theory now becomes manifest. Had we not worked in the light-cone gauge, states could have been generated by $a_0^{\mu\dagger}$. Then, one would find

$$\langle 0;\, p|\, a_0^\mu a_0^{\mu\dagger}\, |0;\, p\rangle = \langle 0;\, p|\, [a_0^\mu, a_0^{\mu\dagger}]\, |0;\, p\rangle = -1, \tag{16}$$

i.e. negative-norm states. In the old covariant way, one has to take care of these states, *ghosts*, to make sure that they effectively do not appear in the theory.

Problem 1.2. *The last problem set discussed ordering ambiguity for the Virasoro generators. Show that the quantum mass formula for the closed string states is given by*

$$M^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\tilde{N} - a), \tag{17}$$

where

$$N = \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i \tag{18}$$

and likewise for $\tilde{N}$. N is called the level, hence the name level-matching.

In the previous problem set, we discussed an additional parameter of the theory: the central extension of the Virasoro algebra. In the light-cone gauge, determining a and D, the so-called *critical dimension*, is not an easy feat, on the contrary. The path we now take is a shortcut - far away from rigour.

Let us remind ourselves of the representation theory of the Poincaré group. For massive particles, it is always possible to find a reference frame in which the particle momentum is written as

$$p^\mu = (M, 0, ..., 0). \tag{19}$$

Then, massive particles form representations of the $SO(D-1)$ rotation group. On the other hand, one cannot boost to a frame in which a massless particle is at rest. We can however find a frame in which

$$p^\mu = (E, 0, ..., 0, E). \tag{20}$$

Thus, massless particle states form representations of the $SO(D-2)$ rotation group.

Problem 1.3. *Let us look at the first excited state of the closed bosonic string: $\tilde{\alpha}_{-1}^j \alpha_{-1}^k\, |0;\, p\rangle$.*

a) What group does this state form a representation of?

b) What is the consequence of demanding Lorentz invariance for the constant a?

Now that the value of a has been determined, let us work out the dimension of the target space D. A way to do this is to compute a explicitly by ordering the level operator manually.

Problem 1.4. *a) Write the operator*

$$\frac{1}{2} \sum_{i=1}^{D-2} \sum_{n\neq 0} \alpha_{-n}^i \alpha_n^i \tag{21}$$

in normal order.

b) Using the unorthodox shortcut $\sum_{n=1}^{\infty} n = -\frac{1}{12}$*, compute the value of the critical dimension* D*.*

2 The string spectrum

2.1 The open string

For the open string, the mass-shell condition is

$$M^2 = \frac{1}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i - a \equiv \frac{1}{\alpha'}(N-a). \tag{22}$$

when only Neumann boundary conditions are imposed, and

$$M^2 = \frac{1}{\alpha'}\left(\sum_{i=1}^{p-1} + \sum_{i=p+1}^{D-1}\right)\sum_{n>0}\alpha_{-n}^i\alpha_n^i - a \equiv \frac{1}{\alpha'}(N-a). \tag{23}$$

whenever the open string ends on a $Dp-$brane, in which case the lightcone coordinates are defined as $X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^p)$. Again, Lorentz invariance imposes $D = 26$, $a = 1$.

Problem 2.1. *Consider the most general open string state with only Neumann boundary conditions*

$$\alpha_{-m}^j\alpha_{-p}^k...\alpha_{-q}^l\,|0;\,p\rangle\,. \tag{24}$$

Show that

$$N = (m + p + ... + q). \tag{25}$$

The squared-mass of such a state is thus

$$M^2 = \frac{1}{\alpha'}\left[(m + p + ... + q) - 1\right] \tag{26}$$

Given (25), we can now analyse the open string spectrum at each level:

- The vacuum $|0;\,p\rangle$ has $N = 0$ and thus

$$M_0^2 = -\frac{1}{\alpha'}. \tag{27}$$

  It is a *tachyon.* This merely signifies the instability of the vacuum the theory is expanded around, as for the Higgs field potential, for example.

- At level $N = 1$, we either have
  - a vector boson $\alpha_{-1}^a\,|0;\,p\rangle$, with $a = 1, ..., p-1$ along the brane. This is a massless field in a vector representation of $SO(1,p)$, the Lorentz group on the Dp-brane.

– $\alpha^I_{-1}|0;p\rangle$, with $I = p+1, ..., D-1$ transverse to the brane. With respect to $SO(1,p)$, these are scalar fields. They can be interpreted as fluctuations of the brane in the transverse direction, which indicates that branes are dynamical objects. Note that from the point of view of the rotation group $SO(26-p-1)$, these states sit in a vector representation.

- At level $N = 2$, with no Dirichlet boundary conditions, we find the first massive excited states: $\alpha^i_{-2}|0;p\rangle$ and $\alpha^i_{-1}\alpha^j_{-1}|0;p\rangle$, with $\alpha' M^2 = 1$. These form $24 + (24 \times 25)/2 = 324$ states. This is the dimension of the symmetric traceless tensor representation of SO(25).

- At each level, all massive states fit into representations of SO(25), as expected.

When Dirichlet boundary conditions are involved, one should be more careful about the indices of the creation operators, i.e. if these live on the brane or not[2].

2.2 The closed string

For the closed string, we must not forget the level-matching conditions.

Problem 2.2. *Consider a state*

$$|\psi\rangle = \tilde{\alpha}^{j_1}_{-m_1}...\tilde{\alpha}^{j_q}_{-m_q}\alpha^{k_1}_{-n_1}...\alpha^{k_r}_{-n_r}|0;p\rangle\,. \tag{28}$$

Show that the level-matching conditions imply

$$N = m_1 + ... + m_q = n_1 + ... + n_r = \tilde{N} \tag{29}$$

As a result, the states of the closed string Fock space will be the following:

- Again, a tachyon with mass $\alpha' M^2 = -4$.

- By virtue of the previous exercise, the first excited state is the massless $\mathbf{24} \otimes \mathbf{24}$ representation of $SO(24)$ we have discussed in the context of Lorentz invariance. This representation is not irreducible: we have

$$\mathbf{24} \otimes \mathbf{24} = \text{traceless symmetric} \oplus \text{anti-symmetric} \oplus \text{trace}.$$

  Each of these irreps is associated to a field. Most interestingly, the traceless symmetric representation is the *graviton*: **string theory results in gravitation**. The antisymmetric irrep. corresponds to the *Kalb-Ramond* field, and the trace is the *dilaton*.

- The level-two states are the following:

$$\left(\tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1} \oplus \tilde{\alpha}^k_{-2}\right) \otimes \left(\alpha^i_{-1}\alpha^j_{-1} \oplus \alpha^k_{-2}\right)$$

  In each sector, one finds $\frac{1}{2}(24 \times 25) + 24 = \frac{1}{2}(26 \times 25) - 1$ states. This is exactly the number of states in the traceless symmetric tensor representation of SO(25), as expected. All of these states have mass $\alpha' M^2 = 4$.

[2] See D. Tong, *String Theory* (lecture notes), p. 55.

We end up with a theory in 26 dimensions with tachyons... problematic? The "issue" of the 26 dimensions is not one per se. For example, we have seen that it is possible to realise Lorentz invariance on a subset of the dimensions with D-branes. This is the basis of the so-called *brane-world* scenarios. The tachyons, however, remain a problem in bosonic string theory. If we wish to describe some model of reality, then bosonic string theory is not enough: matter is fermionic! The natural evolution of bosonic string theory is *superstring theory*, where fermions and supersymmetry are included. In this framework, the ground state tachyons are suppressed by the so-called *GSO projection*. The number of extra dimensions is also reduced: instead of $D = 26$, superstring theory predicts $D = 10$.