# Problem Set 9: String Interactions, pt. 1 **PHYS-F483**

*April 15, 2025*

# 1 The State-Operator map

In a generic QFT, states and local operators are two very different objects. Quite magically, in a CFT, there is an isomorphism between states and operators: the *state-operator map*. This is a useful conceptual pit-stop on our way to string interactions.

In the path integral formalism, different states are obtained by imposing different boundary conditions. In ordinary quantum mechanics, the propagator for a particle moving from a time and position $(\tau_i, x_i)$ to $(\tau_f, x_f)$ is given by

$$G(x_i, x_f) = \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} \mathcal{D}x \, e^{iS}. \tag{1}$$

Then, the initial state evolves according to

$$\psi_f(x_f, \tau_f) = \int dx_i \, G(x_i, x_f) \, \psi_i(x_i, \tau_i). \tag{2}$$

This is the usual story: the wavefunction is evolved by considering all paths ending at $x_f$ when $\tau = \tau_f$ weighted by the initial wavefunctions. In the language of (Euclidean) field theory, on the cylinder, this is simply

$$\Psi[\phi_f(\sigma), \tau_f] = \int \mathcal{D}\phi_i \int_{\phi(\tau_i)=\phi_i}^{\phi(\tau_f)=\phi_f} \mathcal{D}\phi \; e^{-S[\phi]} \, \Psi[\phi_i(\sigma), \tau_i]. \tag{3}$$

We now want to express this on the complex plane. Remember that time has been mapped to radial distance, so states are now defined on circles around the origin. Writing $|z| = r$, this becomes

$$\Psi[\phi_f(\sigma), \tau_f] = \int \mathcal{D}\phi_i \int_{\phi(r_i)=\phi_i}^{\phi(r_f)=\phi_f} \mathcal{D}\phi \; e^{-S[\phi]} \, \Psi[\phi_i(\sigma), r_i]. \tag{4}$$

If the initial state is brought to the far past, i.e. $z = 0$, then the integration goes from an annulus of radii $r_i$ and $r_f$ to the whole disk $|z| \leq r_f$. In (4), the initial state is now weighting the path integral at $z = 0$. This is equivalent to having a local operator inserted at the origin: each local operator $\mathcal{O}(z = 0)$ defines a different state in the theory. In a regular QFT, the infinite past corresponds to a whole spatial slice. Here, it is actually a single point. We have

$$\Psi[\phi_f(\sigma), \tau_f] = \int^{\phi(r_f)=\phi_f} \mathcal{D}\phi \; e^{-S[\phi]} \, \mathcal{O}(z = 0), \tag{5}$$

where we integrate over all field configurations within the disk, including all the values of the field at the origin.

Let us look at our favourite example, the free boson. During the last exercise session, we have described the space of states and in particular, we saw that these were created by the negative mode operators $a_m$ and $\bar{a}_m$ acting on the vacuum. First, let us see that this vacuum is created by the insertion of the identity operator. In order to make contact with the usual notations of string theory, let us set $g = \frac{1}{2\pi\alpha'}$ and $\varphi(z) \equiv X(z)$.

**Problem 1.1.** *Check that*

$$\Psi_0[X_f] = \int^{X_f(r)} \mathcal{D}X \, e^{-S[X]} \tag{6}$$

*is indeed the vacuum state with zero momentum. For this, remember that*

$$i\partial_z X(z) = \sqrt{\frac{\alpha'}{2}} \sum_n a_n z^{-n-1}. \tag{7}$$

For the excited states, we have the following:

**Problem 1.2.** *Show that $a_{-m} |0\rangle$ is equivalent to the state $|\partial^m X\rangle$.*

You may ask yourself if and how all of this also works for the open string. It does! As for how, let us remind ourselves that the worldsheet of the open string is an infinite strip with spatial coordinate $\sigma \in [0, \pi]$. As for the closed string, we can also work with the conformal map

$$z = e^{-iw}, \quad w = \sigma + i\tau. \tag{8}$$

This takes the worldsheet to the upper half-plane, defined by $\mathrm{Im}(z) \geq 0$. The endpoints of the string are mapped to the real axis whereas the infinite past still lies at the origin. Thus, the origin is on the boundary and the state-operator map now relates states to local operators defined on the boundary. A noteworthy consequence is that the open string contains less states than the closed string: for example, Neumann boundary conditions impose $\partial X = \bar{\partial} X$ on the real axis: the operators $\partial X$ and $\bar{\partial} X$ thus describe the same state, unlike what happened before for the free boson on the cylinder.

## 2 The string spectrum, again

Remember the Virasoro constraints on physical states:

$$\begin{aligned} L_n |\Psi\rangle &= 0 \qquad \text{for } n > 0 \\ L_0 |\Psi\rangle &= a |\Psi\rangle , \end{aligned} \tag{9}$$

and likewise for the left-moving generators. In the language of conformal field theory, these are nothing but highest-weight states generated by primary fields of conformal weights $(h, \bar{h}) = (a, \bar{a})$. We can determine $(a, \bar{a})$ by means of the state-operator map.

Consider a theory with a dynamical metric. We have two gauge symmetries: diffeomorphism invariance and Weyl symmetry. Respecting diffeomorphism invariance implies that operators should now be integrated over the whole worldsheet: up to a constant, the operator insertions become

$$V \sim \int d^2 z \, \mathcal{O}. \tag{10}$$

Under a rescaling, the operator $\mathcal{O}$ has to behave as a field of conformal dimensions $(1,1)$. Then, this fixes the normal ordering coefficients $a = \bar{a} = 1$. The physical states of the theory are the states created by primary fields of conformal weights $(1,1)$.

**Problem 2.1.** *In Problem 1.1, you showed that the (absolute) vacuum is equivalent to the identity. How would you give momentum to the vacuum, i.e. create a state $|0;p\rangle$? From your answer, derive the mass of the tachyon.*

**Problem 2.2.** *Consider the first excited level of the closed string: $\xi_{\mu\nu} a^{\mu}_{-1} \bar{a}^{\nu}_{-1} |0;p\rangle$. $\xi_{\mu\nu}$ is the polarisation tensor: it is traceless and symmetric for the graviton, antisymmetric for the $B_{\mu\nu}$ field, and pure trace for the dilaton. Show that these states are transversely polarised and massless.*

The same story applies to the open string, with a few differences. First, since the state-operator map now assigns states to operators on the boundary, vertex operators are defined as

$$V \sim \int ds \ \mathcal{O}(z, \bar{z}). \tag{11}$$

Second, the propagator for the free scalar field is affected by the boundary of the worldsheet. We have

$$\langle X(z) X(w) \rangle = -\frac{\alpha'}{2} \ln |z - w|^2 - \frac{\alpha'}{2} \ln |z - \bar{w}|^2. \tag{12}$$

**Problem 2.3.** *a) Check that the operator $e^{ipX}$ is primary and determine the mass of the open string tachyon.*

*b) The vertex operator for the photon is*

$$V \sim \int_{\partial \mathcal{M}} ds \ {:}\partial X^a e^{ipX}{:} \, \xi_a \tag{13}$$

*with $a = 0, ..., p$ labelling the directions obeying Neumann-Neumann boundary conditions (thus with a properly defined momentum), parallel to the brane. Show that the photon is massless and transversely polarised.*

# 3 Topology on the Worldsheet

We now begin our discussion of string interactions. You may ask, how do interactions come about in a free theory? In a generic QFT, one would add interaction terms, such as $\lambda\phi^4$. Here, interactions terms will not comply with the gauge symmetries of the action. However, there is another way to create interactions off of a theory of 26 free scalars on the worldsheet. Remember why gravity in two dimensions cannot be realised by the Einstein-Hilbert action: the Einstein equations are trivial and do not offer any constraints on the geometry because $2R_{\alpha\beta} = Rg_{\alpha\beta}$, independently of the Einstein equations. There is, however, a topological meaning to the Einstein-Hilbert action, given by the Gauss-Bonnet theorem:

$$\chi(\mathcal{M}) = \frac{1}{4\pi} \int d^2\sigma \sqrt{g}\, R^{(2)}. \tag{14}$$

It is the Euler characteristic of the worldsheet, a topological invariant. This will prove very useful, since the path integral instructs us to integrate over all metrics, but also sum over topologies. What would a string interaction diagram look like? Fig. 1 shows the "four-point" interaction of closed strings. In a Feynman diagram, interactions occur at vertices. Here, there are no such points, so locally, Fig. 1 is the worldsheet of a free string. Only *globally* does it say anything about an interaction. We now consider the string action to be

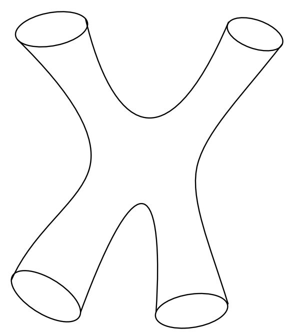

Figure 1: Worldsheet of interacting closed strings.

the Polyakov action complemented by the Euler characteristic of the worldsheet:

$$S_{\text{string}} = S_{\text{Pol.}} + \lambda\chi \tag{15}$$

For manifolds with no boundaries, the Euler characteristic is given by

$$\chi = 2(1-g), \tag{16}$$

where $g$ is the genus of the surface, that is the number of "handles": 0 for the sphere, 1 for the torus. The genus is a topological invariant: topological manifolds having the same genus are homeomorphic. The string path integral is given by

$$\sum_{\text{topologies}} e^{-2\lambda(1-g)} \int \mathcal{D}g\, \mathcal{D}X\, e^{-S_{\text{Pol.}}}. \tag{17}$$

For $e^\lambda \ll 1$, we have a good perturbative expansion (an asymptotic series, just as in a regular QFT) in terms of the genus of the worldsheet. One thus defines the string coupling constant as

$$g_s = e^\lambda. \tag{18}$$

Summing over topologies, i.e. considering a perturbative expansion of the string interactions, implies the existence of the "loop" diagrams shown in Fig. 2.

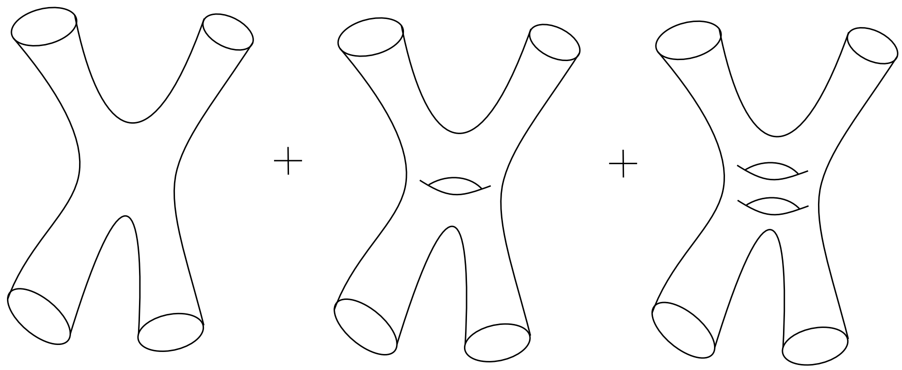

Figure 2: Worldsheet of interacting closed strings with different topologies.

'

In the next session, we will look at the details of some tree-level processes: the Veneziano amplitude - one of the historical motivations of string theory! - and the Virasoro-Shapiro amplitude. The main object that we will want to compute is the S-matrix, obtained by considering the correlation functions of asymptotic states. The state-operator map instructs us that these states are generated by appropriate vertex operators on the worldsheet.