Problem Set 8: Central Charge, Free Boson — PHYS-F483

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1 The central charge, or the reason why D is 26

At the beginning of the semester, we argued that for Lorentz invariance to hold, string theory had to be played on a 26-dimensional surface, corresponding to 26 free bosons locating the string in target space. Since then, we have spent quite some time and energy trying to formalise what we had sloppily constructed by studying conformal field theory, of which string theory is one of the most basic examples.

Remember the mass spectrum for the closed string states: at level one, we found a massless spin-two particle: the graviton. The gravitons arising from the closed string excitations will build up a background metric $g_{\mu\nu}$: starting from the Polyakov action in flat space, string theory embeds *itself* in a dynamical spacetime. The Polyakov action becomes an interacting theory, and you have seen – or will very soon see – that one-loop conformal invariance implies the Einstein equations.

What does conformal invariance imply for the worldsheet? Conformal symmetry is a residual gauge symmetry and as such, it must not be broken in the quantum theory. If it is, then the theory is anomalous: configurations related by gauge transformations are not physically equivalent anymore. Remember that conformal invariance boiled down to having a traceless energy-momentum tensor. It is possible to show[1], and we will not do so for the sake of saving time, that in the quantum theory,

$$\left\langle T^{\mu}_{\ \mu}(x)\right\rangle = \frac{c}{24\pi}R(x), \tag{1}$$

where R is the scalar curvature on the worldsheet[2]. This is called the *Weyl*, or conformal, *anomaly*. A perhaps naive way to look at this equation comes from the choice of gauge on the worldsheet: we went to the conformal gauge, where the worldsheet metric is flat, so should we really care about the Weyl anomaly? Well, actually, yes: under a Weyl transformation $g_{\alpha\beta} \to \omega^2 g_{\alpha\beta}$, the Ricci scalar becomes[3]

$$R \to \omega^{-2}R - 2(D-1)g^{\alpha\beta}\omega^{-3}\nabla_\alpha\nabla_\beta\omega - (D-1)(D-4)g^{\alpha\beta}\omega^{-4}\nabla_\alpha\omega\nabla_\beta\omega, \tag{2}$$

where the choice of gauge, i.e. ω appears explicitly: different choices of gauge, which should simply be redundancies in the description of the theory, now directly impact an observable. For this reason, we have to request that our theory has a zero *total* central charge.

[1] See for example Tong, Ch. 4 or Appendix 5.A. in Di Francesco for two very different proofs.

[2] The prefactor 1/24 depends on the conventions used all throughout our CFT voyage. This is the result from Di Francesco et al.

[3] See Appendix G in Carroll for a derivation.

Let us now come back to the comments that were made at the end of the previous session. Exactly like energy eigenstates of a rotation-invariant system (e.g. the hydrogen atom) fall into representations of $su(2)$, we expect the energy eigenstates of a conformally-invariant system to fall into representations of the Virasoro algebra. Starting from a highest-weight state $|h\rangle$, defined as

$$L_0 |h\rangle = h |h\rangle , \quad L_n |h\rangle = 0, \; n > 0. \tag{3}$$

Applying the negative-mode Virasoro generators on this state gives us a basis for the other states of the representation, the *descendant states*. One finds a *Verma module* generated by the states

$$L_{-k_1} L_{-k_2} ... L_{-k_n} |h\rangle , \quad 1 \leq k_1 \leq ... \leq k_n = h + N, \tag{4}$$

where N is called the *level*, which should ring a bell. This Verma module depends on h and c, the highest weight and the central charge. A representation of the Virasoro algebra will be unitary if does not contain negative-norm states, or *ghosts*. On the Verma module, since $L_m^\dagger = L_{-m}$, the inner product of two states will be given by

$$\langle h| L_{k_m} ... L_{k_1} L_{-l_1} ... L_{-l_n} |h\rangle . \tag{5}$$

In particular, this can be used to compute the norm of a state. For the very simple case of $L_{-n} |h\rangle$, we have

$$\begin{aligned} \langle h| L_n L_{-n} |h\rangle &= \langle h| \left(L_{-n} L_n + 2n L_0 + \frac{c}{12} n(n^2 - 1) \right) |h\rangle \\ &= \left(2nh + \frac{c}{12} n(n^2 - 1) \right) \langle h|h\rangle . \end{aligned} \tag{6}$$

There are two very strong constraints imposed by unitarity: for $c < 0$, the norm of this state becomes negative for arbitrarily large n. Therefore, all the $c < 0$ representations are non-unitary. Likewise, taking $n = 1$ shows that a negative conformal weight also implies a violation of unitarity. Thus, we know that unitarity requires $h \geq 0$, $c \geq 0$. This does not mean that any theory with such parameters will be unitary: it is a necessary condition, not a sufficient one. Furthermore, a theory with $c = 0$ is trivial, it does not contain excited states.

So what's the catch? On one hand, the Weyl anomaly is telling us to cancel the central charge, but on the other hand, a theory with $c = 0$ is trivial. How does string theory fit in? As mentioned many times, covariant quantisation of the string action requires the introduction of a so-called *ghost system*, via the Faddeev-Popov method of gauge-fixing. For these ghosts b and c, the energy-momentum tensor OPEs disclose a central charge $c = -26$. We have also seen that the simplest example of CFT, the free boson comes with a $c = 1$ central charge. Thus, we see the simplest way to comply with the cancellation of the Weyl anomaly is to have a theory of 26 free bosons, i.e. 26 coordinates for the string in target space. Consistency of the theory predicts the number of dimensions of spacetime: this is something unique, that neither General Relativity nor Quantum Field Theory can tell you. We end up with two decoupled conformal field theories: string theory and the ghost CFT, decoupled from physical processes and thus whose non-unitarity is of no importance.

2 The free boson

Consider a free boson φ on a cylinder of circumference L. The Fourier expansion for such a field is

$$\varphi(x,t) = \sum_n e^{2\pi inx/L}\varphi_n(t). \tag{7}$$

Problem 2.1. *a) Rewrite the free field Lagrangian*

$$\frac{1}{2}g\int dx\,\left\{(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right\}. \tag{8}$$

b) Show that the Hamiltonian is given by

$$H = \frac{1}{2gL}\sum_n \left\{\pi_n\pi_{-n} + (2\pi ng)^2\varphi_n\varphi_{-n}\right\} \tag{9}$$

with $\pi_n = gL\dot{\varphi}_{-n}$. Argue from the expression of H that $\varphi_n^\dagger = \varphi_{-n}$, and likewise for its conjugate momentum π_n.

The usual way to define creation and annihilation operators is as follows:

$$\tilde{a}_n = \frac{1}{\sqrt{4\pi g|n|}}\left(2\pi g|n|\varphi_n + i\pi_{-n}\right), \tag{10}$$

with commutation relations $\left[\tilde{a}_n, \tilde{a}_m^\dagger\right] = \delta_{mn}$. This is of course a problem for the zero-mode, so we adopt the following definition:

$$a_n = \begin{cases} -i\sqrt{n}\,\tilde{a}_n & n>0 \\ i\sqrt{-n}\,\tilde{a}_{-n}^\dagger & n<0 \end{cases} \qquad \bar{a}_n = \begin{cases} -i\sqrt{n}\,\tilde{a}_{-n} & n>0 \\ i\sqrt{-n}\,\tilde{a}_n & n<0 \end{cases} \tag{11}$$

and treat the zero-mode separately. These operators obey the commutation relations

$$[a_n, a_m] = n\delta_{n+m}, \quad [a_n\bar{a}_m] = 0, \quad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m}. \tag{12}$$

Problem 2.2. *Rewrite the Hamiltonian in terms of these operators and show that*

$$[H, a_{-m}] = \frac{2\pi}{L}ma_{-m}. \tag{13}$$

The creation and annihilation operators can be used to express the field modes:

$$\varphi_n = \frac{i}{n\sqrt{4\pi g}}(a_n - \bar{a}_{-n}). \tag{14}$$

It is then rather straightforward to show that the mode expansion at an arbitrary time t is given by

$$\varphi(x,t) = \varphi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{4\pi g}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi in(x-t)/L} - \bar{a}_n e^{2\pi in(x+t)/L}\right). \tag{15}$$

Problem 2.3. *a) Go to the Euclidean complex plane by setting*

$$z = e^{2\pi(\tau - ix)/L}, \quad \bar{z} = e^{2\pi(\tau + ix)/L}. \tag{16}$$

Give the expression of the free boson expansion in these coordinates.

b) The free boson field itself is not a primary field because of the logarithm in the $\varphi\varphi$ propagator, but its derivative is. Write $i\partial\varphi(z)$ as a mode expansion.

The conformal dimension of the free boson is ill-defined, but it is possible to construct local primary fields out of φ: *vertex operators*. We define a vertex operator as

$$\mathcal{V}_\alpha(z,\bar{z}) \equiv :e^{i\alpha\varphi(z,\bar{z})}:. \tag{17}$$

As you will now show, these fields are primary.

Problem 2.4. *a) Compute the OPEs of $\partial\varphi$ with $\mathcal{V}_\alpha$ and show that $\mathcal{V}_\alpha$ is a primary field of conformal dimension $h(\alpha) = \frac{\alpha^2}{8\pi g}$.*

b) Compute the OPE of two vertex operators using the identity

$$:e^{a\varphi_1}::e^{b\varphi_2}: = :e^{a\varphi_1 + b\varphi_2}: e^{ab\langle\varphi_1\varphi_2\rangle}. \tag{18}$$

Actually, special conformal transformations constrain two-point functions to be zero for different conformal dimensions. Furthermore, we require that correlators cannot grow with distance. You can convince yourself that this implies that for two vertex operators with phase coefficients α and β, the only non-trivial OPE is for $\alpha = -\beta$. We will discuss vertex operators in more details when we deal with string interactions.

Let us end with a discussion of the Fock space. Since the Hamiltonian does not depend on φ_0, its conjugate momentum π_0 is a good quantum number to label eigenstates of H. Furthermore, you can check from the definition of a_n that π_0 commutes with all the ladder modes: these operators cannot change the eigenvalue of π_0 and the Fock space is built upon a one-parameter family of vacua $|\alpha\rangle$:

$$a_n|\alpha\rangle = \bar{a}_n|\alpha\rangle \ (n>0), \quad a_0|\alpha\rangle = \bar{a}_0|\alpha\rangle = \alpha|\alpha\rangle. \tag{19}$$

Problem 2.5. *From the definition of the holomorphic energy-momentum tensor,*

$$T(z) = -2\pi g :\partial\varphi(z)\partial\varphi(z):, \tag{20}$$

derive the Virasoro generators acting on the Fock space. Show that the Hamiltonian is indeed proportional to $L_0 + \bar{L}_0$.

Problem 2.6. *Each vacuum state $|\alpha\rangle$ is obtained by applying a vertex operator on the "absolute vacuum". Show that $|\alpha\rangle = \mathcal{V}_\alpha(0)|0\rangle$ does respect the conditions* (19) *by using the Hausdorff formula*

$$\left[B, e^A\right] = e^A [B, A] \tag{21}$$

for constant $[B, A]$.