Problem Set 7: Operator Formalism, part 2 **PHYS-F483**

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In the previous problem set, we discussed the basics of radial quantisation. Namely, we saw that time is now put in correspondence with the radial distance from the origin. This allows us to define *radial ordering*: within correlators, the usual time-ordering prescription is now translated to ordering operators with respect to their distance from the origin of the complex plane:

$$\mathcal{R}\left[\Phi(z)\Phi(w)\right] = \begin{cases} \Phi(z)\Phi(w) \text{ if } |z| > |w|, \\ \Phi(w)\Phi(z) \text{ if } |w| > |z|. \end{cases} \tag{1}$$

Remember that fermionic fields take an additional minus sign when an ordering commutation has to be performed. This radial ordering is also used in the OPEs we have computed in the previous session: they make sense for $|z| > |w|$. In practice, we consciously omit the radial ordering symbol $\mathcal{R}$.

Radial quantisation allows us to relate contour integrals to commutators. One can show that for two operators A and B, each the closed contour integral of a holomorphic function, we have

$$[A, b(w)] = \oint_w dz\, a(z)b(w) \tag{2}$$

and

$$[A, B] = \oint_0 dw \oint_w dz\, a(z)b(w). \tag{3}$$

Thus, in a 2D CFT, one can make statements about commutators of operators by evaluating contour integrals, which will only require the knowledge of operator product expansions.

1 The Virasoro algebra

Problem 1.1. *a) Remember that the conserved charge associated to a symmetry is the integral of the Noether current. What is the conserved charge associated to infinitesimal conformal transformations?*

b) Using the conformal Ward identity, show that

$$\delta_\epsilon \Phi(w) = -\left[Q_\epsilon, \Phi(w)\right]. \tag{4}$$

We ended the last problem set by expanding conformal fields as an infinite sum of modes. According to the transformation law of the stress-energy tensor, it is a conformal (non-primary) field of conformal weights (2,2).

Problem 1.2. *a) Give the mode expansion of the holomorphic and antiholomorphic stress-energy tensor components. In hindsight, write the modes as* L_n *and* $\bar{L}_n$*.*

b) Invert this relation to express the modes L_n is terms of the stress-energy tensor.

c) Expand the infinitesimal conformal transformation parameter $\epsilon(z)$ in Laurent series to show that the modes L_n generate local transformations on the Hilbert space.

Hence, the L_n modes are the counterpart, acting on the Hilbert space, of the ℓ_n acting on the space of functions. The following problem takes you to the famous Virasoro algebra, which is obviously the reason why we called the modes L_n.

Problem 1.3. *Using the stress-energy tensor operator product expansions, show that*

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \tag{5}$$

$$\left[L_n, \bar{L}_m\right] = 0, \tag{6}$$

$$\left[\bar{L}_n, \bar{L}_m\right] = (n-m)\bar{L}_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{7}$$

Hence, the generators of local conformal transformations form an algebra, the Virasoro algebra. Despite the central extension, a $SL(2,\mathbb{C})$ subalgebra still exists and can interpreted in the same way as the various combinations of ℓ_{-1}, ℓ_0, ℓ_1, but now acting on states. In particular, $L_0 + \bar{L}_0$ generates dilatations. In the radial quantisation language, dilatations are simply time translations: this combination of Virasoro generators is proportional to the Hamiltonian!

2 A quick look at the Hilbert space

We require the vacuum to be invariant under global conformal transformations, which implies that it be annihilated by the $SL(2,\mathbb{C})$ subalgebra. Even more, the well-definedness of the stress-energy tensor acting on the vacuum implies

$$L_n \left|0\right\rangle = 0, \quad \bar{L}_n \left|0\right\rangle = 0 \tag{8}$$

for $n \geq -1$. We can show that primary fields acting on the vacuum create states that are eigenstates of the Hamiltonian.

Problem 2.1. *a) Consider a primary field ϕ with conformal weights $(h, \bar{h})$. Show that*

$$[L_n, \phi(w,\bar{w})] = h(n+1)w^n\phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w}) \quad (n \geq -1). \tag{9}$$

b) Show that

$$L_0 \left|h,\bar{h}\right\rangle = h\left|h,\bar{h}\right\rangle, \quad \bar{L}_0 \left|h,\bar{h}\right\rangle = h\left|h,\bar{h}\right\rangle \tag{10}$$

and

$$L_n \left|h,\bar{h}\right\rangle = \bar{L}_n \left|h,\bar{h}\right\rangle = 0 \quad \textit{if}\ n > 0. \tag{11}$$

c) Expanding the holomorphic field in modes, show that the modes ϕ_m are ladder operators for the L_0 eigenstates. Likewise, show that acting with Virasoro generators on these eigenstates also modifies the conformal weight.

We have shown that states generated by primary fields are eigenstates of the Hamiltonian. Moreover, we found that these are the "lowest eigenvalue states", or in the language of representation theory, *highest-weight states*. Starting from this highest-weight state, it is then possible to generate a tower of so-called *descendant states* by acting with the Virasoro generators L_{-n}, $n > 0$:

$$L_{-k_1}...L_{-k_n}\left|h\right\rangle \tag{12}$$

with weight $h' = h + k_1 + k_2 + ... + k_n = h + N$, where N is called the *level* of the descendant. The set of such states is closed under the action of the Virasoro generators, and thus forms a representation of the Virasoro algebra: this is called a *Verma module*. This module depends on two quantities: the conformal weight h of the primary state upon which the descendant tower is constructed, and the central charge c of the theory.

Unitarity of the theory restricts the central charge and the conformal weight to be non-negative, but this is not a sufficient condition. It is then necessary to analyse every model individually to isolate the regions of its parameter space where it is indeed unitary. This would be covered in a proper course dedicated to conformal field theory. For our purposes however, we are almost done with this conformal detour: after a deeper look at the free boson, we will come back to strings. After all, string theory is a theory of 26 free bosons.