Problem Set 1: Operator Formalism, part 1 **PHYS-F484**

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1 Operator Product Expansions (OPEs)

When discussing Ward identities, we highlighted the power of 2D CFTs by exploiting the residue theorem. As emphasized then, within contour integrals, only the singular part of the correlators is of interest. This is why we now focus on writing correlation functions as a sum of singular terms: this is the basic principle of operator product expansions, or OPEs. For example, we saw that for a primary field Φ,

$$T(z)\Phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}). \tag{1}$$

For convenience, we drop the correlator brackets whenever we talk about OPEs. However, they are not operator relations but rather strictly defined within correlation functions. Our toy example will be the massless free boson, the simplest conformal system there is.

Problem 1.1. *Consider the massless free boson*

$$S = \frac{g}{2}\int d^2x\ \partial_\mu\varphi\partial^\mu\varphi. \tag{2}$$

a) Compute the propagator $\langle\varphi(x)\varphi(y)\rangle$. This can be done via the path integral, or by looking at the Green functions of the theory.

b) Show that in complex coordinates, the operator product expansion of $\partial\varphi\partial\varphi$ is

$$\partial\varphi(z)\partial\varphi(w) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2}. \tag{3}$$

Remember the definition of the energy-momentum tensor for a free boson:

$$T_{\mu\nu} = g\left(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi\right) \tag{4}$$

Problem 1.2. *a) Show that*

$$T(z) = -2\pi T_{zz} = -2\pi g :\partial\varphi(z)\partial\varphi(z):. \tag{5}$$

b) Show that $\partial\varphi(z)$ is a primary field with conformal weight $h=1$.

c) Show that the energy-momentum tensor is not a primary field. It is actually a quasi-primary field.

In the free fermion model, the same structure appears in the OPEs. This is also the case for the *ghost system*, i.e. the ghosts introduced to perform the BRST quantisation of string theory. We can sum this up in the following way:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \tag{6}$$

For the free boson, we found $c = 1$. For the free fermion, $c = 1/2$. For the ghost system, $c = -26$. The constant c is called the *central charge*. The central charge is an extensive measure of the number of degrees of freedom of the system. If it was not for the c-term, the energy momentum tensor would be a primary field with conformal weight $h = 2$.

Problem 1.3. *Remember the conformal Ward identity, painfully derived in the previous problem set. Apply this formula to the energy-momentum tensor and show that*

$$\delta_\epsilon T(w) = -\frac{c}{12}\partial_w^3\epsilon(w) - 2T(w)\partial_w\epsilon(w) - \epsilon(w)\partial_w T(w). \tag{7}$$

For a finite transformation $z \to w(z)$, we have

$$T'(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2}\left(T(z) - \frac{c}{12}\{w;z\}\right), \tag{8}$$

where $\{\cdot\,;\cdot\}$ is the *Schwartzian derivative*, defined as

$$\{w;z\} \equiv \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right). \tag{9}$$

Under two successive transformations $z \to w \to u$, we find that $T(z) \to T'(u)$ agrees with $T(z) \to T'(w) \to T''(u)$ if and only if

$$\{u;z\} = \{w;z\} + \left(\frac{dw}{dz}\right)^2 \{u;w\}. \tag{10}$$

The Schwartzian derivative is the only addition to the energy-momentum tensor transformation law that satifies a group property and vanishes for global conformal transformations.

2 Radial quantisation

Now that we know how to deal with operators in a smart ways, we must describe some notion of state space on which these operators act. This will be achieved by *radial quantisation*. Consider the theory of a closed string in Minkowski space. The worldsheet is a cylinder with periodicity L, described by a timelike component τ and a spacelike component σ.

Problem 2.1. *Continue the cylinder to Euclidean signature $t = i\tau$ and consider the mapping*

$$z = \exp\left(\frac{2\pi(t+i\sigma)}{L}\right). \tag{11}$$

What is the cylinder mapped to? In particular, argue that Euclidean time is now measured by the distance from the origin.

Let us construct the state space: we start by assuming the existence of a vacuum state $|0\rangle$ on which creation operators act to construct a Hilbert space. If we wish to consider interactions at some point - and we will want to -, then we have to specify a notion of asymptotic states. We define an incoming state as follows:

$$\varphi_{\text{in}} \propto \lim_{z,z\to 0} \varphi(z,\bar{z})\,|0\rangle\,. \tag{12}$$

Defining asymptotic outgoing states amounts to defining a bilinear product on the Hilbert space and specifying the action of hermitian conjugation on conformal fields. We define Hermitian conjugation on the real surface $\bar{z} = z^*$ as

$$[\varphi(z,\bar{z})]^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \varphi\left(1/\bar{z}, 1/z\right). \tag{13}$$

There are two noticeable ingredients: first, the arguments of the field are modified from z to $1/\bar{z}$, and vice-versa. This has to be understood as a consequence of Hermitian conjugation after having performed a Wick rotation, which is not simply a change of coordinates! Second, the prefactors are necessary for having well-behaved correlators, as the next problem shows.

Problem 2.2. *We define the outgoing asymptotic states $\langle\varphi_{out}| = |\varphi_{in}\rangle^\dagger$. Check that the prefactors introduced in the definition of the hermitian conjugate lead to a regular behaviour for the inner product $\langle\varphi_{out}|\varphi_{in}\rangle$. To show this, consider the following steps:*

a) Consider a primary field Φ and the two-point function $\langle\Phi(z_1,\bar{z}_1)\Phi(\bar{z}_2)\rangle$. What are the constraints imposed by translation invariance?

b) Consider a dilatation $z_i \to \lambda z_i$. Argue that the two-point function must behave as

$$\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}}, \tag{14}$$

where C_{12} is a constant.

c) Verify that the inner product between the ingoing and outgoing states is indeed well-behaved.

Let us look at a mode expansion for primary fields with conformal weights $(h,\bar{h})$. We can define a Laurent series

$$\varphi(z,\bar{z}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\varphi_{m,n}, \tag{15}$$

with the usual coefficients

$$\varphi_{m,n} = \frac{1}{2\pi i}\oint dz\; z^{m+h-1}\frac{1}{2\pi i}\oint d\bar{z}\;\bar{z}^{m+h-1}\varphi(z,\bar{z}). \tag{16}$$

Problem 2.3. *Show that*

$$\varphi^\dagger_{m,n} = \varphi_{-m,-n} \tag{17}$$

We have now set the scene for radial quantisation, which we will address in the following problem set. The key point is that ordering operators in time is now equivalent to ordering them along a radial direction, where we can use contour integrals.