Problem Set 5: Ward Identities **PHYS-F483**

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In the last problem set, we discussed the basics of two-dimensional conformal field theory. As the name suggests, a conformal field theory is a field theory that possesses conformal symmetry. We have also learned that in two dimensions, the conformal algebra is generated by the holomorphic functions on the complex plane. As previewed, we will now use the tools of complex analysis in our discussion of two-dimensional conformal field theories.

Let us set the notations: in the complex coordinates we chose, the tracelessness of the energy-momentum tensor is expressed as

$$T_{z\bar{z}} = 0. \tag{1}$$

The conservation of the energy-momentum tensor becomes

$$\bar{\partial} T_{zz} = 0, \quad \partial T_{\bar{z}\bar{z}} = 0. \tag{2}$$

This allows us to write

$$T_{zz}(z) \equiv -\frac{1}{2\pi} T(z), \quad T_{\bar{z}\bar{z}}(\bar{z}) \equiv -\frac{1}{2\pi} \bar{T}(\bar{z}). \tag{3}$$

Problem 1.1. *Consider an infinitesimal transformation $z \to z' = z + \epsilon(z)$. Compute the Noether current associated to this conformal transformation.*

Ward identities are the quantum counterpart to Noether's theorem in a classical setting. They constrain the correlators of the theory according to its symmetries or, from a different point of view, express the quantum breaking of classical symmetries. This entire problem set will walk you through the derivation of the *conformal Ward identity.* First, let us remind ourselves, or get acquainted, with the Ward identities of a generic quantum field theory. Noether's theorem tells us that there exists a conserved current associated to any global symmetry. The Ward identities inform us of the whereabouts of this conserved current in a quantum theory, that is inside correlation functions.

Problem 1.2. *a) Consider the path integral of a (Euclidean) quantum field theory with n field insertions Φ and a field transformation $\delta\Phi = -i\omega_a(x) G_a \Phi$. Assuming that the measure is invariant under such a transformation, derive the Ward identity*

$$\partial_\mu \langle J^\mu \Phi(x_1)...\Phi(x_n)\rangle = -i \sum_{i=1}^{n} \delta(x - x_i) \langle \Phi(x_1)...G_a\Phi(x_i)...\Phi(x_n)\rangle \tag{4}$$

b) Show that the Ward identity associated to translation invariance is

$$\partial_\mu \langle T^{\mu\nu} \Phi(x_1)...\Phi(x_n)\rangle = -\delta(x - x_1) \left\langle \partial^\nu_{x_1} \Phi(x_1)...\Phi(x_n)\right\rangle - ... - \delta(x - x_n) \left\langle \Phi(x_1)...\partial^\nu_{x_n} \Phi(x_n)\right\rangle. \tag{5}$$

Let us now switch to the language of conformal field theory. For reasons that will be apparent later, we will be interested in *primary fields*. These fields are defined by their transformation behaviour under conformal transformations. A field that transforms as

$$\varphi'(w,\bar{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h}\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}) \tag{6}$$

under *any local* conformal map $z \to w(z)$ is called a primary field. The exponent h is called the *conformal dimension* or *conformal weight*. It is related to the behaviour of the field under rotations and dilatations in the following way:

$$h = \frac{1}{2}(\Delta + s)\,, \qquad \bar{h} = \frac{1}{2}(\Delta - s)\,, \tag{7}$$

with s the spin of the field and Δ its *scaling dimension*.

Problem 1.3. *Show that the definition* (6) *is equivalent to the transformation law*

$$\delta_{\epsilon,\bar{\epsilon}}\varphi = -(h\varphi\partial_z\epsilon + \epsilon\partial_z\varphi) - (\bar{h}\varphi\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\varphi) \tag{8}$$

for small $\epsilon(z)$.

A *quasi-primary* field is a field that transforms as a primary field under the global $SL(2,\mathbb{C})$, or Möbius, transformations. Any primary field is a quasi-primary, but the converse is of course not true. An example of field that is not primary is the energy-momentum tensor, as we will see later.

Now that the framework is set, one can work out the following Ward identities for rotations and scale invariance. In conformal field theory, we do not speak of operators anymore, but refer to everything as fields. Consider a collection of primary fields X. Then, the Ward identity associated to rotations is

$$\varepsilon_{\mu\nu}\langle T^{\mu\nu}X\rangle = -i\sum_{i=1}^{n} s_i\delta(x-x_i)\langle X\rangle \tag{9}$$

and the one associated to scale transformations is

$$\langle T^{\mu\nu}X\rangle = -\sum_{i=1}^{n}\Delta_i\delta(x-x_i)\langle X\rangle \tag{10}$$

To recast these identities in the language of our complex coordinates, we make use of the following result:

Problem 1.4. *Prove that*

$$\delta(x)\delta(y) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}}. \tag{11}$$

Problem 1.5. *Use the complex plane delta representation you have proven to recast the translation Ward identities as*

$$2\pi\partial_z\langle T_{\bar{z}z}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}X\rangle = -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle\,, \tag{12}$$

$$2\pi\partial_z\langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n}\partial_z\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle\,. \tag{13}$$

The other two Ward identities can be rewritten as

$$2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\langle X\rangle\,, \tag{14}$$

$$-2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)s_i\langle X\rangle\,. \tag{15}$$

Problem 1.6. *a) Make appropriate combinations of the preceding four equations to get the following relations:*

$$2\pi\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}h_i\langle X\rangle \tag{16}$$

$$2\pi\langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\bar{h}_i\langle X\rangle \tag{17}$$

b) Plugging these back in the translational Ward identities, show that

$$\partial_{\bar{z}}\left\{\langle T(z,\bar{z})X\rangle - \sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right]\right\} = 0, \tag{18}$$

$$\partial_{z}\left\{\langle \bar{T}(z,\bar{z})X\rangle - \sum_{i=1}^{n}\left[\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2}\langle X\rangle\right]\right\} = 0. \tag{19}$$

The last part of Problem 1.6 shows that the expression within braces is holomorphic (respectively, antiholomorphic). Then, we can write

$$\langle T(z)X\rangle = \sum_{i=1}^{n}\left\{\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right\} + \text{ regular terms.} \tag{20}$$

where the regular terms are holomorphic functions of z regular at $z = w_i$. We now arrive at the conformal Ward identity:

Problem 1.7. *a) Show that for an arbitrary conformal transformation*

$$\partial_\mu(\epsilon_\nu T^{\mu\nu}) = \epsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}\partial_\rho\epsilon^\rho\eta_{\mu\nu}T^{\mu\nu} + \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\epsilon_{\mu\nu}T^{\mu\nu}. \tag{21}$$

b) Consider translations, for which ϵ is a constant. By integration over a domain that contains all the positions of the fields contained in X, show that

$$\delta_\epsilon\langle X\rangle = \int_M d^2x\,\partial_\mu\langle T^{\mu\nu}(x)\epsilon_\nu(x)X\rangle\,. \tag{22}$$

This relation also holds for rotations and scale transformations: it contains the three Ward identities we have discussed.

c) Prove the conformal Ward identity

$$\delta_\epsilon\langle X\rangle = -\frac{1}{2\pi i}\oint_C dz\,\epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\,\bar{\epsilon}(\bar{z})\left\langle\bar{T}(\bar{z})X\right\rangle. \tag{23}$$

To show this, you have to use Stokes' theorem in the complex plane:

$$\int_M d^2x\, \partial_\mu F^\mu = \frac{1}{2} i \int_{\partial M} \left(-F^{\bar{z}} dz + F^z d\bar{z}\right). \tag{24}$$

d) Show that, for a single primary field, we retrieve the transformation law derived in Problem 1.3.