Problem Set 4: Conformal Transformations **PHYS-F483**

February 25, 2025

Conformal Field Theory (CFT) is ubiquitous in modern theoretical physics, from critical phenomena in statistical physics to the AdS/CFT correspondence. In our case, on the string worldsheet, we will be interested in two-dimensional conformal field theories. It turns out that a two-dimensional CFT is very special, as we will see. For now, let us discuss generalities.

1 Conformal symmetry

A conformal transformation is a coordinate transformation $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$ such that the metric is modified according to

$$g_{\alpha\beta} \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma). \tag{1}$$

As mentioned in the first problem set, this subset of coordinate transformations leaves the metric invariant up to a Weyl rescaling. These coordinate transformations form a group, of which the Poincaré group is a subgroup, obviously. For simplicity, let us fix the metric to the standard Euclidean metric $g_{\mu\nu} = \mathrm{diag}(1, 1, ..., 1)$. The Osterwalder-Stracher reconstruction theorem relates Lorentzian CFTs to (reflection-positive) Euclidean CFTs through Wick rotation.

Problem 1.1. *Consider an infinitesimal transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$.*

a) How does the metric change under such a transformation?

b) Let us write $\Omega^2(x) = e^{-f(x)}$. Derive the conformal Killing equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}, \tag{2}$$

with $f(x) = \frac{2}{D}\partial_\rho \epsilon^\rho$.

The smooth function $f(x)$ is called the *conformal Killing factor*. For general (semi-)Riemannian manifolds, one finds

$$\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = f(x) g_{\mu\nu}. \tag{3}$$

This is a straightforward generalisation of the Killing equation: conformal transformations generalise isometries.

Problem 1.2. *Consider for a moment that the metric is not fixed. Remember that a theory possesses Weyl symmetry if and only if its energy-momentum tensor is traceless. Prove that a traceless energy-momentum tensor implies conformal symmetry, but not the converse.*

This last problem shows why we are interested in conformal field theory. Since the Polyakov action is Weyl-invariant, it is also automatically conformally-invariant. Let us now have a look at why the two-dimensional case is so special.

Problem 1.3. *Prove the $\Rightarrow$ implication of the following theorem: the smooth function $f(x)$ is a conformal Killing factor if and only if*

$$(D-2)\partial_\mu \partial_\nu f + g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \partial_\beta f = 0. \tag{4}$$

In two dimensions, this reduces to $\Delta f = 0$. For $D \geq 3$, there are many additional conditions:

$$\partial_\mu \partial_\nu f = \begin{cases} 0 & \text{for } \mu \neq \nu, \\ \pm (D-2)^{-1} \Delta f & \text{for } \mu = \nu. \end{cases} \tag{5}$$

This allows the conformal *algebra* (not the conformal group, as often stated) to be infinite-dimensional in two dimensions, whereas it is $\mathfrak{so}(D+1,1)$ for higher dimensions. From now on, we will focus on the two-dimensional case.

2 Two-dimensional conformal symmetry

As mentioned before, we will work in a Euclidean setting, although our goal is to describe physics on a manifold with Minkowski signature. We will simply be one Wick rotation away from the actual worldsheet. Let us fix the notations: we will work with the Euclidean coordinates $(\sigma^1, \sigma^2) = (\sigma^1, i\sigma^0)$. At this point, it is useful to introduce complex coordinates. The reason is simple: local conformal transformations in two dimensions are generated by the holomorphic functions.

Problem 2.1. *Prove the previous statement.*

This is also why the study of conformal field theories in two dimensions is much different from the higher-dimensional ones: the very powerful weaponry of complex analysis is now available. This will become clearer in the following sessions. Convinced by the last problem, let us now take a linear combination of the worldsheet coordinates to construct the following complex coordinates:

$$z = \sigma^1 + i\sigma^2, \quad \bar{z} = \sigma^1 - i\sigma^2. \tag{6}$$

We define the derivatives

$$\partial_z \equiv \partial = \frac{1}{2}(\partial_1 - i\partial_2) \quad \text{and} \quad \partial_{\bar{z}} \equiv \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2). \tag{7}$$

In these coordinates, the metric becomes

$$ds^2 = dz d\bar{z}. \tag{8}$$

We will consider z and $\bar{z}$ as independent variables. Doing that, the worldsheet is extended from $\mathbb{R}^2$ to $\mathbb{C}^2$. In practice, we must remember that $\bar{z} = z^*$, which selects a real slice $\mathbb{R}^2 \subset \mathbb{C}^2$.

Problem 2.2. *Consider the infinitesimal transformations $z \to z' = z + \epsilon(z)$. The infinitesimal parameter $\epsilon(z)$ admits a Laurent expansion around* 0*:*

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1}. \tag{9}$$

a) Consider the variation of a scalar field $\Phi(z, \bar{z})$. Derive the expression of the generators ℓ_n and $\bar{\ell}_n$ associated to the infinitesimal transformation (9).

b) These generators form two copies of the Witt algebra. Show that the holomorphic and antiholomorphic generators indeed obey the expected commutation relations.

Thus, the conformal algebra in two dimensions is the direct sum of two copies of the Witt algebra. The associated generators are not defined globally on the Riemann sphere $S^2 = \mathbb{C} \cup \infty$.

Problem 2.3. *Consider a vector field generating a holomorphic conformal transformation*

$$v(z) = \sum_{n=-\infty}^{\infty} a_n z^{n+1} \partial_z. \tag{10}$$

a) Check whether this vector field is globally defined by inspecting its behaviour at $z = 0$ and $z \to \infty$.

b) Identify a globally well-defined subalgebra. This generates the global conformal group in two dimensions. The global conformal group is not infinite-dimensional, as many physics textbooks suggest.

c) Argue that ℓ_{-1} and $\bar{\ell}_{-1}$ generate translations on the complex plane, $\ell_0 + \bar{\ell}_0$ generates scale transformations and $i(\ell_0 - \bar{\ell}_0)$ generates rotations. The remaining generator ℓ_1 is related to special conformal transformations.

The finite form of these transformations are called the Möbius transformations

$$z \to \frac{az + b}{cz + d}, \tag{11}$$

with a, b, c, $d \in \mathbb{C}$ and $ad - bc = 1$. This group is $SL(2, \mathbb{C})/\mathbb{Z}_2$, which is isomorphic to the Lorentz group in four dimensions $SO(3, 1)$.