Problem Set 2: from Polyakov to Virasoro **PHYS-F483**

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1 The Polyakov action

In the previous problem set, we have shown that the naive action for a string, the Nambu-Goto action, is classically equivalent to the Polyakov action

$$S_{\text{Pol.}} = -\frac{T}{2}\int d^2\sigma\,\sqrt{-G}\,G^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu. \tag{1}$$

The Einstein-Hilbert action can be supplemented with the addition of a *cosmological constant*. In the case of the Polyakov action, this is not a consistent modification of the action.

Problem 1.1. *Show that the action*

$$S = -\frac{T}{2}\int d^2\sigma\,\sqrt{-G}\,G^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + \Lambda\int d^2\sigma\,\sqrt{-G}, \tag{2}$$

$\Lambda \neq 0$*, leads to inconsistent equations of motion.*

The geometric p-brane action discussed during last session can also be "Polyakov-ised":

$$S^p_{\text{Pol.}} = -\frac{T_p}{2}\int d^{p+1}\sigma\,\sqrt{-G}\,G^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu. \tag{3}$$

Problem 1.2. *a) Compute the equations of motion for the metric* (3)*.*

b) Check that the induced metric

$$G_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X$$

is a solution of the equations of motion if and only if $p = 1$*.*

c) Consider the addition of a cosmological constant term to the action (3)*. Derive a relation between the "brane tension"* T_p *and the cosmological constant* Λ_p*.*

d) Show that the p*-brane action with a cosmological constant is classically equivalent to the geometric* p*-brane action.*

We will now investigate the symmetries of the string Polyakov action, starting with *Weyl symmetry*. A Weyl transformation is a local rescaling of the metric tensor. For the worldsheet, this is

$$G_{\alpha\beta} \mapsto e^{-\Phi(\sigma)}G_{\alpha\beta}. \tag{4}$$

It is *not* a conformal transformation, as it does not affect the coordinates. Conformal transformations, however, are defined as coordinate transformations that leave the metric invariant up to a Weyl rescaling. We will learn more about conformal transformations and conformal field theory in a few weeks.

Problem 1.3. *a) Show that the Polyakov action* (1) *is invariant under a Weyl rescaling* (4)*.*

b) Show that the invariance of the action under a Weyl rescaling implies the tracelessness of the energy-momentum tensor.

The Polyakov action enjoys more symmetries:

- Poincaré invariance, as a result of index saturation. Remember that this sends

$$X^\mu \mapsto \Lambda^\mu_{\ \nu} X^\nu + a^\mu,$$

where $\Lambda \in SO(1, d-1)$, d being the dimension of the target space.

- As the Nambu-Goto action, it enjoys reparametrisation, or diffeomorphism, invariance. You can easily check that the change of worldsheet coordinates

$$\sigma^\alpha \mapsto f^\alpha(\sigma)$$

leaves the action invariant.

- Weyl invariance, which you proved in the last problem.

Poincaré symmetry is a global symmetry on the worldsheet, whereas the two other ones are local symmetries. These can be used to pick a convenient gauge, as we sketch now. Diffeomorphism invariance gives us two arbitrary functions that we can use to shape the metric into the following form:

$$G_{\alpha\beta} \mapsto e^{2\phi} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Mathematically, this is expressed by the statement that every two-dimensional pseudo-Riemannian manifold $(\mathcal{M}, g)$ is conformally flat[1]. Then, Weyl invariance allows us to scale out the exponential, which effectively yields

$$G_{\alpha\beta} \mapsto \eta_{\alpha\beta}. \tag{5}$$

We say that the worldsheet metric has been fixed to the *conformal gauge*. In this gauge, the equations of motion for the embeddings X^μ become the very simple wave equation

$$\Box X^\mu = 0. \tag{6}$$

Note that on a fixed background metric, two-dimensional theories of gravity enjoying Weyl and diffeomorphism invariance will become conformally invariant with respect to the fixed metric. This will motivate our study of conformal field theories in the next problem sets.

[1] See for instance *A Mathematical Introduction to Conformal Field Theory*, Schottenloher, Springer Lecture Notes on Physics, 2008.

2 The road to Virasoro

Problem 2.1. *Consider a closed string, i.e. a string that identifies σ and $\sigma + 2\pi$.*

a) Write down the metric tensor, the equations of motion and the constraints $T_{\alpha\beta} = 0$ for the scalar fields in the light-cone coordinates

$$\sigma^+ = \tau + \sigma, \quad \sigma^- = \tau - \sigma. \tag{7}$$

b) What is the most general solution X^μ of the wave equation that respects the closed string boundary conditions? In the light-cone coordinates, the wave equations implies that the scalar functions X^μ can be split into left-movers and right-movers:

$$X^\mu(\sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-). \tag{8}$$

Split your answer for $X^\mu(\tau, \sigma)$ into a left- and right-moving part.

c) How do the reality conditions on X^μ affect the mode expansion?

d) Show that the zero-modes can be interpreted as the initial center of mass position and the momentum (computing the Noether currents associated to translations might be useful for this second part).

Let us also take a look at the open string, parametrised by $\sigma \in [0, \pi]$.

Problem 2.2. *a) Consider the variation of the Polyakov action in the conformal gauge and light-cone coordinates. There are four different boundary conditions that you can impose to retrieve the wave equation. Write these boundary conditions in the (τ, σ) coordinates.*

b) Consider the Neumann-Neumann open string, i.e. the boundary conditions

$$\partial_\sigma X^\mu(\tau, 0) = 0, \quad \partial_\sigma X^\mu(\tau, \pi) = 0 \tag{9}$$

and the ansatz

$$X^\mu(\tau,\sigma) = x_L^\mu + x_R^\mu + \frac{\alpha'}{2}(p_L^\mu + p_R^\mu)\tau + \frac{\alpha'}{2}(p_L^\mu - p_R^\mu)\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)}\right). \tag{10}$$

Find out what the NN boundary conditions and the reality conditions imply on this ansatz.

c) Consider the same ansatz, but with the Dirichlet-Neumann boundary conditions:

$$X^\mu(\tau, 0) = c^\mu, \quad \partial_\sigma X^\mu(\tau, \pi) = 0. \tag{11}$$

d) Given the Polyakov action, compute the Noether current associated to translations and derive a condition for momentum conservation.

Given the condition we have just derived, it is obvious that Dirichlet conditions will spoil the conservation of momentum. The story is the same for the Lorentz generators. Should we conclude that Dirichlet open strings have to be dropped? In D dimensions, there are D scalar embeddings in target space. Suppose that we pick Neumann boundary conditions for some coordinates $X^0, ..., X^p$, and Dirichlet for the other $X^{p+1}, ..., X^{D-1}$. The Dirichlet endpoints all lie on a $(p+1)$-dimensional hypersurface. This hypersurface is a called a *Dp-brane*: D for Dirichlet, p for its dimension. In the presence of a Dp-brane, the Lorentz group in D dimensions broken into

$$SO(1, D-1) \to SO(1, p) \times SO(D-p-1). \tag{12}$$

Thus, D-branes are a way to obtain Lorentz invariance on a subset of the dimensions.

Back to the closed string: there is some additional information in the constraints imposed by the vanishing stress-energy tensor. Since

$$\partial_- X^\mu = \partial_- X_R^\mu = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{in\sigma^-} = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{in\sigma^-}, \tag{13}$$

provided that we set

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu, \tag{14}$$

the constraint for right-movers reads (show this if you haven't!):

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{n-m} \cdot \alpha_m = 0. \tag{15}$$

For left-movers, we have

$$\tilde{L}_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m = 0. \tag{16}$$

These relations hold for any integer n. Since $L_n = \tilde{L}_n$, this means that the left- and right-movers are not decoupled! Let us look at the $n = 0$ case:

$$L_0 = \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{-m} \cdot \alpha_m = \sum_{m \in \mathbb{N}} \alpha_{-m} \cdot \alpha_m = 0. \tag{17}$$

Let us factor out the zero-mode term in that sum:

$$-\alpha_0^2 = \sum_{m>0} \alpha_{-m} \alpha_m. \tag{18}$$

Since α_0 is essentially the momentum, the left-hand side of this last relation is the mass squared:

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_{-n} \cdot \alpha_n. \tag{19}$$

Likewise, had we proceeded in this way for $\tilde{L}_0$, we would have gotten

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n. \tag{20}$$

This yields the so-called *level matching* condition.

The left-hand side of the constraints defined by the vanishing of the stress-energy tensor are called the *Virasoro generators.* They form two copies of the so-called *Witt algebra.* Let us discuss what this algebra looks like. With the canonical momentum $P^\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}$, we have the canonical Poisson bracket

$$[X^\mu(\sigma,\tau), P^\nu(\sigma',\tau)]_{\text{P.B.}} = \eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{21}$$

You can check that these commutation relations will lead to the following Poisson brackets for the modes:

$$[\alpha^\mu_m, \alpha^\nu_n]_{\text{P.B.}} = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n]_{\text{P.B.}} = im\eta^{\mu\nu}\delta_{m+n,0}. \tag{22}$$

Problem 2.3. *Dropping the P.B. subscript for notational convenience, show that*

$$[L_m, L_n] = i(m-n)L_{m+n}. \tag{23}$$

When quantising the theory, Poisson brackets become $-i[\cdot,\cdot]$. For the modes, this is fairly straightforward. However, for the Virasoro generators, things are not so easy. In the quantum theory, the modes become operators, which obey commutation relations. It is therefore not possible to move them across one another as we please, and one must define an ordering scheme. One way to do so is by *normal ordering*, i.e. moving all the annihilation operators (which are the α_m, $m > 0$) to the right. In that case, the Witt algebra picks up a *central extension*:

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}, \tag{24}$$

which we will refer to as the *Virasoro algebra.* The Virasoro constraints used to be written as $L_n = 0 = \tilde{L}_n$ in the classical theory. Here, this becomes a condition on the matrix elements for the physical states $|\Phi\rangle$:

$$L_n |\Phi\rangle = 0 = \tilde{L}_n |\Phi\rangle . \tag{25}$$

Problem: normal ordering does not define L_0 uniquely. Indeed,

$$L_0 = \frac{1}{2}\sum_{k\in\mathbb{Z}} : \alpha_{-k}\cdot\alpha_k := \sum_{k=1}^{\infty} \alpha_{-k}\cdot\alpha_k + \frac{1}{2}\alpha_0^2. \tag{26}$$

This is actually the only operator for which ordering is important, given the expression of the commutator of two modes. A different ordering scheme would give the expression (26) *up to a constant.* To reflect the fact that the ordering of modes is chosen arbitrary, the level matching condition must be modified appropriately. To this effect, we introduce a constant a, such that the level matching condition becomes

$$(L_0 - a)|\Phi\rangle = 0 = (\tilde{L}_0 - a)|\Phi\rangle \tag{27}$$

for physical states Φ.

Problem 2.4. *How does the constant a influence the mass spectrum of the closed string?*

Problem 2.5. *Consider the Virasoro algebra (24).*

a) Show that if $A(1) \neq 0$, then it is possible to redefine L_0 such that $A(1)$ becomes zero.

b) The commutator obeys the Jacobi identity. Use it to derive an identity for $A(n)$.

c) Assuming $A(1) = 0$, show that

$$A(m) = (m^3 - m)\frac{A(2)}{6}. \tag{28}$$

d) Can you find a closed subalgebra of the Virasoro algebra? Write down its commutation relations, and identify what algebra it is.