String theory problem set 1

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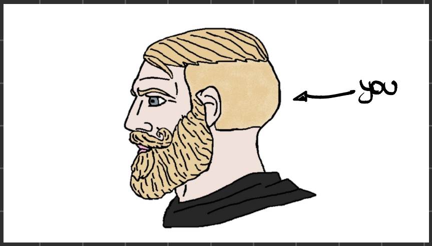
Before we start:

" Hahaha, you just enrolled in a useless course because string theory doesn't exist in reality "



" Maybe string theory as a theory of quantum gravity is not experimentally tested/testable at the moment but it is much more than a try at a physical theory, it is an extremely rich framework that has changed the way we look at fundamental physics. "

References:
- Becker, Becker, Schwarz → most modern
- Zwiebach → is a very simple introduction
- Green, Schwarz, Witten } more difficult
- Polchinski } more difficult
- David Tong's lecture notes] → excellent
- Di Francesco, Sénéchal, Mathieu] → "Yellow pages" Conformal Field Theory

PS 1 : and... action!

I. The relativistic point-particle action

CONVENTIONS: $\eta_{\mu\nu} = \mathrm{diag}(-1, \underbrace{1, \ldots, 1}_{D})$

String theory is a theory of extended objects: let us work our way up recursively on the dimension with 0-dim. objects: point particles.

The action has to be a Lorentz invariant: all Lorentz observers must agree on the value of the action along the particle worldline (ligne d'univers). A simple quantity everyone agrees on is the proper time elapsed, i.e. the time measured by a co-moving clock. Hence, we "guess"

$$S = \alpha \int \frac{ds}{c} = \frac{\alpha}{c} \int d\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} \qquad \dot{} \equiv \frac{d}{d\tau}$$

1.1. Let us set $c = 1$. The NR limit is $v \ll 1$ $\left(v/c \ll 1\right)$.

$$S = \alpha \int ds = \alpha \int \sqrt{dt^2 - d\vec{x}^2} = \alpha \int dt \sqrt{1 - \vec{v}^2}$$

$$\simeq \alpha \int dt \quad 1 - \frac{1}{2}\vec{v}^2 + \mathcal{O}(v^4)$$

In the NR limit, the Lagrangian should be $\underbrace{-m}_{\text{rest mass}} + \underbrace{\frac{1}{2} m \vec{v}^2}_{\text{kinetic energy}}$, i.e. $\alpha = -m$

1.2. a)

Take $S = -m \int d\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}$. Under $\tau \to \tilde{\tau}(\tau)$, $d\tau = \frac{d\tau}{d\tilde{\tau}} d\tilde{\tau}$

$$\frac{dX^\mu}{d\tau} = \frac{dX^\mu}{d\tilde{\tau}} \frac{d\tilde{\tau}}{d\tau}$$

$\longrightarrow$ S is unchanged because $\tilde{\tau}(\tau)$ is monotonic.

b) If τ can be changed to whatever, then time is not a physical degree of freedom, you don't "can move in time", you <u>have</u> to move in time. This gauge symmetry translates to a constraint for the momenta (1st class constraint in Hamiltonian formalism)

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m \dot{X}^\nu \eta_{\mu\nu}}{\sqrt{-\eta_{\alpha\beta}\dot{X}^\alpha \dot{X}^\beta}} \longrightarrow p_\mu p^\mu = -m^2$$

$$\hookrightarrow \boxed{p_\mu p^\mu + m^2 = 0} \quad \text{"mass-shell constraint"}$$

c) Why do we carry a fake degree of freedom? If not, Lorentz symmetry is not so easily apparent.

A Lorentz tf. takes $X^\mu \longrightarrow \Lambda^\mu{}_\nu X^\nu + C^\mu$. Then,

$$\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu \longrightarrow \underbrace{\Lambda^\mu{}_\rho \dot{X}^\rho \Lambda^\nu{}_\sigma \dot{X}^\sigma \eta_{\mu\nu}}_{=\eta_{\rho\sigma} \text{ by def.}} = \eta_{\rho\sigma}\dot{X}^\rho \dot{X}^\sigma$$

<u>1.3.</u> This action is simple, but has downsides:
- $m=0$ gives $S=0$ $\longrightarrow$ not suited for massless
- $\sqrt{\ }$ is problematic for path integral quantisation

Instead, use auxiliary action

$$S_{aux.} = \frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^2 - m^2 e\right)$$

We compute the EOM:

$$\delta_e S_{aux} = \frac{1}{2}\int d\tau \left(-\frac{1}{e^2}\delta e\, \dot{X}^2 - m^2 \delta e\right) \longrightarrow \dot{X}^2 = -m^2 e^2$$

$$\Leftrightarrow e = \frac{1}{m}\sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}$$

Then,

$$S'_{aux} = \frac{1}{2}\int d\tau \left(\frac{m\dot{X}^2}{\sqrt{-\dot{X}^2}} - m\sqrt{-\dot{X}^2}\right) = -m\int d\tau \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}.$$

1.4. a) A scalar field transforms $X'^{\mu}(\tau') = X^{\mu}(\tau)$. Under $\tau \to \tau' = \tau - \xi(\tau)$, we have

$$X'^{\mu}(\tau) = X'^{\mu}(\tau' + \xi(\tau)) = X'^{\mu}(\tau') + \xi(\tau)\dot{X}'^{\mu}(\tau') + \mathcal{O}(\xi^2)$$

$$= X^{\mu}(\tau) + \xi(\tau)\dot{X}^{\mu}(\tau) + \mathcal{O}(\xi^2)$$

$$\to \delta X^{\mu} = \xi(\tau)\dot{X}^{\mu}(\tau)$$

b), c), d) We write $e(\tau) = \sqrt{-G_{\tau\tau}}$

$$\to S_{aux.} = \frac{1}{2}\int d\tau \sqrt{-G_{\tau\tau}}\left(G^{\tau\tau}\partial_\tau X \cdot \partial_\tau X + m^2\right)$$

This is a theory of a scalar field X coupled to one-dimensional gravity. Then,

$$-G_{\tau\tau}(\tau)\, d\tau^2 = ds^2 = -G'_{\tau\tau}(\tau')\, d\tau'^2,$$

i.e. $e(\tau')\, d\tau' = e(\tau)\, d\tau$. Then,

$$e'(\tau) = e'(\tau') + \xi(\tau)\dot{e}'(\tau') + \mathcal{O}(\xi^2)$$
$$= e(\tau)\frac{d\tau'}{d\tau} + \xi(\tau)\dot{e}(\tau) + \mathcal{O}(\xi^2)$$

$$= \frac{d}{d\tau}\left(\xi(\tau)e(\tau)\right) + e(\tau) + \mathcal{O}(\xi^2) \quad \to \delta e(\tau) = \frac{d}{d\tau}\left(\xi(\tau)e(\tau)\right)$$

Then, under this time reparametrisation,

$$\delta S_{aux.} = \frac{1}{2}\int d\tau \left(-\frac{1}{e^2}\delta e\,\dot{X}^2 + 2e^{-1}\dot{X}^{\mu}\delta\dot{X}_{\mu} - m^2\delta e\right)$$

$= 0$ (check for yourself, it is an integral $\int d\tau \frac{d}{d\tau}(\dots)$)

1.5. We fix $e=1$. We already computed $\dot{X}^2 + m^2e^2 = 0$ $\to$ mass-shell condition

The other EOM is

$$\delta_X S_{aux.} = \frac{1}{2}\int d\tau\; -\dot{\left(e^{-1}\left(g_{\mu\nu}\dot{X}^{\nu}\right)\right)}\delta X^{\mu} + e^{-1}\dot{X}^{\mu}\dot{X}^{\nu}\partial_{\rho}g_{\mu\nu}\,\delta X^{\rho} \overset{e=1}{\iff} -\frac{d}{d\tau}\left(g_{\mu\nu}\dot{X}^{\nu}\right) + \partial_{\mu}g_{\rho\nu}\dot{X}^{\rho}\dot{X}^{\nu} = 0$$

You can check that this is equivalent to the geodesic eq.

$$\ddot{X}^\mu + \Gamma^\mu_{\rho\lambda} \dot{X}^\rho \dot{X}^\lambda = 0, \quad \text{with} \quad \Gamma^\mu_{\rho\lambda} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\rho g_{\alpha\lambda} + \partial_\lambda g_{\alpha\rho} - \partial_\alpha g_{\rho\lambda} \right).$$

II. The Nambu-Goto action

We now want to describe strings, i.e. 1-dim. objects. A point draws a worldline in spacetime and its length is the action. Strings describe worldsheets in spacetime, and we take their area! See exercise notes for details. The Nambu-Goto action is

$$S_{NG} = -T \int d^2\sigma \sqrt{-\det G_{\alpha\beta}}, \quad \text{with } G_{\alpha\beta} \text{ the induced metric on the worldsheet,}$$

spacetime coordinates

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}$$

worldsheet coordinates

2.1. We have

$$\begin{cases} G_{\tau\tau} = \dfrac{\partial X^\mu}{\partial \tau} \dfrac{\partial X^\nu}{\partial \tau} \eta_{\mu\nu} = \dot{X}^2, \\ G_{\sigma\sigma} = (X')^2, \quad X' = \dfrac{dX}{d\sigma} \\ G_{\tau\sigma} = G_{\sigma\tau} = \dot{X} \cdot X'. \end{cases}$$

Then, the determinant yields

$$S = -T \int d^2\sigma \sqrt{-\dot{X}^2 X'^2 + (\dot{X} \cdot X')^2}$$

2.2. For the same reasons as for the point-particle action, we want to work with an auxiliary action. Such an action is given by the Polyakov action:

$$S_{Pol.} = -\frac{T}{2} \int d^2\sigma \sqrt{-G}\, G^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$$

a) To prove classical equivalence, we compute the equations of motion.

$$\delta_G S_{Pol.} = -\frac{T}{2}\int d^2\sigma \left(-\frac{1}{2}\sqrt{-G}\, G_{\alpha\beta}\,\delta G^{\alpha\beta}\, G^{\lambda\rho}\,\partial_\lambda X^\mu \partial_\rho X_\mu + \sqrt{-G}\,\delta G^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X_\mu \right)$$

$$\Leftrightarrow \quad -\frac{1}{2} G_{\alpha\beta} G^{\lambda\rho} \partial_\lambda X^\mu \partial_\rho X_\mu + \partial_\alpha X^\mu \partial_\beta X_\mu = 0$$

$$\Leftrightarrow \quad \partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} G_{\alpha\beta} G^{\lambda\rho} \partial_\lambda X^\mu \partial_\rho X_\mu$$

$$\Rightarrow \quad \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)} = \frac{1}{2} G^{\lambda\rho} \partial_\lambda X^\mu \partial_\rho X_\mu \sqrt{-G}$$

b) Trivial by definition.

2.3. a) We compute the momenta: $P^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2\cdot X'^2}}$

$$P^\sigma_\mu = -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X}^2)X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}}$$

b) For this parametrisation, we have

$$\begin{cases} \dot{X}^0 = R, \quad \dot{X}^1 = \dot{X}^2 = 0 \\ X^{0\prime} = 0, \quad X^{1\prime} = R\cos\sigma, \quad X'_2 = -R\sin\sigma \end{cases} \longmapsto \quad \dot{X}^2 = -R^2, \quad X'^2 = R^2, \quad \dot{X}\cdot X' = 0.$$

Then,

$$P^\tau_\mu = T\dot{X}_\mu, \quad P^\sigma_\mu = -TX'_\mu$$

c) The equations of motion are given by $\partial_\tau P^\tau_\mu + \partial_\sigma P^\sigma_\mu = 0$, i.e. $T(\ddot{X}_\mu - X''_\mu) = 0$. Since $\ddot{X}_\mu = 0$ but not X''_μ, the EOMs are not satisfied.

d) Nevertheless, consider

$$P_\mu = T\int_0^{2\pi} d\sigma\, \dot{X}_\mu \longrightarrow P_0 \equiv E = T\int_0^{2\pi} R\, d\sigma = 2\pi R T,$$ then $[T] = \underbrace{\text{energy}}_{=\text{mass}}/\text{length}$, units of tension.

2.4. In arbitrary dimension, we have the brane action

$$S_p = -T_p \int d^{p+1}\sigma \sqrt{-G}$$

Under reparametrisations $\sigma \to \tilde{\sigma}(\sigma)$, we have, with $f^\alpha{}_\beta = \frac{\partial \sigma^\alpha}{\partial \tilde{\sigma}^\beta}$,

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu} = \frac{\partial X^\mu}{\partial \sigma^\gamma}\frac{\partial X^\nu}{\partial \sigma^\delta}\eta_{\mu\nu}\left(f^{-1}\right)^\gamma_\alpha \left(f^{-1}\right)^\delta_\beta$$

The Jacobian is defined as $J = \det f^\alpha{}_\beta$. Then, $\det G_{\alpha\beta} = J^{-2} \det\left(\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\gamma}\frac{\partial X^\nu}{\partial \sigma^\delta}\right)$, which is compensated by

$$d^{p+1}\sigma = J\, d^{p+1}\tilde{\sigma}.$$