Problem Set I: and... action! **PHYS-F483**

February 4, 2025

1 The relativistic point-particle action

As a prelude to our discussion of the string action, let us look at the action for a relativistic point-particle. An action is a Lorentz scalar: for *any* particle world-line, all inertial observers must compute the same value for the action. On a given worldline, such a Lorentz-invariant quantity is given by the proper time elapsed. A suitable action is thus given by

$$S = \frac{\alpha}{c} \int ds = \frac{\alpha}{c} \int d\tau \, \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}, \tag{1}$$

where $\mu = 0, ..., D-1$, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$ is the Minkowski metric and dotted quantities are derivatives with respect to the proper time τ.

Problem 1.1. *From now on, we will set $c = 1$. Take the non-relativistic limit of the action* (1) *to determine the physical expression of the overall constant α.*

Problem 1.2. *a) Show that the relativistic point-particle action has the property of reparametrisation invariance. To do this, consider a monotonic function*

$$\tilde{\tau} = \tilde{\tau}(\tau). \tag{2}$$

b) Reparametrisation invariance implies that the τ degree of freedom is not physical. This also shows in the momenta associated to the Lagrangian described in (1)*. Compute those momenta and verify that they obey the mass-shell constraint for a relativistic particle of mass m,*

$$p_\mu p^\mu + m^2 = 0. \tag{3}$$

c) The advantage of carrying around a "fake" degree of freedom is that Lorentz symmetry appears as a global symmetry on the worldline. Verify that it is indeed a Lorentz-invariant action.

The point-particle action (1) is problematic on multiple levels. First, setting the mass to zero cancels the action, i.e. this action is unable to describe massless particles. Secondly, it possesses a square root: in the path integral formalism, one would deal with physical quantities involving

$$\int \mathcal{D}X \, \mathcal{O} \, e^{iS[X]}$$

where $\mathcal{O}$ is any operator insertion. As you have learned (or will learn), it is quite straightforward to deal with a quadratic action in the path integral, i.e. performing Gaussian integrals, whereas working with a square root action will lead to the ill-definition of the path integral. One way to define a proper path integral is to work in the more fundamental Hamiltonian formalism. In this formalism, it can be shown that the point-particle action is equivalent to a more convenient action that we write as

$$S_{\text{aux.}} = \frac{1}{2} \int d\tau \left(e^{-1} \dot{X}^2 - m^2 e \right), \tag{4}$$

where $e(\tau)$ is an auxiliary field.

Problem 1.3. *Show that the action* (4) *is classically equivalent to the action* (1)*.*

Problem 1.4. *a) Consider an infinitesimal change of parametrisation $\tau \to \tau' = \tau - \xi(\tau)$. How do the scalar fields $X^\mu(\tau)$ vary under this transformation?*

b) The introduction of the auxiliary field e, also called einbein, effectively couples the scalar fields to one-dimensional gravity. By setting $e(\tau) \equiv \sqrt{-G_{\tau\tau}}$, we can rewrite the action as

$$S_{aux.} = \frac{1}{2} \int d\tau \sqrt{-G_{\tau\tau}} \left(G^{\tau\tau} \partial_\tau X \cdot \partial_\tau X + m^2 \right).$$

Knowing this, derive the variation of the auxiliary field under a reparametrisation.

c) Consider now that $g_{\mu\nu} = \eta_{\mu\nu}$. Prove that the variation of the action (4) *under a reparametrisation vanishes. Note that this results holds for a non-flat metric, it is simply more cumbersome to prove.*

Problem 1.5. *This symmetry allows us to fix the gauge and set $e = 1$. Check that the gauge-fixed action generates the mass-shell condition and the geodesic equation.*

2 The Nambu-Goto action

A particle in spacetime describes a line, its *worldline*. A string will therefore sweep a surface in spacetime, the *worldsheet*. This worldsheet is parametrised by a timelike coordinate τ and a spacelike coordinate σ. We will write these coordinates as $\sigma^\alpha = (\tau, \sigma)$. For closed strings, σ is periodic:

$$\sigma \in [0, 2\pi[. \tag{5}$$

As a surface embedded in spacetime, the worldsheet defines mappings from itself to Minkowski space, or *target space*, $X^\mu(\sigma, \tau)$, with μ ranging from 0 to $D - 1$.

An obvious generalisation of the point-particle action is that the string action should now depend on the area of the worldsheet. Although it is embedded in flat spacetime, the worldsheet is a curved surface. The metric for such a surface is called the *induced metric*, and is defined as

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}. \tag{6}$$

In differential geometry terms, the induced metric is the *pullback* of the ambient metric on the submanifold that the worldsheet is.

Back to the action: the area of the worldsheet is simply given by

$$A = \int d^2\sigma \sqrt{-\det G_{\alpha\beta}}. \tag{7}$$

The Nambu-Goto action is then simply

$$S = -T \int d^2\sigma \sqrt{-\det G_{\alpha\beta}}, \tag{8}$$

where T is called the *string tension.*

Problem 2.1. *Unwrap the definition* (8) *of the Nambu-Goto action.*

Deriving the Nambu-Goto action from a geometric point of view is pretty straightforward. Consider an infinitesimal area element dA represented by a parallelogram whose sides are given by two vectors $d\vec{v}_1$ and $d\vec{v}_2$ forming an angle θ, as shown in Fig. 1.

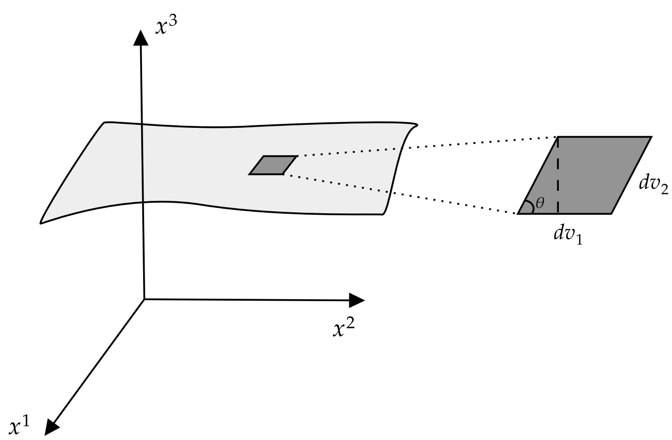

Figure 1: Worldsheet embedded in the target space. An infinitesimal area element is represented by a parallelogram with side lengths $|d\vec{v}_1|$ and $|d\vec{v}_2|$.

We have

$$\begin{aligned} dA &= |d\vec{v}_1||d\vec{v}_2||\sin\theta| = |d\vec{v}_1||d\vec{v}_2|\sqrt{1-\cos^2\theta} \\ &= \sqrt{(d\vec{v}_1\cdot d\vec{v}_1)(d\vec{v}_2\cdot d\vec{v}_2)-(d\vec{v}_1\cdot d\vec{v}_2)^2}. \end{aligned} \tag{9}$$

Let us parametrise the worldsheet by two quantities σ and τ. We choose $d\vec{v}_1$ along the direction of σ, and $d\vec{v}_2$ along the direction of τ, and consider an embedding of the worldsheet in Euclidean space $\vec{X}(\sigma,\tau)$. Then, the two vectors can be rewritten as

$$d\vec{v}_1 = \frac{\partial\vec{X}}{\partial\sigma}d\sigma, \quad d\vec{v}_2 = \frac{\partial\vec{X}}{\partial\tau}d\tau. \tag{10}$$

Then, it follows that the area A almost has the same form as the one from Problem 2.1, up to a sign in the square root. One can show that in Lorentzian signature, the quantity under the square root is negative[1] and thus requires a additional minus sign, which then reproduces the expected result. Although intuitively suitable, the Nambu-Goto action is not the preferred action to work with for quantisation reasons. In practice, one works with the *Polyakov* action.

Problem 2.2. *a) Show that the Nambu-Goto action* (8) *is classically equivalent to the Polyakov action*

$$S_{Pol.} = -\frac{T}{2}\int d^2\sigma\,\sqrt{-G}\,G^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu. \tag{11}$$

b) Show that the equations of motion imply that the worldsheet stress-energy tensor

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-G}}\frac{\delta S_{Pol.}}{\delta G^{\alpha\beta}} \tag{12}$$

vanishes.

Problem 2.3. *a) Compute the canonical momenta*

$$P^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \tag{13}$$

for the Nambu-Goto action (8)*.*

b) Consider a closed string lying at rest in the X^1, X^2 plane with

$$\begin{aligned} X^0 &= R\tau, \\ X^1 &= R\sin\sigma, \\ X^2 &= R\cos\sigma. \end{aligned} \tag{14}$$

Find the momenta P^α_μ.

c) Check whether the mappings X^μ defined above satisfy the string equations of motion.

d) Find the total momentum

$$P_\mu = \int_0^{2\pi} d\sigma\, P^\tau_\mu \tag{15}$$

and show that T really is the tension of the string.

String theory is not just a theory of strings, but rather a theory of extended objects. Just as we evolved from a point-particle action to a string action, we can generalise both cases by considering *p-branes*. In this vocabulary, the string is a one-dimensional brane, or a 1-brane. The p-brane action is given by the hypervolume of the $(p+1)$-dimensional surface swept by the p-brane in target space:

$$S_p = -T_p \int d^{p+1}\sigma\,\sqrt{-G}, \tag{16}$$

with $G_{\alpha\beta}$ the induced metric.

Problem 2.4. *Show that the action* (16) *is invariant under reparametrisations*

$$\sigma \mapsto \tilde{\sigma}(\sigma). \tag{17}$$

[1]Zwiebach, "A first course in string theory", pp. 108-110.