Notes on Quantum Field Theory

(in preparation)

Glenn Barnich*

Physique Théorique et Mathématique
Université libre de Bruxelles
and
International Solvay Institutes
Campus Plaine C.P. 231, B-1050 Bruxelles, Belgique
Bureau 2.O6. 217, E-mail: gbarnich@ulb.ac.be, Tel: 02 650 58 01.

The main objective of the course is to introduce quantum field theory techniques relevant both in the context of elementary particle physics and quantum statistical mechanics.

Contents

Chapter 1

Canonical quantization of free fields

In this chapter, selected chapters of quantum mechanics and of canonical quantization of free fields are reviewed in order to better appreciate the path integral treatment that follows. It follows corrresponding sections of [1] and of [2].

1.1 Time evolution

The dynamics of a quantum system is determined by the Hamiltonian $\widehat{H}[q,p]$. The time dependence of a matrix element of an operator $\widehat{A}[q,p]$ is determined by

$$i\hbar\frac{d}{dt}\langle\psi|\widehat{A}|\phi\rangle = \langle\psi|[\widehat{A},\widehat{H}]|\phi\rangle. \tag{1.1}$$

This equation is sufficient, since physical information is encoded in matrix elements. One may then choose to represent the time evolution of states and operators in different ways, compatible with the above: if $\widehat{H} = \widehat{M} + \widehat{N}$, with $\widehat{M}, \widehat{N}$ hermitian, one may define

$$i\hbar\frac{d}{dt}\widehat{A} = [\widehat{A},\widehat{M}], \quad i\hbar\frac{d}{dt}|\psi\rangle = \widehat{N}|\psi\rangle. \tag{1.2}$$

1.1.1 Schrödinger pricture

In this picture, all time dependence comes from the states, $\widehat{M} = 0, \widehat{N} = \widehat{H}$. It follows that

$$\widehat{A}_S(t) = \widehat{A}_S(t_0), \quad |\psi(t)\rangle_S = e^{\frac{-i}{\hbar}\widehat{H}(t-t_0)}|\psi(t_0)\rangle_S. \tag{1.3}$$

Note that this (formal) solution is valid for an Hamiltonian that does not depend explicitly on time.

1.1.2 Heisenberg picture

In this case, all time dependence lies with the operators, $\widehat{M} = \widehat{H}, \widehat{N} = 0$, so that

$$\widehat{A}_H(t) = e^{\frac{i}{\hbar}\widehat{H}(t-t_0)}\widehat{A}_H(t_0)e^{\frac{-i}{\hbar}\widehat{H}(t-t_0)}, \tag{1.4}$$

$|\psi(t)\rangle_H = |\psi(t_0)\rangle_H$. Note that $\widehat{H}_S = \widehat{H}_H$ and that one may choose to identify $|\psi(t_0)\rangle_S = |\psi(t_0)\rangle_H$, $\widehat{A}_S(t_0) = \widehat{A}_H(t_0)$.

1.1.3 Dirac picture

In case the Hamiltonian may be decomposed in a free (quadratic) piece, and an interaction, $\widehat{H} = \widehat{H}_0 + \widehat{V}$, the Dirac or interaction picture consists in transforming operators with the free Hamiltonian $\widehat{H}_0$, and states with the interacting Hamiltonian $\widehat{V}$, $\widehat{M} = \widehat{H}_0, \widehat{N} = \widehat{V}$. Operators in this picture are written as $\widehat{A}_I(t)$. When $\widehat{H}_0$ is time independent,

$$\widehat{A}_I(t) = e^{\frac{i}{\hbar}\widehat{H}_0(t-t_0)}\widehat{A}_I(t_0)e^{\frac{-i}{\hbar}\widehat{H}_0(t-t_0)}, \tag{1.5}$$

while states $|\psi(t)\rangle_I$ satisfy $i\hbar\frac{d}{dt}|\psi\rangle_I = \widehat{V}_I|\psi\rangle_I$, with

$$\widehat{V}_I(t) = e^{\frac{i}{\hbar}\widehat{H}_0(t-t_0)}\widehat{V}_I(t_0)e^{\frac{-i}{\hbar}\widehat{H}_0(t-t_0)}. \tag{1.6}$$

Again, operators and states in different pictures are identified at t_0.

To find the solution for the evolution equation of states, define

$$\boxed{|\psi(t)\rangle_I = \widehat{U}_I(t,t_0)|\psi(t_0)\rangle_I}. \tag{1.7}$$

This implies that $\widehat{U}_I(t,t) = \widehat{1}$ and $\frac{d}{dt}\widehat{U}_I(t,t_0) = \frac{-i}{\hbar}\widehat{V}_I(t)\widehat{U}_I(t,t_0)$.

Integrating and iterating, one finds

$$\widehat{U}_I(t,t_0) = \widehat{1} - \frac{i}{\hbar}\int_{t_0}^t d\tau\,\widehat{V}_I(\tau)\widehat{U}_I(\tau,t_0) = \sum_{N=0}(-\frac{i}{\hbar})^N\int_{t_0}^t d\tau_1\cdots\int_{t_0}^{\tau_{N-1}} d\tau_N\,\widehat{V}_I(\tau_1)\ldots\widehat{V}_I(\tau_N).$$

To show that this gives

$$\sum_{N=0}\frac{(-\frac{i}{\hbar})^N}{N!}\int_{t_0}^t d\tau_1\cdots\int_{t_0}^t d\tau_N\,\mathrm{T}\{\widehat{V}_I(\tau_1)\ldots\widehat{V}_I(\tau_N)\},$$

one realizes that for $t = t_0$, the expression gives $\widehat{1}$ and that the derivative with respect to t of this expression gives $-\frac{i}{\hbar}\widehat{V}_I(t)$ times the expression itself. The expression thus satisfies the same first order differential equation with the same initial condition than $\widehat{U}_I(t,t_0)$, so that both are equal.

One then finds

$$\boxed{\widehat{U}_I(t,t_0) = Te^{-\frac{i}{\hbar}\int_{t_0}^t d\tau\widehat{V}_I(\tau)}}. \tag{1.8}$$

Identifying $|\psi(t_0)>_I = |\psi(t_0)>_H = |\psi>_H$, one gets from $_H<\psi|\widehat{A}_H(t)|\psi>_H = {}_I<\psi(t)|\widehat{A}_I(t)|\psi(t)>_I$ that $\widehat{A}_I(t) = \widehat{U}_I(t,t_0)\widehat{A}_H(t)\widehat{U}_I(t_0,t)$.

The evolution operator satisfies $\widehat{U}^{-1}(t,t_0) = \widehat{U}(t_0,t) = \widehat{U}^\dagger(t,t_0)$.

1.1.4 Decoupled harmonic oscillators

Consider a collection of n decoupled harmonic oscillators with frequencies ω_a, $a = 1\ldots n$. The system is described by the Lagrangian

$$L = \frac{1}{2}\dot{q}^a\dot{q}_a - \frac{1}{2}\omega^2_{ab}q^aq^b, \quad a,b = 1\ldots n, \quad \omega^2_{ab} = \omega^2_{(a)}\delta_{ab} \tag{1.9}$$

where a sum over repeated indices is understood, while a parenthesis indicates the absence of summation. Indices are lowered and raised with the Kronecker delta δ_{ab} and its inverse δ^{ab}. Canonical momenta are $p_a = \dot{q}_a$, the Hamiltonian is

$$H = \frac{1}{2}p_ap^a + \frac{1}{2}\omega^2_{ab}q^aq^b. \tag{1.10}$$

If oscillator variables are defined as

$$\widehat{a}_a = \frac{\sqrt{\omega_{(a)}}\widehat{q}^a + i\frac{\widehat{p}^a}{\sqrt{\omega_{(a)}}}}{\sqrt{2\hbar}}, \tag{1.11}$$

with inverse transformation

$$\widehat{q}^a = \sqrt{\frac{\hbar}{2\omega_{(a)}}}(\widehat{a}^a + \widehat{a}^{a\dagger}), \quad \widehat{p}_a = -i\sqrt{\frac{\hbar\omega_{(a)}}{2}}(\widehat{a}_a - \widehat{a}^\dagger_a), \tag{1.12}$$

the canonical commutation relations

$$[\widehat{q}^a, \widehat{p}_b] = -i\hbar\delta^a_b, \quad [\widehat{q}^a, \widehat{q}^b] = 0 = [\widehat{p}_a, \widehat{p}_b], \tag{1.13}$$

are equivalent to

$$[\widehat{a}^a, \widehat{a}^{\dagger b}] = \delta^{ab}, \quad [\widehat{a}^a, \widehat{a}^b] = 0 = [\widehat{a}^{\dagger a}, \widehat{a}^{\dagger b}], \tag{1.14}$$

while the Hamiltonian is given by

$$\boxed{\widehat{H} = \hbar\omega_{ab}(\widehat{a}^{\dagger a}\widehat{a}^b + \frac{1}{2}\delta^{ab})}, \quad \omega_{ab} = \omega_{(a)}\delta_{ab} \tag{1.15}$$

The system may be quantized in terms of a complete set of orthonormal states of the form

$$|n_1, \dots n_n\rangle = \frac{1}{\sqrt{n_1!}} \dots \frac{1}{\sqrt{n_n!}}(\widehat{a}^\dagger_1)^{n_1} \dots (\widehat{a}^\dagger_n)^{n_n}|0\rangle, \quad \widehat{a}_a|0\rangle = 0. \tag{1.16}$$

1.2 Canonical quantization of the real scalar field

In this section, we briefly recall canonical quantization of the free scalar field. It is not really needed for the course itself, but it allows one to better appreciate the relation between path integral and operator quantization. We set $\hbar = 1$.

1.2.1 Mode expansion and Hamiltonian

Action:

$$S^{KG} = \int d^4x[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2]. \tag{1.17}$$

Canonical momenta: $\pi(\vec{x}, t) = \partial_0\phi(\vec{x}, t)$,
Poisson brackets: $\{\phi(\vec{x}, t), \pi(\vec{y}, t)\} = \delta(\vec{x} - \vec{y})$.
Hamiltonian:

$$H_0 = \int d^3x[\frac{1}{2}\pi^2(\vec{x}, t) + \frac{1}{2}\partial_k\phi(\vec{x}, t)\partial^k\phi(\vec{x}, t) + \frac{1}{2}m^2\phi(\vec{x}, t)^2] \tag{1.18}$$

Fourier transform:

$$\widetilde{\phi}(\vec{k}, t) = \int d^3x\, e^{-i\vec{k}\vec{x}}\, \phi(\vec{x}, t), \tag{1.19}$$

$$\widetilde{\pi}(\vec{k}, t) = \int d^3x\, e^{-i\vec{k}\vec{x}}\, \pi(\vec{x}, t). \tag{1.20}$$

Oscillator variables:

$$a(\vec{k},t) = \frac{1}{(2\pi)^{3/2}}[\sqrt{\frac{\omega(\vec{k})}{2}}\,\widetilde{\phi}(\vec{k},t) + \frac{i}{\sqrt{2\omega(\vec{k})}}\,\widetilde{\pi}(\vec{k},t)], \tag{1.21}$$

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(\vec{k})}}[a(\vec{k},t)e^{i\vec{k}\cdot\vec{x}} + \text{h.c.}], \tag{1.22}$$

$$\pi(\vec{x},t) = -i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega(\vec{k})}{2}}[a(\vec{k},t)e^{i\vec{k}\cdot\vec{x}} - \text{h.c.}] \tag{1.23}$$

with $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$.
quantization : $[\widehat{a}(\vec{k},t), \widehat{a}^+(\vec{k}',t)] = \delta(\vec{k} - \vec{k}')$.

$$\boxed{:\widehat{H}_0 := \int d^3k\,\omega(\vec{k})\widehat{a}^+(\vec{k},t)\widehat{a}(\vec{k},t)}. \tag{1.24}$$

Why does one need to normal order ? When keeping expressions symmetric in a and a^* before quantization, one finds $\widehat{H}_0 = \int d^3k\,\omega(\vec{k})[\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}) + \frac{1}{2}]$. The last term does not depend on the oscillators. When going to spherical coordinates in momentum space, it is given by $\frac{1}{2}(4\pi)\int_0^\infty dk k^2\sqrt{k^2+m^2}$, which diverges. If the system would be in a box with periodic boundary conditions, one would find instead $\widehat{H}_0 = \sum_{\vec{k}}\omega_{\vec{k}}[\widehat{a}^\dagger_{\vec{k}}\widehat{a}_{\vec{k}} + \frac{1}{2}]$. The $\frac{1}{2}$ corresponds to so-called "zero point energy" of an oscillator. In this case, the sum diverges. Normal ordering avoids the appearance of these divergences. For most questions we are interested in, only energy differences are relevant and normal ordering is a choice that we can do.

1.2.2 Dynamics

Equations of motion :

$$\frac{d}{dt}\widehat{a}(\vec{k},t) = -i[\widehat{a}(\vec{k},t), :\widehat{H}_0:] = -i\omega(\vec{k})\widehat{a}(\vec{k},t) \quad \Longrightarrow \quad \widehat{a}(\vec{k},t) = e^{-i\omega(\vec{k})t}\,\widehat{a}(\vec{k}), \tag{1.25}$$

$$\boxed{\widehat{\phi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(\vec{k})}}[\widehat{a}(\vec{k})e^{ik\cdot x} + \text{h.c.}]} \tag{1.26}$$

$$= \widehat{\phi}^{(+)}(x) + \widehat{\phi}^{(-)}(x), \tag{1.27}$$

$$\widehat{\phi}^{(+)}(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(\vec{k})}}\widehat{a}(\vec{k})e^{ik\cdot x}, \quad \widehat{\phi}^{(-)}(x) = (\widehat{\phi}^{(+)}(x))^\dagger, \tag{1.28}$$

where $k\cdot x = k_\mu x^\mu$ and $k^0 = \omega(\vec{k}) = -k_0$. One says that $\widehat{\phi}^{(+)}(x)$ contains positive frequencies, while $\widehat{\phi}^{(-)}(x)$ contains negative frequencies.

1.2.3 Two-point function

2-point function:

$$\boxed{\frac{1}{i}\Delta_F(x-y) \equiv < 0|T\widehat{\phi}(x)\widehat{\phi}(y)|0 >} =< 0|\widehat{\phi}^{(+)}(x)\widehat{\phi}^{(-)}(y)|0 > \theta(x^0-y^0) + (x \leftrightarrow y),$$
$$=< 0|\left[\widehat{\phi}^{(+)}(x), \widehat{\phi}^{(-)}(y)\right]|0 > \theta(x^0-y^0) + (x \leftrightarrow y). \quad (1.29)$$

$$[\widehat{\phi}^{(+)}(x), \widehat{\phi}^{(-)}(y)\Big] = \frac{1}{(2\pi)^3}\int d^3k d^3k' e^{ik\cdot x}e^{-ik'\cdot y}\frac{1}{2\sqrt{\omega(\vec{k})\omega(\vec{k}')}}\delta(\vec{k}-\vec{k}')$$
$$= \frac{1}{(2\pi)^3}\int d^3k \frac{1}{2\omega(\vec{k})}e^{ik\cdot(x-y)}. \quad (1.30)$$

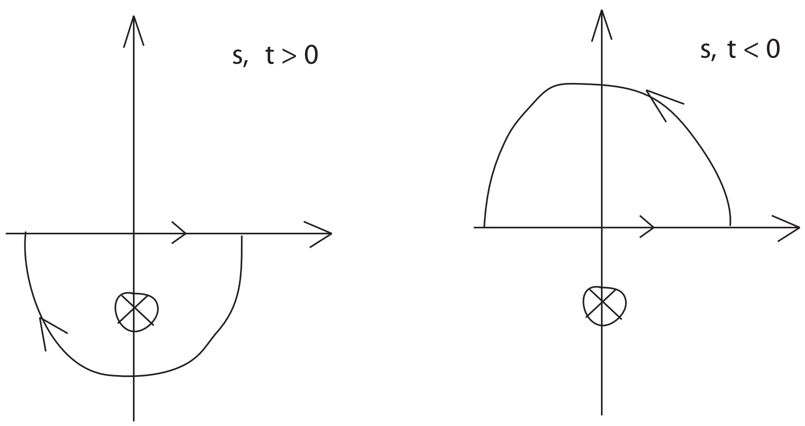
It follows that

$$< 0|T\widehat{\phi}(x)\widehat{\phi}(y)|0 >= \frac{1}{(2\pi)^3}\int d^3k \frac{1}{2\omega(\vec{k})}[e^{ik\cdot(x-y)}\theta(x^0-y^0) + (x \leftrightarrow y)]. \quad (1.31)$$

Note that

$$\theta(t) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} ds \frac{e^{-ist}}{s+i\epsilon}.$$

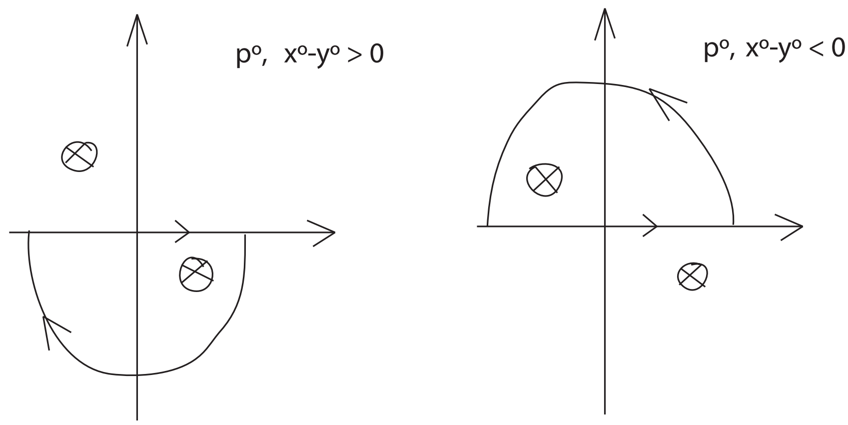
Indeed, if $t > 0$ one closes the integration contour in the lower half-plane of the complex plane of s, and one catches the contribution from the pole in $s = -i\epsilon$. If $t < 0$, one closes the contour in the upper-half plane, there is no pole and the result is zero.



Injecting this expression gives

$$\boxed{\Delta_F(x-y)} = -\frac{1}{(2\pi)^4}\int d^4k \frac{1}{2\omega(\vec{k})}\frac{1}{k^0+i\epsilon}e^{[-i(k^0+\omega(\vec{k}))(x^0-y^0)+i\vec{k}\cdot(\vec{x}-\vec{y})]} + (x \leftrightarrow y)$$
$$= -\frac{1}{(2\pi)^4}\int d^4p \frac{1}{2\omega(\vec{p})}\frac{1}{p^0-\omega(\vec{p})+i\epsilon}e^{ip\cdot(x-y)} + (x \leftrightarrow y)$$
$$= -\frac{1}{(2\pi)^4}\int d^4p \frac{1}{2\omega(\vec{p})}\left(\frac{1}{p^0-\omega(\vec{p})+i\epsilon} + \frac{1}{-p^0-\omega(\vec{p})+i\epsilon}\right)e^{ip\cdot(x-y)}$$
$$= \frac{1}{(2\pi)^4}\int d^4p \frac{1-i\epsilon/\omega(\vec{p})}{p^2+m^2-2i\epsilon\omega(\vec{p})}e^{ip\cdot(x-y)}$$
$$= \boxed{\frac{1}{(2\pi)^4}\int d^4p \frac{1}{p^2+m^2-i\epsilon}e^{ip\cdot(x-y)}}. \quad (1.32)$$

The last line follows from the fact that only the pole for p^0 matter. These poles are determined by $(p^0)^2 = \vec{p}^2 + m^2 - i\epsilon$ and thus $p^0 = \sqrt{\vec{p}^2 + m^2} - i\epsilon$ and $p^0 = -\sqrt{\vec{p}^2 + m^2} + i\epsilon$. If $x^0 - y^0 > 0$, one closes the contour in the lower half-plane, and one catches the first pole. If $x^0 - y^0 < 0$, one closes the contour in the upper half-plane, and one catches the second pole. This gives back the expression from two lines before.



The expression (1.32) allows one to easily verify that $\Delta_F(x-y)$ is a Green's function for the Klein-Gordon operator, $(\Box - m^2)\Delta_F(x-y) = -\delta^4(x-y)$. Furthermore, if $x^0 - y^0 > 0$, one sees from (1.31) that $\Delta_F(x-y)$ only contains positive frequencies, while if $x^0 - y^0 < 0$, there are only negative frequencies. One says that $\Delta_F(x-y)$ satisfies boundary conditions of radiation type.

1.3 S-matrix

In an interacting theory, $\widehat{H} = \widehat{H}_0 + \widehat{V}$, the problem is in general no longer linear and cannot be solved exactly. In particle physics, one is interested in scattering processes, with free particles far from each other, that interact, and that become free again later on.

To describe this situation, one assumes that the Fock space of the interacting theory is the same as the Fock space of the free theory associated with $\widehat{H}_0$. In Heisenberg picture, one assumes that, inside matrix elements, the field operator is proportional to the one of the free field if one goes sufficiently far to the past or the future:

$$\widehat{\phi}(\vec{x},t) \overset{t\to\pm\infty}{\longrightarrow} Z^{1/2}\widehat{\phi}_{\substack{out\\in}}(\vec{x},\pm\infty),$$

where $Z^{1/2}$ is a proportionality factor and $\widehat{\phi}_{\substack{out\\in}}$ are the free fields explicitly defined in the previous paragraph[1].

Let us denote by $|\alpha, {}^{out}_{in} >$ orthonormal basis vectors of Fock space in the far future and far past, respectively, that is to say the Fock space created by $(\widehat{a}^+)^{out}_{in}(\vec{k})$. In Heisenberg picture, these states describe the system for all times, but an observer analyzing a state $|\alpha, in>$ at $t \to -\infty$ does so with $\widehat{a}_{in}(\vec{k})$. This state may appear for instance as describing a simple set of free particles, prepared in advance. If an observer analyzes the same state at $t \to +\infty$, he will do so with operators $\widehat{a}_{out}(\vec{k})$, and now it can appear like a complicated superposition of free particles that is the result of a scattering process at finite t.

[1]A way to implement this idea is to consider spacetime dependent "coupling constants" $g(x)$ in $\widehat{V}$ that vanish in the far past and the far future. In a next step this requires one to discuss the "adiabatic" limit $g(x) \to g\,, cste$ [3].

The S matrix is defined by

$$S_{\beta\alpha} =< \beta; out|\alpha; in > . \tag{1.33}$$

It describes the probability amplitude to go from a free particle state in the far past to a free particle state in the far future. Unitarity of the S matrix is a direct consequence of the fact that the bases $|\alpha, {}^{out}_{in} >$ are orthonormal: $\sum_\gamma < \beta; in|\gamma; out >< \gamma; out|\alpha; in >= \delta_{\beta\alpha}$ implies that $\sum_\gamma S^\dagger_{\beta\gamma} S_{\gamma\alpha} = \delta_{\beta\alpha}$.

Let us introduce in addition $|\alpha\rangle$, a basis for the Fock space of the free theory with Hamiltonian $\widehat{H}_0$ in Heisenberg picture. The states $|\alpha\rangle$, $\langle\beta|$ have the same particle content as the states $|\alpha; in\rangle$, $\langle\beta; out|$. The $\widehat{S}$ operator is then defined as the operator of the Fock space of the free theory that satisfies

$$S_{\beta\alpha} =< \beta|\widehat{S}|\alpha > . \tag{1.34}$$

NB: As will appear in the discussion below, there are now two types of identifications: the identification of states for different pictures at a given time as discussed before, and also how to identify states of the free theory with states of the interacting theory.

It is then natural to use the Dirac picture for the interacting theory and the Heisenberg picture for the free theory, because then the time evolution of operators is the same in the free and the interacting theory. One chooses to identify states of the Heisenberg and Dirac picture of the interacting theory at time $t = 0$. In the absence of interactions, at $t \to |\infty|$, states in the Dirac picture no longer move, so that $|\alpha(-\infty)\rangle_I = |\alpha\rangle$, ${}_I\langle\beta(+\infty)| = \langle\beta|$. This implies

$$\begin{aligned} < \beta|\widehat{S}|\alpha > &=< \beta; out|\alpha; in >= {}_I\langle\beta(0)|\alpha(0)\rangle_I \\ &= {}_I\langle\beta(+\infty)|\widehat{U}_I(+\infty, -\infty)|\alpha(-\infty)\rangle_I = \langle\beta|\widehat{U}_I(+\infty, -\infty)|\alpha\rangle. \end{aligned} \tag{1.35}$$

An alternative proof in Schrödinger picture proceeds as follows. One identifies $|\alpha; in(-\infty)\rangle_S = |\alpha(-\infty)\rangle_S$, ${}_S\langle\beta; out(+\infty)| = {}_S\langle\beta; (+\infty)|$ with $i\frac{\partial}{\partial t}|\alpha; in(t)\rangle_S = \widehat{H}|\alpha; in(t)\rangle_S$ and $i\frac{\partial}{\partial t}|\alpha; (t)\rangle_S = \widehat{H}_0|\alpha; (t)\rangle_S$. Introducing the notation $\widehat{U}(t', t) = e^{-i\widehat{H}(t'-t)}$ and $\widehat{U}_0(t', t) = e^{-i\widehat{H}_0(t'-t)}$, we have

$$\begin{aligned} < \beta|\widehat{S}|\alpha >&=< \beta; out|\alpha; in >= {}_S\langle\beta; out(0)|\alpha; in(0)\rangle_S = {}_S\langle\beta; out(+\infty)|\widehat{U}(\infty, 0)\widehat{U}(0, -\infty)|\alpha; in(-\infty)\rangle_S \\ &= {}_S\langle\beta; (+\infty)|\widehat{U}(+\infty, -\infty)|\alpha(-\infty)\rangle_S = {}_S\langle\beta(0)|\widehat{U}_0(0, \infty)\widehat{U}(+\infty, -\infty)\widehat{U}_0(-\infty, 0)|\alpha(0)\rangle_S. \end{aligned} \tag{1.36}$$

This gives

$$\widehat{S} = \widehat{U}_I(+\infty, -\infty) = \lim_{t\to\infty, t_0\to-\infty} e^{i\widehat{H}_0 t} e^{-i\widehat{H}(t-t_0)} e^{-i\widehat{H}_0 t_0}. \tag{1.37}$$

The two expressions are compatible since

$$\widehat{U}_I(t, t_0) = e^{i\widehat{H}_0 t} e^{-i\widehat{H}(t-t_0)} e^{-i\widehat{H}_0 t_0} = \widehat{\Omega}^\dagger(t)\widehat{\Omega}(t_0), \quad \widehat{\Omega}(t) = e^{i\widehat{H}t} e^{-i\widehat{H}_0 t}. \tag{1.38}$$

Indeed, the right hand side satisfies the same initial condition and the same first order differential equation as $\widehat{U}_I(t, t_0)$,

$$\frac{d}{dt}\widehat{U}_I(t, t_0) = e^{i\widehat{H}_0 t}[i(\widehat{H}_0 - \widehat{H})]e^{-i\widehat{H}(t-t_0)} e^{-i\widehat{H}_0 t_0} = -i\widehat{V}_I(t)\widehat{U}_I(t, t_0).$$

In particular, we deduce that

$$|\beta; out >= \widehat{\Omega}(+\infty)|\beta >, \quad |\alpha; in >= \widehat{\Omega}(-\infty)|\alpha > . \tag{1.39}$$

Combining with the first equality of (1.37) with (1.8), we have also shown the Dyson series for $\widehat{S}$,

$$\boxed{\widehat{S} = Te^{-i\int_{-\infty}^{+\infty} d\tau \widehat{V}_I(\tau)}}. \tag{1.40}$$

1.4 Partition function for massless scalar in d spatial dimensions

1.4.1 Mode expansion

Consider a massless scalar field ϕ in d spatial dimensions, whose Lagrangian action reads

$$S[\phi] = -\frac{1}{2}\int dx^0 \int_{V_d} d^d x\, \partial_\mu\phi\partial^\mu\phi, \tag{1.41}$$

while the first order Hamiltonian action is

$$S_H[\phi,\pi] = \int dx^0 [\int_{V_d} d^d x\, \dot{\phi}\pi - H], \quad H = \frac{1}{2}\int_{V_d} d^d x\big(\pi^2 + \partial_i\phi\partial^i\phi\big), \tag{1.42}$$

with associated canonical Poisson brackets

$$\{\phi(x^0,x), \pi(x^0,x')\} = \delta^d(x-x'). \tag{1.43}$$

Our conventions for indices and their ranges are as follows: the spacetime and spatial indices are $\mu = 0,...,d$, and $i = 1,...,d$, respectively. Besides the Hamiltonian, the other observables that are relevant for us here are linear momenta in the x^i direction,

$$P_i = -\int_{V_d} d^d x\, \pi\partial_i\phi. \tag{1.44}$$

Our aim is to discuss analytic expressions in terms of Eisenstein series and associated modular properties of partition functions of the type

$$Z_d(\beta,\mu^j) = \mathrm{Tr}\, e^{-\beta(\widehat{H} - i\mu^j\widehat{P}_j)}, \tag{1.45}$$

and generalizations thereof. In order to take advantage of techniques in complex analysis, we use purely imaginary chemical potentials $i\mu^j, \mu^j \in \mathbb{R}$, as in 2-dimensional conformal field theory and also for instance in [4, 5, 6].

In order to evaluate such partition functions, a standard procedure is to put the system in a hyperrectangular box of volume $V_d = \prod_{i=1}^d L_i$ and to choose periodic boundary conditions in all spatial dimensions,

$$\phi(x^0, x^1, ..., x^i, ..., x^d) = \phi(x^0, x^1, ..., x^i + L_i, ..., x^d). \tag{1.46}$$

The appropriate orthonormal basis vectors $\{e_{k_i}\}$ are plane waves,

$$e_{k_i}(x) = \frac{1}{\sqrt{V_d}} e^{ik_i x^i}, \quad k_i = \frac{2\pi n_i}{L_i}, \quad \big(e_{k_i}, e_{k'_i}\big) = \int_{V_d} d^d x\, e^*_{k_i}(x) e_{k'_i}(x) = \prod_i \delta_{n_i,n'_i}. \tag{1.47}$$

The fields ϕ, π admit the mode decomposition

$$\phi(x) = \sum_{n_i\in\mathbb{Z}^d} \phi_{k_i} e_{k_i}(x), \quad \pi(x) = \sum_{n_i\in\mathbb{Z}^d} \pi_{k_i}(x) e_{k_i}, \tag{1.48}$$

with reality conditions $\phi_{k_i} = \phi^*_{-k_i}, \pi_{k_i} = \pi^*_{-k_i}$. In terms of the Fourier components, the observables are

$$H = \frac{1}{2}\sum_{n_i\in\mathbb{Z}^d}(\pi_{k_i}\pi^*_{k_i} + \omega^2_{k_i}\phi_{k_i}\phi^*_{k_i}), \quad P_i = -i\sum_{n_i\in\mathbb{Z}^d} k_i\phi_{k_i}\pi^*_{k_i}, \tag{1.49}$$

where $\omega_{k_i} = \sqrt{k_j k^j}$, while the non-vanishing canonical Poisson brackets become

$$\{\phi_{k_i}, \pi^*_{k'_i}\} = \prod_i \delta_{n_i,n'_i}. \tag{1.50}$$

The zero mode is denoted by $\phi_0 = q, \pi_0 = p$ and, for the Fourier components with $n_i \neq (0, \dots, 0)$, one defines standard oscillator variables,

$$a_{k_i} = \sqrt{\frac{\omega_{k_i}}{2}} \big[\phi_{k_i} + \frac{i}{\omega_{k_i}} \pi_{k_i}\big]. \tag{1.51}$$

In these terms,

$$\begin{aligned} \phi &= V_d^{-\frac{d-1}{2d}} q + \frac{1}{\sqrt{V_d}} \sum_{n_i \in \mathbb{Z}^d}{}' \frac{1}{\sqrt{2\omega_{k_i}}} (a_{k_i} e^{ik_j x^j} + \text{c.c.}), \\ \pi &= V_d^{-\frac{d+1}{2d}} p - \frac{i}{\sqrt{V_d}} \sum_{n_i \in \mathbb{Z}^d}{}' \sqrt{\frac{\omega_{k_i}}{2}} (a_{k_i} e^{ik_j x^j} - \text{c.c.}), \end{aligned} \tag{1.52}$$

where the prime indicates that the term with $n_i = (0, \dots, 0)$ is excluded from the sum and q, p are dimensionless. The observables are given by

$$H = V_d^{-\frac{1}{d}} \frac{p^2}{2} + \frac{1}{2} \sum_{n_i \in \mathbb{Z}^d}{}' \omega_{k_i} (a^*_{k_i} a_{k_i} + a_{k_i} a^*_{k_i}), \quad P_i = \frac{1}{2} \sum_{n_i \in \mathbb{Z}^d}{}' k_i (a^*_{k_i} a_{k_i} + a_{k_i} a^*_{k_i}), \tag{1.53}$$

while the non-vanishing Poisson brackets become

$$\{q, p\} = 1, \quad \{a_{k_i}, a^*_{k_i}\} = -i \prod_i \delta_{n_i, n'_i}. \tag{1.54}$$

1.4.2 Canonical approach to partition function in large volume limit

Throughout this section, where we will consider all spatial dimensions to be large, we will also take the chemical potentials μ^j to vanish. How the final result changes when they are turned on will be discussed from a more general perspective later.

1.4.2.1 Zero mode contribution

It follows from the expression of the Hamiltonian (1.53) that the zero mode sector of the theory corresponds to a free particle with Hamiltonian $H_0 = V_d^{-\frac{1}{d}} \frac{1}{2} p^2$. Therefore one may quantize the full theory in a Fock space for which the vacua $|p\rangle$ are labelled by the continuous eigenvalues p of $\widehat{p}$,

$$\widehat{p}|p\rangle = p|p\rangle, \quad \widehat{a}_{k_i}|p\rangle = 0. \tag{1.55}$$

It follows that

$$Z_0(\beta) = \mathrm{Tr}\, e^{-\beta \widehat{H}_0} = \int_{-\infty}^{+\infty} \mathrm{d}p\, e^{-\frac{b}{2}p^2} \langle p|p\rangle = \sqrt{\frac{2\pi}{b}} \delta(0), \quad b = V_d^{-\frac{1}{d}} \beta. \tag{1.56}$$

$\ln Z_0(\beta)$ contains a finite and an infinite part. In most of the problems, one discards both of them, but in some problems, one keeps the finite part.

1.4.2.2 Oscillator contribution

For the oscillator modes quantized in Fock space, one may directly evaluate the trace using the normal ordered Hamiltonian $:\widehat{H}':$ so that no divergences occur. If N_{k_i} denotes the occupation number of the oscillator associated to k_i, we have,

$$Z'_d(\beta) = \mathrm{Tr}\, e^{-\beta :\widehat{H}':} = \prod_{n_i \in \mathbb{Z}^d}{}' \sum_{N_{k_i} \in \mathbb{N}} e^{-\beta \omega_{k_i} N_{k_i}} = \prod_{n_i \in \mathbb{Z}^d}{}' \frac{1}{1 - e^{-\beta \omega_{k_i}}}, \tag{1.57}$$

and thus

$$\ln Z'_d(\beta) = -\sum_{n_i\in\mathbb{Z}^d}{}' \ln(1-e^{-\beta\omega_{k_i}}). \tag{1.58}$$

For large L_i, the sums can be approximated by integrals using $\mathrm{d}k_i = \frac{2\pi}{L_i}\mathrm{d}n_i$. After performing the integrals over the angles in hyperspherical coordinates, one finds

$$\ln Z'_d(\beta) = -\frac{V_d}{(2\pi)^d}\int d^dk \ln(1-e^{-\beta\omega_{k_i}}) = -\frac{V_d}{(2\pi)^d}\mathrm{Vol}(\mathbb{S}^{d-1})\int_0^\infty dk k^{d-1}\ln(1-e^{-\beta k}). \tag{1.59}$$

Since the contribution of the zero mode $\ln Z_0(\beta)\sim \ln V_d$ is subdominant in the large volume limit, it may be neglected unless otherwise specified.

On the one hand, $\mathrm{Vol}(\mathbb{S}^{d-1}) = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ while, on the other, the integral becomes after the change of variables $x=\beta k$ and a standard integrations by parts,

$$\int_0^\infty dk k^{d-1}\ln(1-e^{-\beta k}) = -\frac{1}{d\beta^d}\int_0^\infty dx\frac{x^d}{e^x-1} = -\frac{1}{d\beta^d}\Gamma(d+1)\zeta(d+1). \tag{1.60}$$

One thus ends up with the "scalar black body" partition function

$$\ln Z_d(\beta) = \frac{\Gamma(d)\zeta(d+1)}{2^{d-1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}\frac{V_d}{\beta^d}. \tag{1.61}$$

Furthermore, using the reduplication formula

$$\Gamma(z)\sqrt{\pi} = 2^{z-1}\Gamma(\frac{z}{2})\Gamma(\frac{z+1}{2}), \tag{1.62}$$

at $z=d$, and the definition of the completion of the zeta function,

$$\xi(z) = \frac{\Gamma(\frac{z}{2})\zeta(z)}{\pi^{\frac{z}{2}}}, \tag{1.63}$$

at $z=d+1$, the result can be written as

$$\boxed{\ln Z_d(\beta) = \xi(d+1)\frac{V_d}{\beta^d}} = \begin{cases} \frac{\pi}{6}\frac{L_1}{\beta} & \text{for } d=1 \\ \frac{\zeta(3)}{2\pi}\frac{L_1L_2}{\beta^2} & \text{for } d=2 \\ \frac{\pi^2}{90}\frac{L_1L_2L_3}{\beta^3} & \text{for } d=3 \\ & \vdots \end{cases}. \tag{1.64}$$

Remarks:

This result for the partition function is independent of the choice of boundary conditions in the large volume limit. For instance, for Dirichlet conditions $\phi(t,x^1,...,0,...,x^d) = 0 = \phi(t,x^1,...,L_i,...,x^d)$, the appropriate basis is $e_{k_i} = \sqrt{\frac{2^d}{V_d}}\prod_i \sin k_i x = e^*_{k_i}$, $k_i = \frac{\pi n_i}{L_i}$. In particular, there is no mode at $n_i = (0,\dots,0)$ that has to be dealt with. All mode expansions are the same except that sums and products are restricted to $n_i>0$. In the evaluation of the partition function after (1.58), this is compensated by the fact that $\mathrm{d}k_i = \frac{\pi}{L_i}\mathrm{d}n_i$. Equivalently, the sums can be extended to $\mathbb{Z}^d/(0,\dots,0)$ while dividing by 2^d, because the summands are even functions of the n_i, and the computation proceeds as before.

1.5 Canonical quantization of the free electromagnetic field

1.5.1 Gauge potentials and Lagrangian formulation

When using units such that $c = 1 = \epsilon_0$, Maxwell's equations for the electric and magnetic fields $\vec{E}(x^\mu)$, $\vec{B}(x^\mu)$ are

$$\vec{\nabla}\cdot\vec{E} = \rho, \tag{1.65}$$
$$\vec{\nabla}\times\vec{B} - \partial_0\vec{E} = \vec{j}, \tag{1.66}$$
$$\vec{\nabla}\cdot\vec{B} = 0, \tag{1.67}$$
$$\vec{\nabla}\times\vec{E} + \partial_0\vec{B} = 0, \tag{1.68}$$

where $\rho(x^\mu)$ is the electric charge density and $\vec{j}(x^\mu)$ the electric current density.

In particular, taking the gradient of (1.66), i.e., applying $\vec{\nabla}\cdot$ to this equation gives $-\partial_0\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\vec{j}$. When using (1.65), this gives the continuity equation $\partial_0\rho + \vec{\nabla}\cdot\vec{j} = 0$, which expresses local conservation of electric charge.

When introducing the electric four-vector current density j^μ and the antisymmetric electromagnetic field $F^{\mu\nu}$ tensor defined through

$$j^\mu = \begin{pmatrix} \rho \\ j^1 \\ j^2 \\ j^3 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}, \tag{1.69}$$

Maxwell's equations and the continuity equation can be written in the equivalent (manifestly Lorentz covariant) form as

$$\boxed{\partial_\nu F^{\mu\nu} = j^\mu, \quad \epsilon^{\mu\nu\lambda\rho}\partial_\nu F_{\lambda\rho} = 0, \quad \partial_\mu j^\mu = 0}, \tag{1.70}$$

where $\epsilon^{\mu\nu\lambda\rho}$ is the Levi-Civita pseudo-tensor defined to be completely antisymmetric in the exchange of μ, ν, λ, ρ with $\epsilon^{0123} = 1$, and one remembers that indices are lowered and raised with $\eta_{\mu\nu}$ and its inverse $\eta^{\nu\lambda}$ (so that $\eta_{\mu\nu}\eta^{\nu\lambda} = \delta_\mu^\lambda$), where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta^{\mu\nu}. \tag{1.71}$$

First, we note that

$$\epsilon^{0ijk} = \epsilon^{ijk}, \quad \epsilon_{ijk}\epsilon^{lmk} = \delta_i^l\delta_j^m - \delta_i^m\delta_j^l, \quad \epsilon_{ijk}\epsilon^{ljk} = 2\delta_i^l.$$

In components, the electromagnetic field tensor is determined by $F^{0i} = E^i$, $F^{ij} = \epsilon^{ijk}B_k$, and $\partial_i F^{0i} = j^0$ is (1.65), while $\partial_0 F^{i0} + \partial_j F^{ij} = j^i$ is equivalent to $-\partial_0 E^i + \epsilon^{ijk}\partial_j B_k = j^i$ which is equivalent to (1.66). For the second set of equations, one starts with $\mu = 0$. In this case, they reduce to $0 = \epsilon^{ijk}\partial_i F_{jk} = \epsilon^{ijk}\partial_i(\epsilon_{jkl}B^l) = 2\partial_i B^i$, and thus to (1.67). If $\mu = i$, we get $0 = \epsilon^{i0jk}\partial_0 F_{jk} + \epsilon^{ij0k}\partial_j F_{0k} + \epsilon^{ijk0}\partial_j F_{k0} = \epsilon^{0ijk}(\partial_0 F_{kj} + \partial_j F_{0k} + \partial_k F_{j0}) = 2(-\partial_0 B^i - \epsilon^{ijk}\partial_j E_k)$ which is equivalent to (1.68).

On $\mathbb{R}^3$ with suitable fall-off conditions, every vector field $\vec{v}$ admits a unique decomposition into a longitudinal and a transverse part,

$$\vec{v} = \vec{\nabla}\psi + \vec{\nabla}\times\vec{w}. \tag{1.72}$$

For a divergence-free vector field, the longitudinal part vanishes, while for a curl-free vector field, it is the transverse part that vanishes.

When the Laplacian Δ is invertible (which we assume to be the case and which holds if one considers fields on $\mathbb{R}\times\mathbb{R}^3$ that decrease at least as fact as r^{-1} when $r\to\infty$) the decomposition of a $\vec{v}(x)$ into its longitudinal and transversal components follows from $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})=\vec{\nabla}(\vec{\nabla}\cdot\vec{v})-\Delta\vec{v}$:

$$\vec{v}=\vec{\nabla}\psi+\vec{\nabla}\times\vec{w},\quad \psi=\Delta^{-1}(\vec{\nabla}\cdot\vec{v}),\quad \vec{w}=-\Delta^{-1}(\vec{\nabla}\times\vec{v}).$$

(if Δ^{-1} commutes with the gradient $\vec{\nabla}$ and the curl $\vec{\nabla}\times$). NB: These properties can be explicitly checked when using that $\Delta\phi(\vec{x})=-\delta^{(3)}(\vec{x}-\vec{y})$ iff $\phi(\vec{x})=\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}|}$ so that $\Delta\phi(\vec{x})=j(\vec{x})\iff\phi(\vec{x})=-\frac{1}{4\pi}\int d^3y\frac{j(\vec{y})}{|\vec{x}-\vec{y}|}$, which provides an explicit expression for Δ^{-1}.

Equation (1.67) then implies the existence of a vector potential $\vec{A}$ such that $\vec{B}=\vec{\nabla}\times\vec{A}$. When injecting into (1.68), this implies $\vec{\nabla}\times(\vec{E}+\partial_0\vec{A})=0$, and in turn that $\vec{E}=-\partial_0\vec{A}-\vec{\nabla}\phi$ for some scalar potential ϕ.

NB: the vector and scalar potentials that give rise to a given electric and magnetic field are not uniquely defined. If $\vec{B}=\vec{\nabla}\times\vec{A}'$, $\vec{E}=-\partial_0\vec{A}'-\vec{\nabla}\phi'$, it follows by substraction that $\vec{\nabla}\times(\vec{A}'-\vec{A})=0$, so that $\vec{A}'=\vec{A}+\vec{\nabla}\chi$ for some χ, and then, when subtracting the two expressions for $\vec{E}$, that $\vec{\nabla}(\partial_0\chi+\phi'-\phi)=0$, so that $\phi'=\phi-\partial_0\chi$ up to a function of time alone, which needs to vanish on account of the boundary conditions imposed on the fields.

Defining

$$A_\mu=(-\phi,A_1,A_2,A_3),\tag{1.73}$$

it follows that

$$\boxed{F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu},\tag{1.74}$$

while gauge transformations of the potentials giving rise to identical electric and magnetic fields can be written as

$$\boxed{A'_\mu=A_\mu+\partial_\mu\chi}.\tag{1.75}$$

Indeed, $F_{0i}=-E_i=\partial_0A_i+\partial_i\phi$, as it should. For the other components, we have $F_{ij}=\epsilon_{ijk}B^k$ by definition, and $F_{ij}=\partial_iA_j-\partial_jA_i$ according to (1.74). Contracting both with ϵ^{ljk} on the other gives $2B^l=2\epsilon^{ljk}\partial_jA_k$ as it should.

In these terms, equations (1.65) and(1.66), which have been shown to be equivalent to $\partial_\nu F^{\mu\nu}=j^\mu$ read explicitly $\partial^\mu\partial_\nu A^\nu-\Box A^\mu=j^\mu$, or $\partial^0(\cancel{\partial_0A^0}+\partial_iA^i)-\cancel{\partial_0\partial^0A^0}-\Delta A^0=\rho$ and $\partial_0(\partial^iA^0-\partial^0A^i)+\partial_j(\partial^iA^j-\partial^jA^i)=j^j$. When taking into account that $\partial_j(\partial^iA^j-\partial^jA^i)=\partial_j(\epsilon^{ijk}\epsilon_{klm}\partial^lA^m)$ these can be written as

$$-(\partial_0\vec{\nabla}\cdot\vec{A}+\Delta\phi)=\rho,\quad \partial_0(\vec{\nabla}\phi+\partial_0\vec{A})+\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{j}.\tag{1.76}$$

The important point is that while equations (1.67) and (1.68) have been solved by the introduction of the potentials, the remaining equations (1.65) and(1.66) derive from the action principle,

$$\boxed{S[A_\mu;j^\mu]=\int d^4x\,\mathcal{L}^M=\int d^4x\,(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+j^\mu A_\mu)}.\tag{1.77}$$

Indeed, varying the gauge potentials and neglecting boundary terms in the action gives

$$\int d^4x\,[-\frac{1}{2}F^{\mu\nu}(\partial_\mu\delta A_\nu-\partial_\nu\delta A_\mu)+j^\mu\delta A_\mu]=\int d^4x\,\delta A_\mu(x)[-\partial_\nu F^{\mu\nu}(x)+j^\mu(x)].$$

The action principle then requires that

$$0=\frac{\delta S}{\delta A_\mu(x)}=\frac{\delta\mathcal{L}^M}{\delta A_\mu}(x)=[\frac{\partial\mathcal{L}^M}{\partial A_\mu}-\partial_\lambda\frac{\partial\mathcal{L}^M}{\partial\partial_\lambda A_\mu}](x)=[-\partial_\nu F^{\mu\nu}+j^\mu](x).\tag{1.78}$$

1.5.2 Hamiltonian formulation of electromagnetism

In the electromagentic case, it is not so easy to directly pass to the Hamiltonian formulation. Indeed, the canonical momentum associated to A_0 vanishes identically, $\pi^0(\vec{x}) = \frac{\delta L}{\delta \partial_0 A_0(\vec{x})} = 0$, so that one cannot perform the standard Legendre transform. We will first get rid of A_0 altogether before passing to the Hamiltonian formalism. Note first that

$$\begin{aligned} S &= \int d^4x\, [-\frac{1}{2}F^{0i}F_{0i} - \frac{1}{4}F^{ij}F_{ij} + A_\mu j^\mu] \\ &= \int d^4x\, [\frac{1}{2}\partial_0 A_i \partial_0 A^i - \frac{1}{2}B^i B_i + A_i j^i + A_0(\partial_0\partial_j A^j + j^0 - \frac{1}{2}\Delta A_0)], \end{aligned} \tag{1.79}$$

by using spatial integrations by parts and dropping the boundary terms under the assumptions that fields decrease at infinity.

The Euler-Lagrange equations of motions for A_0 imply that

$$A_0 = \frac{1}{\Delta}(\partial_0\partial_j A^j + j^0). \tag{1.80}$$

Because the equation of motion for A_0 can be solved "algebraically" for A_0 without invoking initial conditions, one is allowed to inject the solution for A_0 into the action. This gives rise to an equivalent, reduced, action principle whose solutions agree with the original one. More concretely, the reduced action reads

$$\begin{aligned} S &= \int d^4x\, [\frac{1}{2}\partial_0 A_i \partial_0 A^i + \frac{1}{2}\frac{1}{\Delta}(\partial_0\partial_j A^j + j^0)(\partial_0\partial_k A^k + j^0) - \frac{1}{2}B^i B_i + A_i j^i] \\ &= \int d^4x\, [\frac{1}{2}\partial_0 A^T_i \partial_0 A^i_T - \frac{1}{2}B^i B_i + \frac{1}{2}j^0\frac{1}{\Delta}j^0 + A^T_i j^i_T] = \int dt L, \end{aligned} \tag{1.81}$$

where we have used that $\int d^3x\, v_i w^i = \int d^3x (v^T_i w^i_T + v^L_i w^i_L)$ and the continuity equation $\partial_i j^i = -\partial_0 j^0$. Note also that $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}_T$, so that the reduced action principle only involves the transverse vector potentials. The associated equations of motion are

$$\partial_0^2 \vec{A}^T = -\vec{\nabla} \times (\vec{\nabla} \times \vec{A}_T) + \vec{j}_T = \Delta \vec{A}^T + \vec{j}_T, \tag{1.82}$$

and agree with the equations obtained when inserting the solution of the first into the second set of equations in (1.76).

From the point of view of gauge transformations, the above formulation, which does no longer involve $A_0 = -\phi$ nor $\vec{A}_L$, is referred to as the Coulomb gauge. When choosing the time dependence of χ such that $\partial_0\chi = -A_0$ and also $\chi = -\frac{1}{\Delta}\vec{\nabla} \cdot \vec{A}$ in (1.75), it follows that

$$A'_0 = 0, \quad \vec{\nabla} \cdot \vec{A}' = 0 \Longrightarrow \vec{A}' = \vec{A}'_T. \tag{1.83}$$

It is now straightforward to pass to the Hamiltonian formulation. The conjugate momentum is

$$\pi^i_T(t, \vec{x}) = \frac{\delta L}{\delta \partial_0 A^T_i(t, \vec{x})} = \partial_0 A^T_i(t, \vec{x}), \tag{1.84}$$

and

$$\boxed{H = \int d^3x\, [\frac{1}{2}\pi^i_T \pi^T_i + \frac{1}{2}B^i B_i - \frac{1}{2}j^0\frac{1}{\Delta}j^0 - A^T_i j^i_T]}. \tag{1.85}$$

Note also that $\vec{E} = -\partial_0\vec{A} + \vec{\nabla}A_0$ so that $\vec{E}_T = -\vec{\pi}_T$, $\vec{E}_L = \vec{\nabla}(\frac{1}{\Delta}j^0)$, which implies that the Hamiltonian may also be written as

$$H = \int d^3x\, [\frac{1}{2}E^i E_i + \frac{1}{2}B^i B_i - A^T_i j^i_T]. \tag{1.86}$$

1.5.3 Electromagnetic radiation in a box

In the absence of sources, electromagnetic theory reduces on the classical level to the free wave equation for the transverse vector potential. We now proceed to canonically quantize this system in the case of periodic boundary conditions.

When using that

$$\delta H = \int d^3x\, [\delta\pi_i^T \pi_T^i - \delta A_T^k(\Delta A_k^T - \cancel{\partial_k(\vec{\nabla}\cdot\vec{A}_T)})], \tag{1.87}$$

the Hamiltonian equations of motion are

$$\dot{A}_T^i(x) = \{A_T^i(x), H\} = \frac{\delta H}{\delta \pi_i^T(x)} = \pi_T^i(x), \quad \dot{\pi}_i^T(x) = \{\pi_i^T(x), H\} = -\frac{\delta H}{\delta A_T^i(x)} = \Delta A_i^T(x). \tag{1.88}$$

Substituting the first into the second gives the wave equation $\partial_\mu\partial^\mu A_T^i = 0$, which is also what one gets when taking the transverse part of the second of equation (1.76) in the absence of sources.

In a box with sides of lenght L_i and periodic boundary conditions, one decomposes $A_i(x)$ in Fourier components,

$$A_i(x) = \bar{A}_i(t) + \sum_{\vec{k}\neq\vec{0}} (\frac{\hbar}{2\omega(\vec{k})L^3})^{\frac{1}{2}} \widetilde{A}_i(\vec{k},t) e^{i\vec{k}\cdot\vec{x}}, \tag{1.89}$$

with $k^i = \frac{2\pi n^i}{L_{(i)}}$, $n^i \in \mathbb{Z}$, and $\omega(\vec{k}) = k = \sqrt{\vec{k}^2}$. The unusual factor $(\frac{\hbar}{2\omega(\vec{k})V})^{\frac{1}{2}}$ in the definition of $\widetilde{A}_i(\vec{k},t)$ is chosen in such as a way as to simplify subsequent formulae. Imposing the condition $\partial_i A^i = 0$ implies $k^i\widetilde{A}_i(\vec{k},t) = 0$, whereas the fact that A_i is real implies $\widetilde{A}_i(-\vec{k},t) = \widetilde{A}_i^*(\vec{k},t)$. Before quantizing the system, note that the general solution to the wave equation, which in modes translates to $\frac{\partial^2}{\partial t^2}\bar{A}^i(t) = 0$, $(\frac{\partial^2}{\partial t^2} + \omega^2)\widetilde{A}_i(\vec{k},t) = 0$, is given by $\widetilde{A}_i(\vec{k},t) = c_i(\vec{k})e^{-i\omega t} + c_i^*(-\vec{k})e^{i\omega t}$, with $c_i(\vec{k}) \in \mathbb{C}$ arbitrary for $\vec{k} \neq \vec{0}$, and $\bar{A}_i(t) = \bar{A}_i + \bar{\pi}_i t$ with $\bar{A}_i, \bar{\pi}_i \in \mathbb{R}$ arbitrary. In what follows we will discard the zero mode, $\bar{A}_i, \bar{\pi}_i = 0$.

The divergence-free condition is implemented by introducing polarization vectors, that is to say an orthonormal frame $e_i^m(\vec{k})$, $m = 1,\dots,3$, with $e_i^3(\vec{k}) = \frac{k_i}{k}$ parallel to $\vec{k}$ and e_i^s, $s = 1,2$ two orthormal vectors orthogonal to $\vec{k}$, in such a way that $\sum_m e_i^m(\vec{k})e_j^m(\vec{k}) = \delta_{ij}$: $c_i(\vec{k}) = a_s(\vec{k})e_i^s(\vec{k})$. Explicitly, one may for instance choose $e_i^1 = (k_\perp)^{-1}(k_2, -k_1, 0)$, $e_i^2 = (k_\perp k)^{-1}(k_1k_3, k_2k_3, -k_\perp^2)$, with $k_\perp = \sqrt{k_1^2 + k_2^2}$. The classical solution can then be written as

$$A_i(x) = \sum_{\vec{k}\neq\vec{0}} (\frac{\hbar}{2\omega(\vec{k})V})^{\frac{1}{2}} [a_s(\vec{k})e_i^s(\vec{k})e^{ik\cdot x} + c.c.]. \tag{1.90}$$

with $k\cdot x = k_\mu x^\mu = -\omega t + \vec{k}\cdot\vec{x}$. The associated electric and magnetic fields are

$$E_i(x) = i\sum_{\vec{k}\neq\vec{0}} (\frac{\hbar}{2\omega(\vec{k})V})^{\frac{1}{2}} [\omega(\vec{k})a_s(\vec{k})e_i^s(\vec{k})e^{ik\cdot x} - c.c.], \tag{1.91}$$

$$B^i(x) = i\sum_{\vec{k}\neq\vec{0}} (\frac{\hbar}{2\omega(\vec{k})V})^{\frac{1}{2}} [\epsilon^{ijk}k_j a_s(\vec{k})e_k^s(\vec{k})e^{ik\cdot x} - c.c.]. \tag{1.92}$$

The energy of the electromagnetic field in vacuum is given by

$$H(t) = \frac{1}{2}\int_{cube} d^3x [E^i(x)E_i(x) + B^i(x)B_i(x)]. \tag{1.93}$$

By substituting the previous expansions and using that

$$\int_{cube} d^3x e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = V\delta^3_{\vec{k},\vec{k}'}, \tag{1.94}$$

one gets

$$H = \sum_{\vec{k}\neq\vec{0}} \hbar\omega a_s^*(\vec{k})a^s(\vec{k}). \tag{1.95}$$

The time dependence of $a_s(\vec{k},t) \equiv a_s(\vec{k})e^{-i\omega t}$, $\frac{d}{dt}a_s(\vec{k},t) = -i\omega a_s(\vec{k},t)$ is compatible with the Hamiltonian equations

$$\frac{d}{dt}a_s(\vec{k},t) = \{a_s(\vec{k},t), H\}, \tag{1.96}$$

if the following Poisson brackets

$$\{a_s(\vec{k}), a^*_{s'}(\vec{k}')\} = -\frac{i}{\hbar}\delta_{\vec{k},\vec{k}'}\delta_{s,s'}, \quad \{a_s(\vec{k}), a_{s'}(\vec{k}')\} = 0 = \{a^*_s(\vec{k}), a^*_{s'}(\vec{k}')\} \tag{1.97}$$

hold. They can be shown to be equivalent to the canonical Poisson brackets of $A_i^T(x)$ and $\pi^i_T(y)$.

The free electromagnetic field in a box appears as an infinite countable collection of non-interacting harmonic oscillators; for each vector $\vec{k}$ there are two separate oscillators, one for each transverse polarizationion s.

1.5.3.1 Canonical quantization

Canonical quantization is now straightforward: one substitutes the oscillator variables by operators, $a_s(\vec{k}) \to \widehat{a}_s(\vec{k})$ with non-vanishing commutation relations given by

$$[\widehat{a}_s(\vec{k}), \widehat{a}^\dagger_{s'}(\vec{k}')] = \delta_{\vec{k},\vec{k}'}\delta_{s,s'}. \tag{1.98}$$

The Hilbert space $\mathcal{H}$ is the Fock space generated by the creation operators $\widehat{a}^\dagger_s(\vec{k})$ for each mode $\vec{k}, s$: $\mathcal{H} = \otimes_{\vec{k},s}\mathcal{H}_{\vec{k},s}$, a basis of $\mathcal{H}_{\vec{k},s}$ being

$$|n_{\vec{k},s}\rangle = \frac{(\widehat{a}^\dagger_s(\vec{k}))^{n_{\vec{k},s}}}{\sqrt{n_{\vec{k},s}!}}|0\rangle. \tag{1.99}$$

For the Hamiltonian operator, we choose the normal ordered form:

$$\boxed{\widehat{H} = \sum_{\vec{k},s} \widehat{H}_{\vec{k},s}, \quad \widehat{H}_{\vec{k},s} = \hbar\omega(\vec{k})\widehat{a}^\dagger_{(s)}(\vec{k})\widehat{a}_{(s)}(\vec{k})}. \tag{1.100}$$

1.5.3.2 Black body radiation

Description Black body radiation (see e.g. [7, 8] is the name given to an electromagnetic field in thermal equilibrium with a reservoir at a temperature T.

The black body itself is an idealized object that absorbs all electromagnetic radiation. In a laboratory, it can be modelled by a small hole in a cavity whose walls are at temperature T. Before being able to emerge, light that enters the hole is reflected multiple times by the wall and will be absorbed in the process.

According to the rules of statistical mechanics, if the coupling between the field and the reservoir is sufficiently weak, the electromagnetic field can be considered as free and the thermodynamical properties can be derived from the partition function

$$\boxed{Z(\beta) = \mathrm{Tr}\, e^{-\beta\widehat{H}}}, \tag{1.101}$$

where $\beta = 1/(k_B T)$, k_B is the Boltzmann constant and $\widehat{H}$ is the Hamiltonian operator of the free electro-magentic field discussed above.

Partition function and thermodynamics Since the Hamiltonian is a sum of non-interacting terms and the Hilbert space a direct product, the partition function factorizes,

$$\begin{aligned} Z(\beta) &= \prod_{\vec{k},s}\sum_{n_{\vec{k},s}} \langle n_{\vec{k},s}| e^{-\beta\hbar\widehat{H}_{\vec{k},s}} |n_{\vec{k},s}\rangle \\ &= \prod_{\vec{k},s}\sum_{n_{\vec{k},s}} e^{(-\beta\hbar\omega(\vec{k})n_{\vec{k},s})} \\ &= \prod_{\vec{k},s}(1-e^{-\beta\hbar k})^{-1}, \end{aligned} \tag{1.102}$$

where the last equality follows by summing the geometrical series. It follows that

$$\ln Z(\beta) = -2\sum_{\vec{k}} \ln(1-e^{-\beta\hbar k}). \tag{1.103}$$

In order to perform the sum over the integers $n^i \in \mathbb{Z}$, one notes that, when the box becomes large, $L_1 \to \infty$, $\sum_{n^1} \to \int_{-\infty}^{+\infty} dn^1 = \frac{L}{2\pi}\int dk^1$. One then finds

$$\begin{aligned} \ln Z(\beta) &= -\frac{2V}{(2\pi)^3}\int d^3k \ln(1-e^{-\beta\hbar k}) \\ &= -\frac{V}{\pi^2}\int_0^\infty dk\, k^2 \ln(1-e^{-\beta\hbar k}) \\ &= -\frac{\beta^{-3}V}{\hbar^3\pi^2}\int_0^\infty dx\, x^2 \ln(1-e^{-x}) \\ &= \frac{\beta^{-3}V}{3\hbar^3\pi^2}\int_0^\infty dx\, \frac{x^3}{e^x-1}. \end{aligned} \tag{1.104}$$

The last line follows from an integration by parts and the fact that $\lim_{x\to+\infty}\frac{x^3}{3}\ln(1-e^{-x}) = 0$. Using that

$$\int_0^\infty dx \frac{x^3}{e^x-1} = \Gamma(4)\zeta(4) = 3!\frac{\pi^4}{90}, \tag{1.105}$$

we finally get

$$\boxed{\ln Z(\beta) = \frac{\beta^{-3}V\pi^2}{45\hbar^3}} = \frac{1}{3}bV\beta^{-3}, \quad b = \frac{\pi^2}{15\hbar^3}. \tag{1.106}$$

The internal energy U is given by,

$$U = \langle H \rangle = -\frac{\partial}{\partial\beta}\ln Z(\beta) = bV\beta^{-4}. \tag{1.107}$$

Inverting gives β as a function of U,

$$\beta(U) = (\frac{U}{bV})^{-\frac{1}{4}}. \tag{1.108}$$

The entropy is the Legendre transform of the logarithm of the partition function with respect to β,

$$S(U,V) = (1-\beta\frac{\partial}{\partial\beta})\ln Z(\beta)|_{\beta=\beta(U)} = \frac{4}{3}bV\beta^{-3}|_{\beta=\beta(U)} = \frac{4}{3}(bV)^{\frac{1}{4}}U^{\frac{3}{4}}. \tag{1.109}$$

The free energy $F(\beta,V)$, defined by

$$e^{-\beta F(\beta,V)} = Z(\beta), \tag{1.110}$$

is given by

$$F(\beta,V) = -\beta^{-1}\ln Z(\beta) = -\frac{bV}{3}\beta^{-4}, \tag{1.111}$$

and the radiation pressure

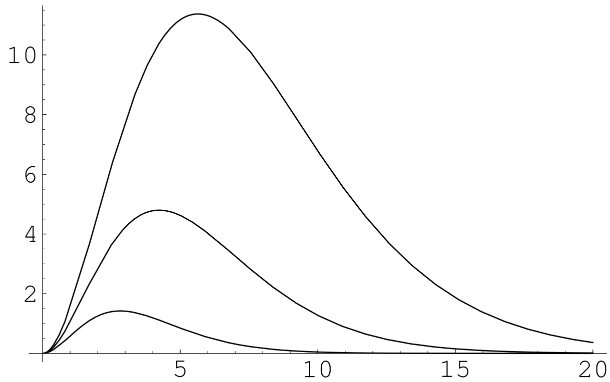
$$P(T,V) = -\frac{\partial F}{\partial V} = \frac{U}{3V}. \tag{1.112}$$

Re-introducing factors of c, the velocity of light, and starting from

$$U = \frac{(\partial\beta F)}{\partial\beta} = \frac{\partial}{\partial\beta}\frac{V}{\pi^2}\int_0^\infty dk\, k^2\ln(1-e^{-\beta\hbar ck}) = \frac{V\hbar c}{\pi^2}\int_0^\infty dk\,\frac{k^3}{e^{\beta\hbar ck}-1}, \tag{1.113}$$

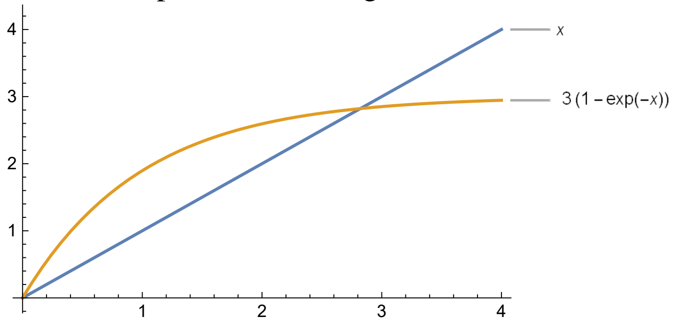
one finds that the energy by unit volume of photons in an interval dk is given by $u(k,T)dk = \frac{\hbar c}{\pi^2}\frac{k^3 dk}{e^{\beta\hbar ck}-1}$. In terms of frequency, $\nu = \frac{ck}{2\pi}$, this gives Planck's famous law of radiation,

$$\boxed{u(\nu,T)d\nu = \frac{8\pi h\nu^3 d\nu}{c^3(e^{\beta h\nu}-1)}}. \tag{1.114}$$



$u(x) = \frac{x^3}{e^{x/T}-1}$ for $T = 1, 1.5, 2$

The maximum is determined by $3 = \frac{h\beta\nu_M}{1-e^{-\beta h\nu_M}}$, which gives $\nu_M = \frac{2{,}82k_BT}{h}$. This maximum is a linear function of temperature, which goes under the name of *Wien's displacement law*.



Graphical solution to $x = 3(1-e^{-x})$.

Historical note [7]

Planck's formula apears here as a direct consequence of quantum statistical mechanics. Historically, the explanantion of the experimental curves $u(\nu)$ was problematic because classical statistical mechanics gave the Rayleigh-Jeans formula (1900) $u(\nu) \to \frac{8\pi\nu^2}{\beta c^3}$, which can be obtained from Planck's formula at low frequency or high temperature, $\beta h\nu << 1$, and by expanding the exponential. At high frequency or low temperature, $\beta h\nu >> 1$, one finds $u(\nu) \to \frac{8\pi h\nu^3}{c^3} e^{-\beta h\nu}$ which is Wien's formula known empirically since 1893. Planck (1901) writes his formula for $u(\nu)$ to interpolate between those two limits and explain the experimental curves. Einstein (1905) justifies this formula by postulating that the total energy of each stationary state of radiation is quantized, and by associating a particle, the photon, to the elementary unit of energy of each electromagnetic mode. The explanation of black body radiation has played a crucial role in the developement of the ideas leading to quantum mechanics.

Chapter 2

Path integrals

Paths integrals allow for an intuitive reformulation of quantum mechanics with a rather straightforward generalization to quantum field theory. They are well suited to set up perturbation theory (derivation of Feynman rules, Ward identities, etc.). They are also instrumental in the non-perturbative context, but this is beyond the scope of the current course.

This chapter is based on [2], [1], [9], [10], [11], [12], [13].

We also use [14], chapter 2.10, and [15], section 10.2.

2.1 Hamiltonian formulation

cf [2], [11].

2.1.1 Evolution operator

Consider a quantum mechanical system in Schrödinger representation, with Hermition "position" operators $\widehat{Q}^a$ and conjugate momenta $\widehat{P}_b$ satisfying

$$[\widehat{Q}^a, \widehat{P}_b] = i\delta^a_b,\ [\widehat{Q}^a, \widehat{Q}^b] = 0,\ [\widehat{P}_a, \widehat{P}_b] = 0\ \ a, b = 1, \ldots, n, \tag{2.1}$$

where $\hbar = 1$. The position eignestates $\widehat{Q}^a|q >= q^a|q >$ are orthonormal and complete

$$< q'|q >= \prod_a \delta(q'^a - q^a) \equiv \delta(q' - q), \tag{2.2}$$

$$1 = \int \prod_a dq^a\, |q >< q| \equiv \int dq\, |q >< q|, \tag{2.3}$$

and similarly for the momentum eigenstates $\widehat{P}_b|p >= p_b|p >$,

$$< p'|p >= \prod_b \delta(p'_b - p_b) \equiv \delta(p' - p), \tag{2.4}$$

$$1 = \int \prod_p dp_b\, |p >< p| \equiv \int dp\, |p >< p|. \tag{2.5}$$

As a consequence,

$$< q|p >= \frac{1}{\sqrt{2\pi}^n} e^{iq^a p_a} = \prod_a \frac{1}{\sqrt{2\pi}} e^{iq^{\bar{a}} p_{\bar{a}}}, \tag{2.6}$$

(with no summation on barred indices).

Indeed, in position representation, $\psi(q) =< q|\psi >$, $< q|\widehat{Q}^a|\psi >= q^a\psi(q)$, $< q|\widehat{P}_p|\psi >= -i\frac{\partial}{\partial q^b}\psi(q)$. Hence, $< q|\widehat{P}_b|p >= -i\frac{\partial}{\partial q^b} < q|p >$, but also $< q|\widehat{P}_b|p >= p_b < q|p >$, which implies that $< q|p >= \alpha \exp iq^a p_a$. Normalisation: $\delta(p'-p) =< p'|p >= \int dq \, < p'|q >< q|p >= \alpha\alpha^* \int dq\, e^{iq^a(p_a - p'_a)}$. Explicitly, $\prod_a \delta(p'_a - p_a) = \alpha\alpha^* \prod_a \int dq^a\, e^{iq^{\bar a}(p_{\bar a} - p'_{\bar a})}$. This implies $\alpha\alpha^* = \frac{1}{2\pi}^n$ and we can choose $\alpha = \frac{1}{\sqrt{2\pi}^n}$, because for a sinngle copie $\delta(p'_a - p_a) = \frac{1}{2\pi}\int dq^a\, e^{iq^{\bar a}(p_{\bar a} - p'_{\bar a})}$, (see exercices on Gaussian integration).

For a time-independent Hamiltonian in Heisenberg picture, we have

$$\widehat{Q}^a(t) = e^{i\widehat{H}t}\,\widehat{Q}^a\, e^{-i\widehat{H}t}, \tag{2.7}$$

$$\widehat{P}_b(t) = e^{i\widehat{H}t}\,\widehat{P}_b\, e^{-i\widehat{H}t}. \tag{2.8}$$

Eigenstates: $\widehat{Q}^a(t)|q;t >= q^a|q;t >$, $\widehat{P}_b(t)|p;t >= p_b|p;t >$, which implies $|q;t >= e^{i\widehat{H}t}|q >$, $|p;t >= e^{i\widehat{H}t}|p >$. We also have

$$< q';t|q;t >= \delta(q'-q), \quad 1 = \int dq\,|q;t >< q;t|, \tag{2.9}$$

$$< p';t|p;t >= \delta(p'-p), \quad 1 = \int dp\,|p;t >< p;t|, \tag{2.10}$$

$$< q;t|p;t >= \prod_a \frac{1}{\sqrt{2\pi}} e^{iq^{\bar a}p_{\bar a}}, \tag{2.11}$$

we want to compute the probability amplitude between the initial state $|q;t>$ and the final state $|q';t'>$:

$$< q';t'|q;t >=< q'|\widehat{U}(t',t)|q >,\ \widehat{U}(t',t) = e^{-i\widehat{H}(t'-t)}, \tag{2.12}$$

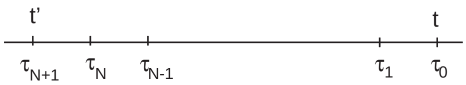
where $\widehat{U}(t',t)$ is the evolution operator. Note that $\widehat{H}(\widehat{Q},\widehat{P}) = \widehat{H}(\widehat{Q}(t),\widehat{P}(t))$.

Indeed, suppose that $\widehat{H}$ is a polynomial. One uses $\widehat{H} = e^{i\widehat{H}t}\,\widehat{H}\, e^{-i\widehat{H}t}$ and inserts $1 = e^{i\widehat{H}t}e^{-i\widehat{H}t}$ as many times as needed. For what follows, let us order all $\widehat{Q}$ to the left of the $\widehat{P}$ using the commuttaions relations (2.1): $\widehat{H} = \sum_{m,n} k^{b_1\ldots b_m}_{a_1\ldots a_n}\widehat{Q}^{a_1}\ldots\widehat{Q}^{a_n}\widehat{P}_{b_1}\ldots\widehat{P}_{b_m}$.

For an infinitesimal time interval, $t' = \tau + d\tau$, $t = \tau$, we have

$$\begin{aligned}
< q';\tau + d\tau|q;\tau >&=< q';\tau|e^{-i\widehat{H}d\tau}|q;\tau >=< q';\tau|\mathbf{1} - i\widehat{H}(\widehat{Q}(\tau),\widehat{P}(\tau))d\tau|q;\tau > \\
&= \int dp\ < q';\tau|\mathbf{1} - i\widehat{H}(\widehat{Q}(\tau),\widehat{P}(\tau))d\tau|p;\tau >< p;\tau|q;\tau > \\
&= \int dp\ < q';\tau|p;\tau > \Big(1 - i(\sum_{m,n} k^{b_1\ldots b_m}_{a_1\ldots a_n} q'^{a_1}\ldots q'^{a_n}p_{b_1}\ldots p_{b_m})d\tau\Big) < p;\tau|q;\tau > \\
&= \int (\prod_a \frac{dp_a}{2\pi})\ e^{i[p_a(q'^a - q^a) - H(q',p)d\tau]}
\end{aligned} \tag{2.13}$$

where the function $H(q',p)$ is obtained by replacing operators through functions in $\widehat{H}$ in which all $\widehat{Q}$'s had previously been ordered to the left of the $\widehat{P}$'s. This function is called the $q-p$ symbol of the operator $\widehat{H}$.



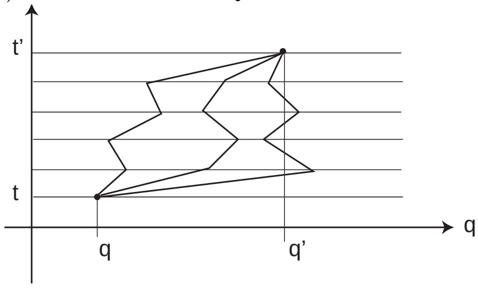
For a finite time interval, one cuts into pieces: $\Delta\tau = \tau_{k+1} - \tau_k = \frac{t'-t}{N+1}$ and

$$\begin{aligned}
< q';t'|q;t >= \int dq_1\ldots dq_N\ < q';t'|q_N;\tau_N >< q_N;\tau_N|q_{N-1};\tau_{N-1} > \ldots \\
\cdots < q_1;\tau_1|q;t > .
\end{aligned} \tag{2.14}$$

If $N \to \infty$, these intervals become infinitesimally small and one may use (2.13) for each piece, yielding

$$< q'; t'|q; t >= \lim_{N\to\infty} \int [\prod_{k=1}^{N} \prod_{a} dq_k^a][\prod_{k=0}^{N} \prod_{a} \frac{dp_{ak}}{2\pi}] \\ e^{i\sum_{k=1}^{N+1}[(q_k^a - q_{k-1}^a)p_{ak-1} - H(q_k, p_{k-1})d\tau]} \quad (2.15)$$

with $q_0^a = q^a$ and $q_{N+1}^a = q'^a$. Introducing interpolating functions $q(\tau), p(\tau)$ between the different points, $q^a(\tau_k) = q_k^a, p_a(\tau_k) = p_{ak}$, the integration may be seen as an integration over all paths $q(\tau)$ going from $q(t)$ to $q(t')$ and over all paths $p(\tau)$ with no boundary conditions.



If the interpolating functions are sufficiently regular, the argument of the exponential in (2.15) becomes

$$\sum_{k=1}^{N+1} [(q^a(\tau_k) - q^a(\tau_{k-1})p_a(\tau_{k-1}) - H(q(\tau_k), p(\tau_{k-1}))d\tau] =$$

$$= \sum_{k=1}^{N+1} [(\dot{q}^a(\tau_k)p_a(\tau_k) - H(q(\tau_k), p(\tau_k))d\tau + O(d\tau^2)],$$

by using $q^a(\tau_k) = q^a(\tau_{k-1}) + \frac{dq^a}{d\tau}(\tau_k)d\tau + O(d\tau^2)$, and $p_a(\tau_k) = p_a(\tau_{k-1}) + O(d\tau)$, and in the limit $N \to \infty$, this gives

$$\int_t^{t'} d\tau\, [\dot{q}^a(\tau)p_a(\tau) - H(q(\tau), p(\tau)) \equiv S_H[q, p].$$

In the limit $N \to \infty$, the measure in the path integral is denoted as follows:

$$\lim_{N\to\infty} \int [\prod_{k=1}^{N} \prod_{a} dq_k^a][\prod_{k=0}^{N} \prod_{a} \frac{dp_{ak}}{2\pi}] = \int_{q(t)=q}^{q(t')=q'} \prod_{a,\tau} dq^a(\tau) \prod_{a,\tau}{}' \frac{dp_a(\tau)}{2\pi}.$$

This measure is thus defined through a limit from a discretized measure containing one more integration over the p's than the q's. A mathematically rigorous definition of the measure is in general problematic.

Going to the limit, we thus have

$$\boxed{< q'; t'|q; t >= \int_{q(t)=q}^{q(t')=q'} \prod_{a,\tau} dq^a(\tau) \prod_{a,\tau}{}' \frac{dp_a(\tau)}{2\pi} e^{iS_H[q,p]}}, \quad (2.16)$$

where $S_H[q, p]$ is the first order Hamiltonian action.

When extrapolating the result for the stationary phase approximation of such integrals in the finite dimensional case, the classical path, which is an extremum of S_H will give the dominant contribution to this integral.

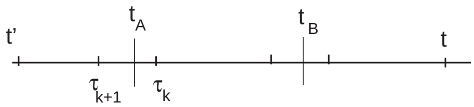
2.1.2 Matrix elements

One now would like to compute matrix elements with insertions of operators $\widehat{O}$ (for which one uses the commutation relations (2.1) to order the $\widehat{P}$ to the left of the $\widehat{Q}$).

For an infinitesimal interval, we have

$$< q'; \tau + d\tau|\widehat{O}(\widehat{P}(\tau), \widehat{Q}(\tau))|q; \tau >= \int dp < q'; \tau|e^{-i\widehat{H}d\tau}|p; \tau >< p; \tau|\widehat{O}(\widehat{P}(\tau), \widehat{Q}(\tau))|q; \tau >$$
$$= \int (\prod_a \frac{dp_a}{2\pi})\, e^{i[p_a(q'^a - q^a) - H(q',p)d\tau]} O(p(\tau), q(\tau)).$$

where $O(p, q)$ is the $p-q$ symbol of the operator $\widehat{O}$. Consider now the matrix element with an insertion of a product $\widehat{O}_A(\widehat{P}(t_A), \widehat{Q}(t_A))\widehat{O}_B(\widehat{P}(t_B), \widehat{Q}(t_B))\dots$, where one assumes that $t_A > t_B > \dots$. In the decomposition of the time interval, if $\tau_{k+1} > t_A > \tau_k$,



the operator $\widehat{O}_A$ is inserted in $< q_{k+1}; \tau_{k+1}|\widehat{O}_A|q_k; \tau_k >$ and $< p_k; \tau_k|\widehat{O}_A|q_k; \tau_k >= O_A(p(t_A), q(t_A)) < p_k; \tau_k|q_k; \tau_k > +O(d\tau)$. Terms of order $d\tau$ vanish in the limit because there are only a finite number of such terms (contrary to the previous case where one needed to keep such terms as their number grew with N). Following the previous reasoning, one finds for $t' > t_A > t_B > \dots > t$,

$$< q'; t'|\widehat{O}_A\widehat{O}_B \dots |q; t >=$$
$$= \int_{q(t)=q}^{q(t')=q'} \prod_{a,\tau} dq^a(\tau) \prod_{a,\tau}{}' \frac{dp_a(\tau)}{2\pi} O_A(p(t_A), q(t_A))O_B(p(t_B), q(t_B)) \dots e^{iS_H[q,p]}.$$

In the right hand side, the ordering of the various times is irrelevant since one is dealing with ordinary functions. Assume then that the times $t_A, t_B, \dots$ are not ordered. In this case, the right hand side corresponds to the left hand side where the times are in decreasing order (from left to right). Denoting through T decreasing time ordering, we thus have shown:

$$\boxed{< q'; t'|\mathrm{T}\{\widehat{O}_A(\widehat{P}(t_A), \widehat{Q}(t_A))\widehat{O}_B(\widehat{P}(t_B), \widehat{Q}(t_B)) \dots\}|q; t >=}$$
$$= \boxed{\int_{q(t)=q}^{q(t')=q'} \prod_{a,\tau} dq^a(\tau) \prod_{a,\tau}{}' \frac{dp_a(\tau)}{2\pi} O_A(p(t_A), q(t_A))O_B(p(t_B), q(t_B)) \dots e^{iS_H[q,p]}}. \quad (2.17)$$

2.1.3 Schwinger-Dyson equations

Take $z^\alpha = (q^a, p_a)$ and let us insert as an operator the left hand sides of the equations of motion, $\widehat{O}_A(z^\alpha(t_A)) = \widehat{\frac{\delta S_H}{\delta z^\alpha(t_A)}}$ with

$$\frac{\delta S_H}{\delta z^\alpha(t_A)} = \begin{cases} (-\dot{p}_a - \dfrac{\partial H}{\partial q^a})(t_A), \\ (\dot{q}^b - \dfrac{\partial H}{\partial p_b})(t_A) \end{cases} \quad (2.18)$$

and $t < t_A < t'$.

We have

$$\frac{\delta S_H}{\delta z^\alpha(t_A)} O_B(z(t_B))\ldots e^{iS_H[z]} = -i\frac{\delta}{\delta z^\alpha(t_A)}\left(O_B(z(t_B))\ldots e^{iS_H[z]}\right) + \left(i\frac{\delta}{\delta z^\alpha(t_A)}[O_B(z(t_B))\ldots]\right)e^{iS_H[z]}.$$

As we are integrating over all intermediary z^α, and thus also over $z^\alpha(t_A)$ from $-\infty$ à $+\infty$, one may argue that the first term, which reduces to $-i[(O_B(z(t_B))\ldots e^{iS_H[z]}]_{z^\alpha(\tau)=-\infty}^{z^\alpha(\tau)=+\infty}$, vanishes (see the exercises for a justification). We thus have:

$$< q'; t|\mathrm{T}\{\widehat{\frac{\delta S_H}{\delta z^\alpha(t_A)}}\widehat{O}_B(\widehat{z}(t_B))\ldots\}|q; t >= = i < q'; t|\mathrm{T}\{\frac{\delta}{\delta z^\alpha(t_A)}[\widehat{O_B(z(t_B))\ldots}]\}|q; t > . \tag{2.19}$$

These are the Schwinger-Dyson equations which express what the classical equations of motion become in the quantum theory.

2.1.4 Exercises

2.1.4.1 Re-introduce $\hbar$

Re-introduce $\hbar$ in previous formulas.

2.1.4.2 Path integral for p-q symbol

What are the $q-p$ and the $p-q$ symbols of $\widehat{H} = \frac{1}{2}(\widehat{Q}\widehat{P} + \widehat{P}\widehat{Q})$? Derive a path integral expression for $< q'; t'|q; t >$ involving the $p-q$ symbol of $\widehat{H}$. Show that the p's have to be evaluated later than the q's.

Hint: Insert the decomposition of the identity at the left rather than at the right.

2.1.4.3 Path integral in momentum space

Derive the path integral representation of $< p'; t'|p; t >$ (directly or by Fourier transforming the expression for $< q'; t'|q; t >$). Show that the action that appears in the integral is

$$S_H - p'q' + pq = \int_t^{t'} d\tau\, [-\dot{p}_a q^a - H].$$

2.1.4.4 Partition function

Check that

$$\mathrm{Tr}\,\widehat{U}(t',t) \equiv \int dq\ < q|e^{-\frac{i}{\hbar}\widehat{H}(t'-t)}|q > = \int_{\text{perodic paths in q,p}} \prod dq(\tau) \prod \frac{dp(\tau)}{2\pi\hbar}\, e^{\frac{i}{\hbar}S_H[q,p]}, \tag{2.20}$$

Hint: One may define $p_{aN+1} = p_{a0}$ because p_{aN+1} does not appear in the discretized version of S_H.

Show that

$$\mathrm{Tr}\,\widehat{U}(t',t) = \sum_n N_n e^{-iE_n(t'-t)},$$

where the sum is over the different energy levels E_n and N_n is the degeneracy of a given level (assuming that the spectrum of $\widehat{H}$ is discret).

Putting $t' - t = -i\hbar\beta$, one gets

$$\mathrm{Tr}\,\widehat{U}(t',t) = \mathrm{Tr}\, e^{-\beta\widehat{H}} \equiv Z(\beta),$$

where $Z(\beta)$ is the quantum statistical partition function. Derive by direct computation a path integral representation for $Z(\beta)$.

Answer:

$$\boxed{Z(\beta) = \int dq\ < q|e^{-\beta\widehat{H}}|q > = \int_{\text{periodic paths in q,p}} \prod dq(\tau) \prod \frac{dp(\tau)}{2\pi\hbar}\, e^{-\frac{1}{\hbar}S^e_H[q,p]}}, \tag{2.21}$$

with $\boxed{S^e_H[q,p] = \int_0^{\hbar\beta} d\tau'[-i\dot{q}^a(\tau')p_a(\tau') + H(q(\tau'),p(\tau'))]}$. The integration is over all periodic paths of period $\hbar\beta$.

Show that this expression may be obtained from the one for $\mathrm{Tr}\,\widehat{U}(t',t)$ by putting $\tau' = i\tau$ and by rotating the integration path through 90 degrees in the complex plane of τ'.

Conclusion: Path integrals may also be used in the context of quantum statistical mechanics.

Details on the solution can be found below.

Partition function

Thursday, September 26, 2019 11:10 AM

$$\mathrm{Tr}\,\hat{U}(t',t) = \int dq \, \langle q | e^{-i/\hbar \, \hat{H}(t'-t)} | q \rangle$$

$$= \int dq \lim_{N\to\infty} \prod_{k=1}^{N} \prod_a dq_k^a \prod_{k=0}^{N} \prod_a \frac{dp_{ak}}{2\pi\hbar} \, e^{\frac{i}{\hbar}\sum_{k=1}^{N+1} (q_k^a - q_{k-1}^a) p_{a\,k-1} - H(q_k, p_{k-1})\delta t}$$

$$q_0^a = q^a \qquad q_{N+1}^a = q'^a = q \quad \Big\{ \text{because we are doing a trace}$$

no $p_{a\,N+1}$ appears

$$\to \int \prod_{a,t} dq^a(t) \prod_{a,t} \frac{dp_a(t)}{2\pi\hbar} \, e^{i/\hbar \, S_H[q,p]}$$

$q^a(t) = q^a(t')$, $p_a(t) = p_a(t')$ (periodic paths)

<u>Direct computation:</u>

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\left(e^{-\frac{\beta}{N+1}H}\right)^{N+1}$$

$$= \int dq_0 \left(\int dq_N \dots dq_1 \, \langle q_{N+1} | e^{-\frac{\beta}{N+1}\hat{H}} | q_N \rangle \dots \langle q_1 | e^{-\frac{\beta}{N+1}\hat{H}} | q_0 \rangle \right)\Bigg|_{q_{N+1}=q_0}$$

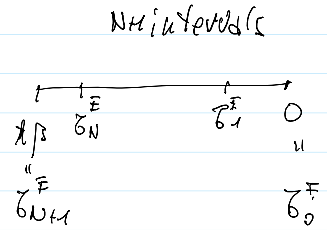
$$= \int dp_N \dots dp_0 \int dq_N \dots dq_0 \, \langle \underset{q_0}{\overset{q_{N+1}}{\shortparallel}} | e^{-\beta/N \hat{H}} | p_N \rangle \langle p_N | q_N \rangle \dots$$

$$\langle q_1 | e^{-\beta/N \hat{H}} | p_0 \rangle \langle p_0 | q_0 \rangle$$

$$= \prod_{k=0}^{N} \prod_a \int \frac{dq_k^a \, dp_{ak}}{2\pi\hbar} \, e^{-\beta/N \, H(q_k, p_{k-1})} \, e^{+i/\hbar (q_{k+1}^a - q_k^a) p_{ak}}$$

$$\tau^E \in [0, \hbar\beta] \qquad \tau_N^E = \tau_{N-1}^E + \Delta\tau^E$$

$$\Delta\tau^E = \frac{\hbar\beta}{N+1}$$



$q_{N+1} = q_0$, no p_{N+1} appearing $\Rightarrow$ one may choose these paths periodic as well

$$\xrightarrow[N\to\infty]{} \int_{\substack{q(0)=q(\hbar\beta)\\ p(0)=p(\hbar\beta)}} \prod_{a,\tau_E} \frac{dq^a(\tau_E)\, dp_a(\tau_E)}{2\pi\hbar}\, e^{-\frac{1}{\hbar}\underbrace{\int_0^{\hbar\beta} d\tau_E \left[-i\dot{q}^a p_a + H(q,p)\right]}_{S_E}}$$

relation: exponents in the PI:

$$\frac{i}{\hbar} S[q,p] = \frac{i}{\hbar}\int_t^{t'} d\tau \left[\dot{q}^a(\tau)p_a(\tau) - H(q(\tau),p(\tau))\right] \qquad (a)$$

$$-\frac{1}{\hbar} S_E[q,p] = -\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau_E \left(-i\dot{q}^a(\tau_E)p_a(\tau_E) + H(q(\tau_E),p(\tau_E))\right) \qquad (b)$$

(b)

Change of variables in (a) gives (b)

$$\tau_E = i(\tau - t) \qquad t'-t = -i\hbar\beta$$

$$\tau = -i\tau_E + t \qquad \frac{dq}{d\tau} = i\frac{dq}{d\tau_E}$$

$$d\tau = -i\, d\tau_E \qquad \begin{array}{l}\tau = t \Rightarrow \tau_E = 0\\ \tau = t' \Rightarrow \tau_E = \hbar\beta\end{array}$$

Suppose now that one takes a complete set of states that diagonalizes the Hamiltonian

$$\hat{H}\, |E_i, r_{k_1} \dots r_{k_m}\rangle = E_i\, |E_i, r_{k_1}, \dots r_{k_m}\rangle$$

and assume for simplicity that $i, k_1 \dots k_m$ are discrete (discrete energy spectrum, as for the harmonic oscillator)

$$\langle E_j, r_{l_1} \dots r_{l_m} | E_i, r_{k_1} \dots k_m\rangle = \delta_{ij}\, \delta_{l_1 k_1} \dots \delta_{l_m k_m}$$

$$\mathbb{1} = \sum_{i, k_1 \dots k_m} |E_i, r_{k_1} \dots r_{k_m}\rangle\langle E_i, r_{k_1} \dots r_{k_m}|$$

$_{i, \alpha_1 \ldots \alpha_m}$

It follows that

$$\mathrm{Tr}\ \hat{U}(t'-t) = \sum_{i\alpha_1 \ldots \alpha_m} \langle E_i, \gamma_{\alpha_1} \ldots \gamma_{\alpha_m} | e^{-i/\hbar \hat{H}(t'-t)} | E_i, \gamma_{\alpha_1} \ldots \gamma_{\alpha_m} \rangle$$

$$= \sum_i e^{-i/\hbar (t'-t) E_i} N_i$$

where $N_i = \sum_{\alpha_1 \ldots \alpha_m} \langle E_i\ \gamma_{\alpha_1} \ldots \gamma_{\alpha_m} | E_i\ \gamma_{\alpha_1} \ldots \gamma_{\alpha_m} \rangle$

is the degeneracy of an eigenstate of energy E_i

$$\mathrm{Tr}\ e^{-\beta \hat{H}} = \sum_i e^{-\beta E_i} N_i$$

2.1.4.5 More on Schwinger-Dyson equations

Derive the Schwinger-Dyson equations from the path integral representation of

$$< q';t'|\mathrm{T}\{\widehat{O}_B(\widehat{z}(t_B))\dots\}|q;t >$$

by performing the change of variables $z^\alpha(\tau) \to z^\alpha(\tau)+\xi^\alpha(\tau)\delta(\tau-t_A)$ with infinitesimal ξ^α and assuming that the path integral measure is invariant under translations.

Derive the Schwinger-Dyson equations

$$\mathrm{T}\{\widehat{\frac{\delta S_H}{\delta z^\alpha(t_A)}}\widehat{O}(\widehat{z}(t))\} = i\mathrm{T}\widehat{\frac{\delta O(z(t))}{\delta z^\alpha(t_A)}}$$

in the operator formalism. Here $\widehat{z}^\alpha(t)$ are operators in Heisenberg picture.

Hints:

- Use the correspondance rule $[\widehat{A},\widehat{B}] = i\hbar\widehat{\{A,B\}}$ to check that the equations of motion in Heisenberg picture may be written as

$$\widehat{\frac{\delta S_H}{\delta z^\alpha(t_A)}} = 0.$$

- Beware: Due to the presence of $\dot{\widehat{z}}^\beta(t_A)$ in the field equations, this does not mean that the left hand side of the Schwinger-Dyson equations vanishes.

- From the definition of the chronological product and the derivative,

$$\mathrm{T}\{\widehat{A}(t_1)\widehat{B}(t_2)\} = \widehat{A}(t_1)\widehat{B}(t_2)\theta(t_1-t_2) + \widehat{B}(t_2)\widehat{A}(t_1)\theta(t_2-t_1),$$
$$\dot{\widehat{z}}^\beta(t_A) = \lim_{\epsilon\to} \frac{\widehat{z}^\beta(t_A+\frac{\epsilon}{2}) - \widehat{z}^\beta(t_A-\frac{\epsilon}{2})}{\epsilon},$$

  show that

$$\mathrm{T}\{\dot{\widehat{z}}^\beta(t_A)\widehat{A}(t)\} \equiv \frac{d}{dt_A}\mathrm{T}\{\widehat{z}^\beta(t_A)\widehat{A}(t)\}$$
$$= \dot{\widehat{z}}^\beta(t_A)\widehat{A}(t)\theta(t_A-t) + \widehat{A}(t)\dot{\widehat{z}}^\beta(t_A)\theta(t-t_A) + \delta(t_A-t)[\widehat{z}^\beta(t_A),\widehat{A}(t)].$$

- Use the correspondence rule to conclude.

Remark: in order to use the correspondence rule without correction terms of higher order in $\hbar$, one has to assume that the functions H and O are at most quadratic phase space functions.

This computation justifies to some extent translation invariance of the path integral measure and the assumption that $-i[(O_B(z(t_B))\dots e^{iS_H[z]}]^{z^\alpha(\tau)=+\infty}_{z^\alpha(\tau)=-\infty} = 0$.

2.2 Passage to the S-matrix

cf. [2], [16].

2.2.1 Field theory

For a field theory,

$$\widehat{Q}^a \to \widehat{Q}^{\vec{x},m} \equiv \widehat{Q}^m(\vec{x}), \quad \widehat{P}_b \to \widehat{P}_{\vec{y},n} \equiv \widehat{P}_n(\vec{y}), \tag{2.22}$$
$$\delta^a_b \to \delta^{\vec{x},m}_{\vec{y},n} \equiv \delta^m_n\delta^3(\vec{x}-\vec{y}). \tag{2.23}$$

The matrix element with operator insertions can be represented as

$$
\begin{aligned}
&< q';t'|\mathrm{T}\{\widehat{O}_A[\widehat{z}(t_A)]\widehat{O}_B[\widehat{z}(t_B)]\dots\}|q;t>= \\
&\int_{\substack{q^m(\vec{x},t)=q^m(\vec{x})\\ q^m(\vec{x},t')=q'^m(\vec{x})}} \prod_{m,\vec{x},\tau} dq^m(\vec{x},\tau) \prod_{m,\vec{x},\tau}{}' \frac{dp_m(\vec{x},\tau)}{2\pi} \\
&\qquad \times O_A[z(t_A)]O_B[z(t_B)]\dots e^{\frac{i}{\hbar}S_H[q,p]}, \qquad (2.24)\\
&S_H[q,p] = \int_t^{t'} d\tau \left\{ \int d^3\vec{x}\, \dot{q}^m(\vec{x},\tau)p_m(\vec{x},\tau) - H[z(\tau)] \right\}.
\end{aligned}
$$

2.2.2 Probability amplitudes for in-out transitions

In field theory, one does not want to compute probability amplitudes of eigenstates of position operators, but S matrix elements, that is to say probability amplitudes between states which in the far past $t \to -\infty$ and the far future $t' \to +\infty$ contain a fixed number of particles characterized by a set of properties. These states are called "in" and "out" and denoted by $|\alpha; in>$, $|\beta; out>$. The letters α and β denote sets of particles characterized for exampe by their momentum, their spin, etc.

To represent such matrix elements, one multiplies (2.24) by wave functions $<\beta; out|q';t'>$ and $<q;t|\alpha; in>$ at fixed t in the far past and at fixed t' in the far future, where there are no interactions. One then integrates over the arguments $q^m(\vec{x}) = q^m(\vec{x}, -\infty)$ and $q'^m(\vec{x}) = q^m(\vec{x}, +\infty)$. Using the completeness relation, the left hand side yields

$$
<\beta; out|\mathrm{T}\{\widehat{O}_A[\widehat{z}(t_A)]\widehat{O}_B[\widehat{z}(t_B)]\dots\}|\alpha; in> \qquad (2.25)
$$

In the right hand side, we had a path integral constrained by boundary conditions in t and t', but now we are also integrating over these boundary conditions, which yields a path integral without boundary conditions:

$$
\begin{aligned}
(2.25) = &\int \prod_{m,\vec{x},\tau} dq^m(\vec{x},\tau) \prod_{m,\vec{x},\tau}{}' \frac{dp_m(\vec{x},\tau)}{2\pi\hbar} O_A[z(t_A)]O_B[z(t_B)]\dots \exp\frac{i}{\hbar}S_H[q,p] \\
&\times <\beta; out|q(+\infty); +\infty>< q(-\infty); -\infty|\alpha; in>, \qquad (2.26)\\
&S_H[q,p] = \int_{-\infty}^{+\infty} d\tau \left\{ \int d^3\vec{x}\, \dot{q}^m(\vec{x},\tau)p_m(\vec{x},\tau) - H[z(\tau)] \right\}.
\end{aligned}
$$

2.2.3 Vacuum to vacuum probability amplitude

Let us now take as intial and final states the in and out vacua, $|\beta; out>=|VAC; out>$, $|\alpha; in>=|VAC; in>$, defined through

$$
\widehat{a}_{in}(\vec{p}, m)|VAC; in>= 0, \qquad (2.27)
$$

$$
\widehat{a}_{out}(\vec{p}, m)|VAC; out>= 0. \qquad (2.28)
$$

where m represents the type of particles (photons, electrons,...) and also other characteristics such as spin for instance. We also take $\hbar = 1$

Recall that $Z^{1/2}\widehat{a}_{in}$ and $Z^{1/2}\widehat{a}_{out}$ are the operators appearing as coefficients of $e^{(i\vec{p}\cdot\vec{x}-i\omega(\vec{p})t)}$ in the expansion of $\widehat{Q}^m(\vec{x}, t)$ when $t \to \pm\infty$, where we have free fields.

Let us focus on scalar field theory, where one does not need an index m, and where one uses the standard notation $\widehat{Q}(\vec{x},t) = \widehat{\Phi}(\vec{x},t)$, $\widehat{P}(\vec{x},t) = \widehat{\Pi}(\vec{x},t)$. We have

$$\widehat{\Phi}(\vec{x},t) \overset{t\to\mp\infty}{\longrightarrow} Z^{1/2}(2\pi)^{-3/2}\int d^3p(2\omega(\vec{p}))^{-1/2}[\widehat{a}_{\substack{in\\out}}(\vec{p})e^{ip\cdot x} + \text{ h.c.}] \tag{2.29}$$

$$\widehat{\Pi}(\vec{x},t) \overset{t\to\mp\infty}{\longrightarrow} \frac{d}{dt}\widehat{\Phi}(\vec{x},t)$$
$$\overset{t\to\mp\infty}{\longrightarrow} -Z^{1/2}i(2\pi)^{-3/2}\int d^3p(\frac{\omega(\vec{p})}{2})^{1/2}[\widehat{a}_{\substack{in\\out}}(\vec{p})e^{ip\cdot x} - \text{ h.c.}] \tag{2.30}$$

with $p^0 = \omega(\vec{p}) = \sqrt{\vec{p}^2+m^2}$, $p\cdot x = \eta_{\mu\nu}p^\mu x^\nu$ and $\eta_{\mu\nu} = diag(-1,1,1,1)$. Inverting the Fourier transform to extract creation and destruction operators, one gets

$$Z^{1/2}\widehat{a}_{\substack{in\\out}} = \lim_{t\to\mp\infty}(2\pi)^{-3/2}e^{i\omega(\vec{p})t}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}$$
$$\times[(\frac{\omega(\vec{p})}{2})^{1/2}\widehat{\Phi}(\vec{x},t) + i(2\omega(\vec{p}))^{-1/2}\widehat{\Pi}(\vec{x},t)]. \tag{2.31}$$

For wave functions in the ϕ basis, $\widehat{\Pi}(\vec{x},t)$ acts like $-i\frac{\delta}{\delta\phi(\vec{x})}$, and the definition of the vacua becomes

$$\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\,[\frac{\delta}{\delta\phi(\vec{x})} + \omega(\vec{p})\phi(\vec{x})] < \phi(\vec{x}); \mp\infty|VAC;_{\substack{in\\out}} >= 0. \tag{2.32}$$

Recall that the differential equation $(\frac{d}{dy} + \omega y)f(y) = 0$ is solved by $f(y) = Ne^{-\frac{1}{2}\omega y^2}$ and that functional derivatives are defined by $\frac{\delta\phi^i(\vec{y})}{\delta\phi^j(\vec{x})} = \delta^i_j\delta(\vec{y}-\vec{x})$, $\frac{\delta}{\delta\phi^j(\vec{x})}\frac{\partial}{\partial y^m}\phi^i(\vec{y}) = \delta^i_j\frac{\partial}{\partial y^m}\delta(\vec{y}-\vec{x})$ etc.

One tries the Gaussian ansatz:

$$< \phi(\vec{x}); \mp\infty|VAC;_{\substack{in\\out}} >= \mathcal{N}e^{-\frac{1}{2}\int d^3x d^3y\,\Omega(\vec{x},\vec{y})\phi(\vec{x})\phi(\vec{y})}, \tag{2.33}$$

where one needs to determine the symmetric kernel $\Omega(\vec{x},\vec{y}) = \Omega(\vec{y},\vec{x})$ and the constant $\mathcal{N}$. Substituting in the differential equation, one gets

$$0 = \int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\,[\int d^3y\,\Omega(\vec{x},\vec{y})\phi(\vec{y}) - \omega(\vec{p})\phi(\vec{x})]$$
$$= \int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\int d^3y\Big[\Omega(\vec{x},\vec{y}) - \omega(\vec{p})\int\frac{d^3p'}{(2\pi)^3}e^{i\vec{p'}\cdot(\vec{x}-\vec{y})}\Big]\phi(\vec{y}). \tag{2.34}$$

Since this equation needs to hold for all $\phi(\vec{y})$, one deduces that

$$\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\,\Omega(\vec{x},\vec{y}) = \int d^3x\int\frac{d^3p'}{(2\pi)^3}\,\omega(\vec{p})e^{i\vec{x}\cdot(\vec{p'}-\vec{p})}e^{-i\vec{p'}\cdot\vec{y}}$$
$$= \omega(\vec{p})e^{-i\vec{p}\cdot\vec{y}}. \tag{2.35}$$

Inverting the Fourier transform yields

$$\Omega(\vec{x},\vec{y}) = \int\frac{d^3p}{(2\pi)^3}\,\omega(\vec{p})e^{i\vec{p}\cdot(\vec{x}-\vec{y})}. \tag{2.36}$$

The constant $\mathcal{N}$ may be obtained formally by the normalization of the vacuum, but we will not need an explicit expression here.

The last term of (2.26) is thus given by

$$< VAC; out|\phi(+\infty); +\infty >< \phi(-\infty); -\infty|VAC; in >$$
$$= |\mathcal{N}|^2 e^{-\frac{1}{2}\int d^3x d^3y\,\Omega(\vec{x},\vec{y})[\phi(\vec{x};+\infty)\phi(\vec{y};+\infty)+\phi(\vec{x};-\infty)\phi(\vec{y};-\infty)]}$$
$$= |\mathcal{N}|^2 e^{-\frac{1}{2}\epsilon\int d^3x d^3y\,\int_{-\infty}^{+\infty} d\tau\Omega(\vec{x},\vec{y})\phi(\vec{x};\tau)\phi(\vec{y};\tau)e^{-\epsilon|\tau|}}, \quad (2.37)$$

with $\epsilon > 0$ infinitesimal.

Indeed,

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \epsilon \int_{-\infty}^{+\infty} d\tau\, f(\tau)e^{-\epsilon|\tau|} &= \lim_{\epsilon\to 0^+} \epsilon\Big(\int_0^{+\infty} d\tau\, f(\tau)e^{-\epsilon\tau} + \int_{-\infty}^{0} d\tau\, f(\tau)e^{\epsilon\tau}\Big) \\
&= \lim_{\epsilon\to 0^+} \Big(-\int_0^{+\infty} d\tau\, f(\tau)\frac{d}{d\tau}e^{-\epsilon\tau} + \int_{-\infty}^{0} d\tau\, f(\tau)\frac{d}{d\tau}e^{\epsilon\tau}\Big) \\
&= \lim_{\epsilon\to 0^+} \Big(-\int_0^{+\infty} d\tau\, \frac{d}{d\tau}(f(\tau)e^{-\epsilon\tau}) + \int_0^{+\infty} d\tau\, f'(\tau)e^{-\epsilon\tau} \\
&\qquad + \int_{-\infty}^{0} d\tau\, \frac{d}{d\tau}(f(\tau)\exp\epsilon\tau) - \int_{-\infty}^{0} d\tau\, f'(\tau)e^{\epsilon\tau}\Big) \\
&= f(0) + f(+\infty) - f(0) + f(0) - f(0) + f(-\infty) \\
&= f(+\infty) + f(-\infty).
\end{aligned}$$

We thus find

$$< VAC; out|\mathrm{T}\{\widehat{O}_A\widehat{O}_B \dots\}|VAC; in >$$
$$= |\mathcal{N}|^2 \int \prod d\phi(x) \prod \frac{d\pi(x)}{2\pi}\, O_A O_B \dots \times e^{i(S_H + i\epsilon \text{ term})}, \quad (2.38)$$

$$S_H + i\epsilon \text{ term} = \int_{-\infty}^{+\infty} d\tau \Big[\int d^3\vec{x}\, \dot{\phi}(\vec{x},\tau)\pi(\vec{x},\tau) - H[\phi,\pi]$$
$$+ i\epsilon\frac{1}{2}\int d^3x d^3y\,\Omega(\vec{x},\vec{y})e^{-\epsilon|\tau|}\phi(\vec{x},\tau)\phi(\vec{y},\tau)\Big]. \quad (2.39)$$

As we will see, the $i\epsilon$ terms will give the correct $i\epsilon$ terms in the propagators. To complete the transition to the S matrix, we will need to recall the link between $< \beta; out|\mathrm{T}\{\widehat{O}_A\widehat{O}_B \dots\}|\alpha; in >$ and the S matrix. This is the object of so-called reduction formulas that will be treated later.

2.2.4 External sources

Let us now consider external sources $J^A(\vec{x},t)$ and the potential $\widehat{V}^J(t) = \widehat{V}(t) - \int d^3x\, J^A(\vec{x},t)\widehat{O}_A(\vec{x},t)$. The S-matrix becomes a functional of $J^A(\vec{x},t)$: $\langle\beta; out|\alpha; in\rangle^J$.

Derivatives of the S matrix with respect to the sources are related to transition amplitudes with insertions of operators in Heisenberg picture:

$$\Big[\frac{\delta^r}{\delta J^A(x_A)\delta J^B(x_B)\dots}\langle\beta; out|\alpha; in\rangle^J\Big]\Big|_{J=0} =< \beta; out|\mathrm{T}\{i\widehat{O}_A(x_A)i\widehat{O}_B(x_B)\dots\}|\alpha; in > . \quad (2.40)$$

Take as an Hamiltonian $H^J(t) = H(t) - \int d^3x J^A(\vec{x},t)O_A(\vec{x},t)$ in (2.26) without opertor insertions. Taking derivatives with respect to external sources and putting them to zero, we find in the right hand side a path integral representation with insertions of functions. One concludes by using again (2.26), this time with operator insertions.

Remark: The direct derivation of this result in terms of the operator formula $\widehat{S}[J] = Te^{-i\int_{-\infty}^{+\infty} d\tau\, \widehat{V}_I^J(t)}$ is much more involved than this path integral derivation.

2.2.5 Green's functions

Taking $|\alpha; in >= |VAC; in >$, $|\beta; out >= |VAC; out >$ and considering in particular a coupling to the fundamental fields of the theory, $\int d^3x\, J^A(\vec{x},t)O_A(\vec{x},t) = \int d^3x\, J_m(\vec{x},t)\phi^m(\vec{x},t)$, we have

$$\frac{\delta^r}{\delta J_{m_1}(x_1)\dots\delta J_{m_r}(x_r)}\langle VAC; out|VAC; in\rangle^J\Big|_{J=0} = (\frac{i}{\hbar})^r\langle VAC; out|\mathrm{T}\{\widehat{\Phi}^{m_1}(x_1)\dots\widehat{\Phi}^{m_r}(x_r)\}|VAC; in\rangle$$
$$= (\frac{i}{\hbar})^r|\mathcal{N}|^2\int\prod d\phi(x)\prod\frac{d\pi(x)}{2\pi}\Phi^{m_1}(x_1)\dots\Phi^{m_r}(x_r)\times e^{\frac{i}{\hbar}[S_H+i\epsilon \text{ term}]}. \quad (2.41)$$

The matrix elements $< VAC; out|\mathrm{T}\{\widehat{\Phi}^{m_1}(x_1)\dots\widehat{\Phi}^{m_r}\}(x_r)|VAC; in >$ are also called Green's functions. If $Z[J]$ denotes their generating functional, we have

$$\boxed{Z[J]} \equiv$$
$$\sum_{r=0}\frac{1}{r!}(\frac{i}{\hbar})^r\int dx_1\dots dx_r\, J_{m_1}(x_1)\dots J_{m_r}(x_r) < VAC; out|\mathrm{T}\{\widehat{\Phi}^{m_1}(x_1)\dots\widehat{\Phi}^{m_r}\}(x_r)|VAC; in >$$
$$\boxed{= |\mathcal{N}|^2\int\prod d\phi(x)\prod\frac{d\pi(x)}{2\pi\hbar}e^{\frac{i}{\hbar}[S_H+\int d^4x\, J_m(x)\phi^m(x)+i\epsilon \text{ term}]}}. \quad (2.42)$$

2.2.6 Exercises

2.2.6.1 Gaussian integration

- Show that

$$G(\alpha) = \int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}, \quad \alpha > 0.$$

Hint: Take the square and use polar coordinates.

- If one defines $T_\alpha(x) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2}$ and

$$\delta(x) = \lim_{\alpha\to\infty} T_\alpha(x),$$

show that

$$\int_{-\infty}^{+\infty} dx\, \delta(x) f(x) = f(0).$$

Hint: Taylor expand $f(x)$ around 0 and use

$$\int_{-\infty}^{+\infty} dx\, x^{2n} e^{-\alpha x^2} = (-)^n \frac{d^n}{(d\alpha)^n} G(\alpha).$$

- For $\alpha > 0$, show by completing the square that if

$$J = \int_{-\infty}^{+\infty} dx\, e^{-Q(x)}, \quad Q(x) = \alpha x^2 + \beta x + \gamma,$$

then

$$J = \sqrt{\frac{\pi}{\alpha}} e^{-Q(\bar{x})}, \quad \bar{x} = -\frac{\beta}{2\alpha}.$$

It follows that the integral J is given by $\sqrt{\frac{\pi}{\alpha}}$ times the exponential of the quadratic form evaluated at its extremum.

- Show that

$$\int_{-\infty}^{+\infty} dp\, e^{ipy} = 2\pi\delta(y)$$

Hint: Compute

$$\int_{-\infty}^{+\infty} dx\, e^{-ipx}\, T_\alpha(x)$$

as in the previous point and then compute

$$\int_{-\infty}^{+\infty} dp\, e^{ipy} \int_{-\infty}^{+\infty} dx\, e^{-ipx}\, T_\alpha(x).$$

Take the limit $\alpha \to \infty$.

- Show that if

$$J(A,b,c) \equiv \int_{-\infty}^{+\infty} \prod_i dx^i \, e^{-Q(x)}, \quad Q(x) = \frac{1}{2} A_{ij} x^i x^j + b_i x^i + c,$$

where A_{ij}, b_i, c are real and A_{ij} is symmetric and positive definite, then

$$J(A,b,c) = (\det(\frac{A}{2\pi}))^{-\frac{1}{2}} e^{(\frac{1}{2} b_i (A^{-1})^{ij} b_j - c)} = (\det(\frac{A}{2\pi}))^{-\frac{1}{2}} e^{-Q(\bar{x})},$$

where $\bar{x}^k$ is the extremum of $Q(x)$.

Hint: Diagonalize A using an orthogonal matrix O, $(O^T A O)_{ij} = \lambda^{\bar{i}} \delta_{ij}$, $\lambda^i > 0$ and use the change of variables $x^i = O^i{}_j x'^j$.

2.2.6.2 Fresnel integrals

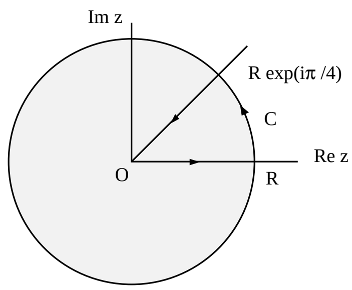
Show that

$$\int_{-\infty}^{+\infty} dx \, e^{\pm i x^2} = \sqrt{\pm i \pi}.$$

Hint: Use

$$\oint_C dz e^{iz^2} = 0,$$

where the integration contour C is



2.2.6.3 Wick theorem

Consider n variables $x^1, \ldots x^n$ and define the Gaussian momenta

$$< x^{k_1} \ldots x^{k_N} >= (J(A,0,0))^{-1} \int_{-\infty}^{+\infty} \prod_{i=1}^{n} dx^i \, x^{k_1} \ldots x^{k_N} e^{-\frac{1}{2} x^i A_{ij} x^j},$$

where $k_1, \ldots k_N$ are indices that take values from 1 to N. In particular, it follows that $< 1 >= 1$. Show that the result is zero for N odd and

$$< x^{k_1} \ldots x^{k_N} >= \sum_{\substack{\text{partitions of } \{k^1,\ldots,k^N\} \\ \text{in pairs of indices}}} \prod_{\text{pairs (i,j)}} (A^{-1})^{ij},$$

for N even.

Hint: Apply $\frac{\partial}{\partial b_{k_1}} \ldots \frac{\partial}{\partial b_{k_N}}$ to $J(A,b,0)$ with $b=0$ to show that

$$< x^{k_1} \ldots x^{k_N} >= [\frac{\partial}{\partial b_{k_1}} \ldots \frac{\partial}{\partial b_{k_N}} e^{(\frac{1}{2} b_i (A^{-1})^{ij} b_j)}]|_{b=0}.$$

2.3 Lagrangian formulation

2.3.1 Legendre transform in classical mechanics

How to go back from a first order Hamiltonian variational principle to a Lagrangian one ? Let

$$S_H[q,p] = \int_{t_1}^{t_2} d\tau\, [\dot{q}^a p_a - H(q,p)], \tag{2.43}$$

$$H = \frac{1}{2} g^{ab}(q) p_a p_b + h^a(q) p_a + V(q), \tag{2.44}$$

with g^{ab} symmetric and positive definite. Variation gives

$$0 = \delta S_H = \int_{t_1}^{t_2} d\tau\, (\dot{q}^b - \frac{\partial H}{\partial p_b})\delta p_b + \frac{d}{dt}(p_a \delta q^a) + (-\dot{p}_a - \frac{\partial H}{\partial q^a})\delta q^a. \tag{2.45}$$

Under the boundary conditions $\delta q^a(t_1) = 0 = \delta q^a(t_2)$ (but no need for boundary conditions on p_b), the field equations following by asking for an extremum of this action are the Hamiltonian equations of motion,

$$\begin{cases} \dot{q}^b - \dfrac{\partial H}{\partial p_b} = 0, \\ -\dot{p}_a - \dfrac{\partial H}{\partial q^a} = 0 \end{cases}. \tag{2.46}$$

The equations

$$\frac{\delta S_H}{\delta p_b(t)} = 0 \Leftrightarrow \dot{q}^b - g^{ba} p_a - h^b = 0 \Leftrightarrow p_b = g_{ba}(\dot{q}^a - h^a) \equiv \pi_b(q, \dot{q})$$

may be solved algebraically for p_b in terms of $q, \dot{q}$. Such Lagrange variables are referred to as auxiliary fields.

Reminder: H is defined as the Legendre transform of L with respect to $\dot{q}^a$:

$$L = L(q, \dot{q}),\ p_b = \frac{\partial L}{\partial \dot{q}^b}(q, \dot{q}).$$

If $|\frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}| \neq 0$, this last relation is invertible, $\dot{q}^a = U^a(q,p)$, which is equivalent to saying that

$$\frac{\partial L}{\partial \dot{q}^b}|_{\dot{q}=U} = p_b \Leftrightarrow U^a|_{p=\partial L/\partial \dot{q}} = \dot{q}^a, \tag{2.47}$$

while

$$H(q,p) = (p_b \dot{q}^b - L)|_{\dot{q}=U}, \tag{2.48}$$

$$\frac{\partial H}{\partial q^a}|_p = p_b \frac{\partial U^b}{\partial q^a} - \frac{\partial L}{\partial q^a}|_{\dot{q}=U} - \frac{\partial L}{\partial \dot{q}^b}|_{\dot{q}=U} \frac{\partial U^b}{\partial q^a} = -\frac{\partial L}{\partial q^a}|_{\dot{q}=U} \tag{2.49}$$

$$\frac{\partial H}{\partial p_a}|_q = U^a + p_b \frac{\partial U^b}{\partial p_a} - \frac{\partial L}{\partial \dot{q}^b}|_{\dot{q}=U} \frac{\partial U^b}{\partial p_a} = U^a. \tag{2.50}$$

This implies that $\dot{q}^a = \frac{\partial H}{\partial p_a}$ is the inverse relation to $p_b = \frac{\partial L}{\partial \dot{q}^b}$. When substituting this last relation in the second Hamiltonian equation $-\dot{p}_a - \frac{\partial H}{\partial q^a} = 0$ and using (2.49), one recovers the Lagrangian equation $-\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^b}) + \frac{\partial L}{\partial q^b} = 0$, which derives from the variational principle

$$0 = \delta S_L = \delta \int_{t_1}^{t_2} d\tau\, L(q, \dot{q}),\ \ \delta q^a(t_1) = 0 = \delta q^a(t_2). \tag{2.51}$$

The Lagrangian action is obtained from the Hamiltonian one by substituting the p_a through their solutions,

$$S_H|_{p=\partial L/\partial\dot{q}} = \int_{t_1}^{t_2} d\tau\, [p_b\dot{q}^b - H]|_{p=\partial L/\partial\dot{q}} \tag{2.52}$$

$$= \int_{t_1}^{t_2} d\tau\, [p_b\dot{q}^b - (p_b\dot{q}^b - L)|_{\dot{q}=U}]_{p=\partial L/\partial\dot{q}} \tag{2.53}$$

$$= \int_{t_1}^{t_2} d\tau\, L(q,\dot{q}). \tag{2.54}$$

Note also that the Lagrangian associated to the Hamiltonian (2.44) is explicitly given by

$$L(q,\dot{q}) = \pi_b(\dot{q}^b - h^b) - \frac{1}{2}\pi_b g^{bc}\pi_c - V(q) \tag{2.55}$$

$$= \frac{1}{2}(\dot{q}^a - h^a)g_{ab}(\dot{q}^b - h^b) - V(q). \tag{2.56}$$

2.3.2 Integrating out the momenta

What happens at the level of the path integral ? The p_b are independent integration variables,

$$\langle q', t'|q, t\rangle = \int \prod_{a,k=1}^{N} dq_k^a \prod_{b,k=0}^{N} \frac{dp_{bk}}{2\pi} e^{iS_H^D}, \tag{2.57}$$

$$S_H^D = \sum_{k=1}^{N+1}[(q_k^a - q_{k-1}^a)p_{ak-1} - H(q_k^a, p_{ak-1})d\tau], \tag{2.58}$$

$$H(q_k^a, p_{ak-1}) = \frac{1}{2}g^{ab}(q_k)p_{ak-1}p_{bk-1} + h^a(q_k)p_{ak-1} + V(q_k). \tag{2.59}$$

Shifting the sum from 0 to N and using $h^a(q_{k+1})d\tau = h^a(q_k)d\tau + O(d\tau^2)$, and similarly for $g^{ab}(q_{k+1}), V(q_{k+1})$, gives

$$S_H^D = \sum_{k=0}^{N}[p_{ak}(q_{k+1}^a - q_k^a + h^a(q_k)d\tau) - \frac{1}{2}g^{ab}(q_k)p_{ak}p_{bk}d\tau - V(q_k)d\tau]. \tag{2.60}$$

The integral over p_{ak} is Gaussian for each k. If $g = \det g_{ab}$, the pre-factor is

$$\frac{1}{(2\pi)^n}[\det\frac{ig^{ab}(q_k)d\tau}{2\pi}]^{-\frac{1}{2}} = [\frac{1}{2\pi^n}\sqrt{\frac{(2\pi)^n}{i^n \det g_{ab}^{-1}(q_k)(d\tau)^n}} = \sqrt{\frac{g}{(i\frac{2\pi(t'-t)}{N+1})^n}}.$$

The extremum is determined by

$$q_{k+1}^a - q_k^a + h^a(q_k)d\tau - g^{ab}(q_k)p_{bk}d\tau = 0, \tag{2.61}$$

$$p_{bk} = g_{ba}(q_k)[\frac{q_{k+1}^a - q_k^a}{d\tau} - h^a] \equiv \pi_{bk}, \tag{2.62}$$

$$S_H^D|_{extr} = \lim_{N\to\infty}\sum_{k=0}^{N}[\pi_{bk}g^{bc}(q_k)\pi_{ck}d\tau - \frac{1}{2}g^{ab}(q_k)\pi_{ak}\pi_{bk}d\tau - V(q_k)d\tau] =$$

$$= \int_t^{t'} d\tau\,[\frac{1}{2}g_{ab}(q)(\dot{q}^a - h^a)(\dot{q}^b - h^b) - V(q)] = \int_t^{t'} d\tau\, L(q,\dot{q}). \tag{2.63}$$

For the derivation of the Feynman rules, it is important to have all dependence on the q^a in the exponential,

$$\prod_{k=0}^{N}\sqrt{g(q_k)} = e^{\sum_{k=0}^{N}\ln\sqrt{g(q_k)}} = e^{\frac{i}{d\tau}[\sum_{k=0}^{N}(-i)\ln\sqrt{g(q_k)}]d\tau}$$

$$_{N\to\infty}\to \exp i\delta(0)\int_t^{t'} d\tau\frac{-i}{2}\ln g(q(\tau)). \quad (2.64)$$

Indeed,

$$f_{k'} = \sum_{k=0}^{N} d\tau f_k\frac{\delta_{kk'}}{d\tau} \to_{N\to\infty} f(\tau') = \int_t^{t'} d\tau f(\tau)\frac{\delta_{kk'}}{d\tau} = \int_t^{t'} d\tau f(\tau)\delta(\tau-\tau'),$$

so that $\delta(\tau-\tau') = \lim_{N\to\infty}\frac{\delta_{kk'}}{d\tau}$, $\delta(0) = \lim_{N\to\infty}\frac{1}{d\tau}$.

We thus find

$$\langle q';t'|q;t\rangle = \mathcal{M}'\int_{q(t)=q}^{q(t')=q'}\prod dq(\tau)e^{\frac{i}{\hbar}S^q}, \quad (2.65)$$

with

$$S^q[q] = \int_t^{t'} d\tau L^q(q(\tau),\dot{q}(\tau)), \quad L^q = L - i\hbar\delta(0)\frac{1}{2}\ln g, \quad (2.66)$$

$$\mathcal{M}' = \lim_{N\to\infty}(\frac{2\pi i\hbar(t'-t)}{N+1})^{-\frac{1}{2}(N+1)n}. \quad (2.67)$$

Remark: If g_{ab} does not depend on q, one usually absorbs the term with g in the measure:

$$\boxed{\langle q';t'|q;t\rangle = \mathcal{M}\int_{q(t)=q}^{q(t')=q'}\prod dq(\tau)e^{\frac{i}{\hbar}S}, \quad S[q] = \int_t^{t'} d\tau L}, \quad (2.68)$$

$$\mathcal{M} = \lim_{N\to\infty}(\frac{g(N+1)^n}{((2\pi i\hbar(t'-t))^n})^{\frac{1}{2}(N+1)}. \quad (2.69)$$

2.3.3 Exercises

2.3.3.1 More on the partition function

For an Hamiltonian of the type $H = \frac{1}{2} g^{ab} p_a p_b + V(q)$ with $g^{ab}(q)$ a positive definite matrix, what does the path integral representation of the partition function become after integration over the momenta ?

Answer:

Starting from (2.21), the extremum is given by $p_b(\tau) = g_{ab} i \dot{q}^b$. The integration over the momenta gives

$$\boxed{Z(\beta) = \int dq \; < q|e^{-\beta \widehat{H}}|q >= \int \prod_{\text{periodic paths in q}} dq^a(\tau) \, [\det(2\pi\hbar\mathcal{G})]^{-1/2} e^{-\frac{1}{\hbar}S^e_L[q]}}. \tag{2.70}$$

The integration is over periodic paths $q^a(\tau)$ with period β and

$$\boxed{S^e_L[q,p] = \int_0^{\hbar\beta} d\tau' [\frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b + V(q)]}, \tag{2.71}$$

$$\mathcal{G}^{a,b}(\tau,\tau') = g^{ab}(q(\tau))\delta(\tau,\tau'). \tag{2.72}$$

2.3.4 Semi-classical expansion of the partition function

Semi-classical expansion of the partition function
Thursday, February 20, 2020 9:47 PM

The semi-classical expansion corresponds to the limit where all quantum numbers are large with respect to $\hbar$, or formally, where $\hbar \to 0$.

Starting from

$$Z(\beta) = \int_{\substack{q(0)=q(\hbar\beta)\\ p(0)=p(\hbar\beta)}} \prod_{a,\tau} dq^a(\tau) \prod_{a,\tau'} \frac{dp_a(\tau')}{2\pi\hbar} \, e^{-\frac{1}{\hbar} S_H^E}$$

$$S_H^E = \int_0^{\hbar\beta} d\tau \left(-i\dot{q}^a p_a + H\right)$$

$$H = \frac{1}{2} g^{ab} p_a p_b + V(q)$$

one first does the change of integration variables

$$\tau \to \frac{\tau}{\hbar}$$

$\Rightarrow$ the boundary condition for the PI become

$$q(0) = q(\beta)$$
$$p(0) = p(\beta)$$

while

$$\frac{1}{\hbar} S_H^E = \int_0^\beta d\tau \left[-i\dot{q}^a \frac{p_a}{\hbar} + H\right]$$

The classical solution is determined by

$$\begin{cases} -i\dot{q}^a + \hbar \dfrac{\partial H}{\partial p_a} = 0 \\[2mm] +i\dot{p}_a + \hbar \dfrac{\partial H}{\partial q^a} = 0 \end{cases}$$

so that, when $\hbar \to 0$, it is the constant paths $q^a = q^a_0$, $p_a = p_{a0}$ that become the classical solutions and that also satisfy the boundary conditions.

Since this is not a unique classical solution, it is better to compute

$$\langle q | e^{-\beta \hat{H}} | p \rangle = \int_{\substack{p(0)=p \\ q(\hbar\beta)=q}} \prod_{a,\tau} dq^a(\tau) \prod_{b,\tau'} \frac{dp_b(\tau')}{2\pi\hbar} e^{-\frac{1}{\hbar}S'^E_H}$$

where

$$\frac{1}{\hbar} S'^E_H = \frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \left[\frac{1}{2}(-i\dot{q}^a p_a + i q^a \dot{p}_a) + H\right] + \frac{1}{\hbar}\frac{1}{2i}\left[q^a(\beta)p_a(\beta) + q^a(0)p_a(0)\right]$$

That this is the correct action to use follows as in the derivation of the kernel for the evolution operator in holomorphic representation.

In this case, there is a unique constant extremum when $\hbar \to 0$ $q^a(\tau) = q^a$ $p_a(\tau) = p_a$ and

$$S'^E_H = +\beta H(q,p) + \frac{1}{i\hbar} q^a p_a$$

It follows that, to leading order in $\hbar$,

$$\langle q | e^{-\beta \hat{H}} | p \rangle = \underbrace{\boxed{e^{-\beta H(q,p) - \frac{1}{i\hbar}q^a p_a} \frac{1}{\sqrt{2\pi\hbar}^{n}}}}_{c}$$

(as if there was only 1 slice)

while

$$Z_{class}(\beta) = \int \prod_a dq^a \prod_b dp_b \langle q | e^{-\beta \hat{H}} | p \rangle \langle p | q \rangle$$

$$= \int \prod_a dq^a \prod_b \frac{dp_b}{2\pi\hbar} e^{-\beta H(q,p) + \cancel{\frac{1}{i\hbar}q^a p_a}} \cancel{e^{-\frac{1}{i\hbar}q^b p_b}}$$

Hence $\boxed{Z_{class}(\beta) = \int \prod_a dq^a \prod_a \frac{dp_a}{2\pi\hbar} e^{-\beta H(q,p)}}$

which is the classical partition function, including the correct $\hbar$ dependent volume of phase space.

In the classical limit, there is thus a dimensional reduction from an integral over all periodic path,

$$q^a(\tau) = q^a_0 + \sum_{n>0} q^a_n e^{\frac{i2\pi}{\hbar\beta}\tau n} \qquad p_a(\tau) = p_{a0} + \sum_{n>0} p_{an} e^{\frac{i2\pi}{\hbar\beta}\tau n}$$

to an integral over just the constant zero modes.

(for a proof in the operator formalism, see e.g. Balian Vol I, 2.3.4).

—

The corrections and the precise nature of the $\hbar \to 0$ limit are determined as follows:

Consider $\langle q| e^{-\beta\hat{H}} |q\rangle$ after having performed the path integral over the p's.

The prefactor is

$$\prod_{\tau'} \frac{1}{(2\pi\hbar)^n} \det\left(\frac{g^{ab}d\tau}{2\pi}\right)^{-1/2} = \left(\sqrt{\frac{g}{2\pi\hbar^2\frac{\beta}{N+1}}}\right)^{N+1} = C'$$

so that

$$\langle q|e^{-\beta H}|q\rangle = C' \int_{\substack{q(0)=q\\ q(\hbar\beta)=q}} \prod_{a,\tau} dq^a(\tau)\, e^{-\int_0^{\hbar\beta} d\tau \left[\frac{1}{2} g_{ab}\dot{q}^a\dot{q}^b + V(q)\right]}$$

$$= C' \int_{\substack{q(0)=q \\ q(\beta)=q}} \prod_{a,\tau} dq^a(\tau)\; e^{-\int_0^\beta d\tau \left[\frac{1}{2\hbar^2} g_{ab} \dot{q}^a \dot{q}^b + V(q)\right]}$$

Consider the change of variables from $q^a(\tau) \to \tilde{q}^a(\tau)$

with $q^a(\tau) = q^a + \tilde{q}^a(\tau)$ around the constant value q^a

It follows that

$$\langle q | e^{-\beta \hat{H}} | q \rangle = C'' \int_{\tilde{q}(0)=0=\tilde{q}(\beta)} \prod_{a\tau} d\tilde{q}^a(\tau)\; e^{-\int_0^\beta d\tau \left[\frac{1}{2} g_{ab} \frac{\dot{\tilde{q}}^a \dot{\tilde{q}}^b}{\hbar^2} + V(q) + \frac{\partial V}{\partial q^a}\Big|_q \tilde{q}^a + \frac{1}{2} \frac{\partial^2 V}{\partial q^a \partial q^b}\Big|_q \tilde{q}^a \tilde{q}^b + O(\tilde{q}^3)\right]}$$

$$= C' C'' e^{-\beta V(q)} \Big(\langle 1 \rangle - \frac{\partial V}{\partial q^a}\Big| \langle \int_0^\beta d\tau\, \tilde{q}^a(\tau) \rangle - \frac{1}{2} \frac{\partial^2 V}{\partial q^a \partial q^b}\Big| \langle \int_0^\beta d\tau\, \tilde{q}^a(\tau) \tilde{q}^b(\tau) \rangle$$
$$+ \frac{1}{2} \frac{\partial V}{\partial q^a}\Big| \frac{\partial V}{\partial q^b}\Big| \langle \int_0^\beta d\tau \int_0^\beta d\tau'\, \tilde{q}^a(\tau) \tilde{q}^b(\tau') \rangle + O(\tilde{q}^3) \Big) \quad (*)$$

where $\langle X \rangle$ denotes the normalized Gaussian average with free particle measure,

$$\langle X \rangle = \frac{1}{C''} \int_{\tilde{q}(0)=0=\tilde{q}(\beta)} \prod_{a,\tau} d\tilde{q}^a(\tau)\; X\; e^{-\int_0^\beta d\tau\, \frac{1}{2} g_{ab} \dot{\tilde{q}}^a \dot{\tilde{q}}^b / \hbar^2}, \qquad \langle 1 \rangle = 1$$

If one concentrates on a single degree of freedom

we evaluate, as usual the integral with a source

$$I_p(\beta, b) = \int_{\tilde{q}(0)=0=\tilde{q}(\beta)} \prod_\tau d\tilde{q}(\tau)\; e^{\int_0^\beta d\tau \left[-\frac{m}{2\hbar^2} \dot{\tilde{q}}^2 + b(\tau)\tilde{q}(\tau)\right]} = C'' \langle e^{\int_0^\beta d\tau\, b(\tau)\tilde{q}(\tau)} \rangle$$

We then need the solution to $-\ddot{\Delta}(t,u) = +\delta(t-u)$

with $\Delta(\beta,u)=0$, $\Delta(0,u)=0$

It is given by $\Delta(t,u) = -\frac{1}{2}|t-u| + \frac{1}{2}(t+u-2ut/\beta) = \begin{cases} u-\frac{ut}{\beta} & t>u \\ t-\frac{ut}{\beta} & t<u \end{cases}$

Indeed:

$$\Delta(\beta,u) = -\frac{1}{2}|\beta-u| + \frac{1}{2}(\beta+u-2u) = 0\,, \qquad \Delta(0,u) = -\frac{1}{2}u+\frac{1}{2}u = 0$$

$$\dot{\Delta}(t,u) = \left.\begin{cases} -\frac{1}{2}+\frac{1}{2}-\frac{u}{\beta} & t>u \\ \frac{1}{2}+\frac{1}{2}-\frac{u}{\beta} & t<u \end{cases}\right\} = +\theta(u-t)-\frac{u}{\beta} \Rightarrow \dot{\Delta}(0,u) = -\frac{u}{\beta}$$

$$\ddot{\Delta}(t,u) = -\delta(t-u)$$

It follows that

$$I_P(b,\beta) = \prod_{\zeta}\left[\frac{\frac{m}{\hbar^2}\frac{\partial^2}{\partial\zeta^2}\delta(\zeta,\zeta')}{2\pi}\right]^{-1/2} e^{\frac{\hbar^2}{2m}\int_0^\beta dt\,du\, b(t)\,\Delta(t,u)\,b(u)}$$

The factor can be found because we know that

$$\langle q=0|e^{-\beta\frac{1}{2m}\hat{p}^2}|q=0\rangle = \int dp\,\langle 0|e^{-\beta\frac{1}{2m}\hat{p}^2}|p\rangle\langle p|0\rangle = \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi m}{\beta}} = \sqrt{\frac{m}{2\pi\beta\hbar^2}}$$

From the first term of (x) with $V(q)=0$ and the expression for C'

we also have

$$\langle q=0|e^{-\beta\frac{1}{2m}\hat{p}^2}|q=0\rangle = C'C''\langle 1\rangle \Leftrightarrow \sqrt{\frac{m}{2\pi\beta\hbar^2}} = C'C''$$

It follows that the one point function vanishes while

$$\langle q|e^{-\beta\hat{H}}|q\rangle = \sqrt{\frac{m}{2\pi\beta\hbar^2}}\left(1-\frac{1}{2}V''(q)\frac{\hbar^2}{m}\int_0^\beta d\zeta\,\Delta(\zeta,\zeta)+\frac{1}{2}(V'(q))^2\frac{\hbar^2}{m}\int_0^\beta d\zeta\int_0^\beta d\zeta'\,\Delta(\zeta,\zeta')+\mathcal{O}(\hbar^3)\right)$$

$$\Delta(\zeta,\zeta) = \zeta-\zeta^2/\beta \qquad \int_0^\beta d\zeta\,\Delta(\zeta,\zeta) = \frac{1}{2}\beta^2-\frac{1}{3}\,\beta^3/\beta = \frac{1}{6}\beta^2$$

$$\int_0^\beta d\tau \int_0^\tau d\tau' \Delta(\tau,\tau') + \int_0^\beta d\tau \int_\tau^\beta d\tau' \Delta(\tau,\tau') = \int_0^\beta d\tau \int_0^\tau d\tau' \left(\tau' - \frac{\tau'\tau}{\beta}\right) + \int_0^\beta d\tau \int_\tau^\beta d\tau' \left(\tau - \frac{\tau'\tau}{\beta}\right)$$

$$= \int_0^\beta d\tau \left[\frac{\tau^2}{2} - \frac{\tau}{\beta}\frac{\tau^2}{2} + \tau(\beta-\tau) - \frac{\tau}{\beta}\frac{1}{2}(\beta^2-\tau^2)\right]$$

$$= \int_0^\beta d\tau \left(\frac{\tau^2}{2} - \cancel{\frac{1}{\beta}\frac{1}{2}\tau^3} + \tau\beta - \tau^2 - \frac{1}{2}\beta\tau + \cancel{\frac{1}{\beta}\frac{1}{2}\tau^3}\right) = \int_0^\beta d\tau \left(-\frac{\tau^2}{2} + \frac{1}{2}\tau\beta\right)$$

$$= -\frac{\beta^3}{6} + \frac{1}{4}\beta\beta^2 = \frac{6-4}{24}\beta^3 = \frac{1}{12}\beta^3$$

$$Z(\beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}} \int dq\, e^{-\beta V(q)} \left[1 + \frac{\hbar^2\beta^3}{24m}(V'(q))^2 - \frac{\hbar^2\beta^2}{12m} V''(q) + O(\hbar^3)\right]$$

$$e^{-\beta V(q)} V'(q) V'(q) = \frac{d}{dq}\left(e^{-\beta V(q)}\right) V'(q)\left(-\frac{1}{\beta}\right) = \frac{d}{dq}\left((e^{-\beta V(q)} V'(q)\left(-\frac{1}{\beta}\right)\right) + e^{-\beta V(q)} V''(q) \frac{1}{\beta}$$

$$\boxed{Z(\beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}} \int dq\, e^{-\beta V(q)} \left[1 - \frac{\hbar^2\beta^2}{24m} V''(q) + O(\hbar^3)\right]}$$

For the harmonic oscillator, $V(q) = \frac{1}{2}kq^2$, $\frac{k}{m} = \omega^2$

$$Z(\beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}} \int_{-\infty}^{+\infty} dq\, e^{-\frac{\beta}{2}kq^2} \left[1 - \frac{\hbar^2\beta^2 k}{24m} + O(\hbar^3)\right]$$

$$= \sqrt{\frac{m}{2\pi\hbar^2\beta}} \left(\sqrt{\frac{2\pi}{\beta k}} \left(1 - \frac{\hbar^2\beta^2\omega^2}{24}\right) + O(\hbar^3)\right)$$

$$= \frac{1}{\hbar\beta\omega}\left(1 - \frac{\hbar^2\beta^2\omega^2}{24} + O(\hbar^3)\right) = \frac{1}{\hbar\beta\omega} - \frac{\hbar\beta\omega}{24} + O(\hbar^2)$$

2.4 Feynman rules

cf. [2]

2.4.1 Perturbation theory

In the case of field theory, one uses the notation $q^m(\vec{x}_a, t_a) \equiv \phi^m(x_a) = \phi^A$, and also $|VAC; {}^{out}_{in}\rangle = |0; {}^{+\infty}_{-\infty}\rangle$. A sum over A includes an integral over x_a and a sum over m. By combining the results of the previous two sections, Green's functions are represented by

$$\langle +\infty; 0|T\{\widehat{\phi}^{A_1} \dots \widehat{\phi}^{A_r}\}|0; -\infty\rangle = |\mathcal{N}|^2\mathcal{M} \int \prod d\phi\, \phi^{A_1} \dots \phi^{A_r} e^{\frac{i}{\hbar}I[\phi]} \tag{2.73}$$

$$\equiv \int \mathcal{D}\phi\, \phi^{A_1} \dots \phi^{A_r} e^{\frac{i}{\hbar}I[\phi]}, \tag{2.74}$$

with

$$I[\phi] = \int_{-\infty}^{+\infty} d\tau L + i\varepsilon,\ L = \int d^3x \mathcal{L}_0[\phi^m(x), \partial_\mu \phi^m(x)] + \mathcal{L}_1[\phi^m(x), \partial_\mu \phi^m(x)]. \tag{2.75}$$

Here $I_0 = \int d^4x \mathcal{L}_0 + i\varepsilon$ is the quadratic part, while $I_1 = \int d^4x \mathcal{L}_1$ denotes interactions that are cubic or of higher order in the fields and their derivatives.

For the generating functional, we have

$$Z[J] \equiv \langle +\infty; 0|0; -\infty\rangle^J = \int \mathcal{D}\phi\, e^{\frac{i}{\hbar}(I[\phi] + J_A\phi^A)}, \tag{2.76}$$

and

$$\langle +\infty; 0|T\{\widehat{\phi}^{A_1} \dots \widehat{\phi}^{A_r}\}|0; -\infty\rangle = (\frac{\hbar}{i})^r \frac{\delta}{\delta J_{A_1}} \dots \frac{\delta}{\delta J_{A_r}} Z[J]|_{J=0}. \tag{2.77}$$

One may then treat the interaction perturbatively

$$Z[J] = e^{\frac{i}{\hbar}I_1[\frac{\hbar}{i}\frac{\delta}{\delta J}]} \int \mathcal{D}\phi\, e^{\frac{i}{\hbar}(I_0[\phi] + J_A\phi^A)}, \tag{2.78}$$

with $I_0 = -\frac{1}{2}\phi^A \mathcal{D}_{AB}\phi^B$.

Performing discretized Gaussian integrals, this gives

$$Z[J] = |\mathcal{N}|^2\mathcal{M}[\det(\frac{i\mathcal{D}}{2\pi\hbar})]^{-\frac{1}{2}} e^{\frac{i}{\hbar}I_1[\frac{\hbar}{i}\frac{\delta}{\delta J}]} e^{\frac{i}{2\hbar}J_A(\mathcal{D}^{-1})^{AB}J_B}, \tag{2.79}$$

and then

$$\boxed{\frac{Z[J]}{Z_0[0]}} = \frac{\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}(I[\phi]+J_A\phi^A)}}{\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}I_0[\phi]}} = \boxed{e^{\frac{i}{\hbar}I_1[\frac{\hbar}{i}\frac{\delta}{\delta J}]} e^{\frac{i}{2\hbar}J_A(\mathcal{D}^{-1})^{AB}J_B}}. \tag{2.80}$$

N.B: This last expression can also be used as a perturbative definition of the path integral, since it can also be proved directly in the operator formalism.

If $I_1 = 0$, one recovers Wick's theorem,

$$\frac{\langle +\infty; 0|T\{\widehat{\phi}^{A_1} \dots \widehat{\phi}^{A_r}\}|0; -\infty\rangle}{\langle +\infty; 0|0; -\infty\rangle} = \sum_{pairs\, of\, indices} \prod (\frac{\hbar}{i}(\mathcal{D}^{-1})^{paired\, indices}. \tag{2.81}$$

If $I_1 \neq 0$, one develops in terms of the interaction. At a given order, there are always the same type of Gaussian integrals to be carried out. Some of the ϕ^A however are not coming from differentiation with respect to an external source J_A, but they are due to $\int d^4x \mathcal{L}_1[\frac{\hbar}{i}\frac{\delta}{\delta J}]$ and involve an integral over x^μ in field theory.

2.4.2 Propagators

The Feynman propagators $\Delta_F^{m_2m_3}(x_2,x_3)\equiv(\mathcal{D}^{-1})^{m_2m_3}(x_2,x_3)$ that appear in (2.80) are thus defined as the inverse of the quadratic part of the Lagrangian (including $i\epsilon$ terms):

$$\int d^dx_2\mathcal{D}_{m_1m_2}(x_1,x_2)\Delta_F^{m_2m_3}(x_2,x_3)=\delta_{m_1}^{m_3}\delta^d(x_1,x_3). \tag{2.82}$$

As we will see, $\mathcal{D}_{m_1m_2}(x_1,x_2)$ can generally be written as an operator acting on $\delta^n(x_1,x_3)$ and its Fourier transform depends only on a single rather than 2 variables,

$$\mathcal{D}_{m_1m_2}(x_1,x_2)=\frac{1}{(2\pi)^d}\int d^dp\; e^{ip(x_1-x_2)}\widetilde{\mathcal{D}}_{m_1m_2}(p)=\mathcal{D}_{m_1m_2}(x_1-x_2). \tag{2.83}$$

As a consequence, the Fourier transform of $\Delta_F^{m_2m_3}(x_2,x_3)$ is the inverse matrix of $\widetilde{\mathcal{D}}_{m_1m_2}(p)$,

$$\widetilde{\mathcal{D}}_{m_1m_2}(p)\widetilde{\Delta}_F^{m_2m_3}(p)=\delta_{m_1}^{m_3}. \tag{2.84}$$

Indeed, this allows one to check equation (2.82):

$$\frac{1}{(2\pi)^{2d}}\int d^dx_2d^dpd^dp'e^{ix_2(p'-p)}e^{(ipx_1-ip'x_3)}\widetilde{\mathcal{D}}_{m_1m_2}(p)\widetilde{\Delta}_F^{m_2m_3}(p')=$$
$$=\frac{1}{(2\pi)^d}\int d^dpe^{ip(x_1-x_3)}\delta_{m_1}^{m_3}=\delta_{m_1}^{m_3}\delta^d(x_1,x_3).$$

For the massive scalar field in 4 dimensions for example, $I_0=-\frac{1}{2}\int d^4xd^4x'\mathcal{D}(x,x')\phi(x)\phi(x')$. Since

$$\int d^4x\;\mathcal{L}_0+i\epsilon=\int d^4x\,(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2)+\frac{1}{2}i\epsilon\int dtd^3xd^3x'\Omega(\vec{x},\vec{x}')e^{-\epsilon|t|}\phi(\vec{x},t)\phi(\vec{x}',t), \tag{2.85}$$

one finds

$$\mathcal{D}(x,x')=(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'_\mu}+m^2)\delta^4(x-x')-i\epsilon\Omega(\vec{x},\vec{x}')e^{-\epsilon|t|}\delta(t-t'). \tag{2.86}$$

At this level, one may replace $e^{-\epsilon|t|}$ by 1 because the difference corresponds to terms of higher order in ϵ. Since

$$\Omega(\vec{x},\vec{x}')=\frac{1}{(2\pi)^3}\int d^3p\,\omega(\vec{p})e^{i\vec{p}(\vec{x}-\vec{x}')},$$

one finds

$$\mathcal{D}(x_1,x_2)=\frac{1}{(2\pi)^4}\int d^4p\;e^{ip(x_1-x_2)}[p^2+m^2-i\epsilon\omega(\vec{p})], \tag{2.87}$$

and thus

$$\Delta_F(x_1,x_2)=\frac{1}{(2\pi)^4}\int d^4p\;e^{ip(x_1-x_2)}\frac{1}{p^2+m^2-i\epsilon\omega(\vec{p})}\;, \tag{2.88}$$

in complete agreement with the computation in terms of operators in (1.32).

2.4.3 Feynman rules for scalar field theory

For the scalar field, let us consider a quartic interaction,

$$I_1[\phi] = -\frac{g}{4!}\int d^4x\,\phi^4. \tag{2.89}$$

The 2 and 4 point functions are defined by

$$G^{(2)}(x_1,x_2) = \frac{1}{Z[0]}\langle +\infty;0|T\{\widehat{\phi}(x_1)\widehat{\phi}(x_2)\}|0;-\infty\rangle, \tag{2.90}$$

$$G^{(4)}(x_1,\ldots,x_4) = \frac{1}{Z[0]}\langle +\infty;0|T\{\widehat{\phi}(x_1)\ldots\widehat{\phi}(x_4)\}|0;-\infty\rangle\,. \tag{2.91}$$

To order 0 in g, $Z[0]_0 = \langle +\infty;0|0;-\infty\rangle_0 = |\mathcal{N}|^2\mathcal{M}[\det(\frac{i\mathcal{D}}{2\pi\hbar})]^{-\frac{1}{2}} \equiv \mathcal{K}$.

$$G_0^{(2)}(x_1,x_2) = (\frac{\hbar}{i})^2\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_1)}\frac{Z_0[J]}{Z_0[0]}|_{J=0} = \frac{\hbar}{i}\mathcal{D}^{-1}(x_1,x_2). \tag{2.92}$$

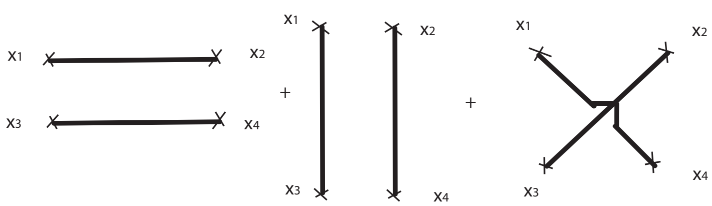
Graphical representation: 1 —— 2 or

For $\int d^4x\mathcal{D}^{-1}(x_1,x)J(x)$, let us introduce the graphical representation 1 ——□.

$$G_0^{(6)}(x_1,\ldots,x_6)=(\tfrac{\hbar}{i})^6\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_6)}\frac{Z_0[J]}{Z_0[0]}|_{J=0}=$$
$$(\tfrac{\hbar}{i})^5\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_5)}\Big[\ 6\text{——}\square\ \ \frac{Z_0[J]}{Z_0[0]}\Big]|_{J=0}=$$
$$(\tfrac{\hbar}{i})^4\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_4)}\Big[\big(6\text{——}\square\ 5\text{——}\square+6\text{——}5\big)\frac{Z_0[J]}{Z_0[0]}\Big]|_{J=0}=$$
$$(\tfrac{\hbar}{i})^3\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_3)}\Big[\Big($$
$$\big(6\text{——}\square\ 5\text{——}\square+6\text{——}5\big)4\text{——}\square+6\text{——}4\ 5\text{——}\square+6\text{——}\square\ 5\text{——}4$$
$$\Big)\frac{Z_0[J]}{Z_0[0]}\Big]|_{J=0}=(\tfrac{\hbar}{i})^2\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\Big[\Big($$
$$\Big(\big(6\text{——}\square\ 5\text{——}\square+6\text{——}5\big)4\text{——}\square+6\text{——}4\ 5\text{——}\square+6\text{——}\square\ 5\text{——}4\Big)3\text{——}\square$$
$$+\big(6\text{——}\square\ 5\text{——}\square+6\text{——}5\big)4\text{——}3$$
$$+\big(6\text{——}3\ 5\text{——}\square+6\text{——}\square\ 5\text{——}3\big)4\text{——}\square$$
$$+6\text{——}4\ 5\text{——}3+6\text{——}3\ 5\text{——}4\Big)\frac{Z_0[J]}{Z_0[0]}\Big]|_{J=0}=(\tfrac{\hbar}{i})\frac{\delta}{\delta J(x_1)}\Big[\Big(\Big($$
$$\Big(\big(6\text{——}\square\ 5\text{——}\square+6\text{——}5\big)4\text{——}\square+6\text{——}4\ 5\text{——}\square+6\text{——}\square\ 5\text{——}4\Big)3\text{——}\square$$
$$+\big(6\text{——}\square\ 5\text{——}\square+6\text{——}5\big)4\text{——}3$$
$$+\big(6\text{——}3\ 5\text{——}\square+6\text{——}\square\ 5\text{——}3\big)4\text{——}\square$$
$$+6\text{——}4\ 5\text{——}3+6\text{——}3\ 5\text{——}4\Big)2\text{——}\square$$
$$+\Big(\big(6\text{——}\square\ 5\text{——}\square+6\text{——}5\big)4\text{——}\square+6\text{——}4\ 5\text{——}\square+6\text{——}\square\ 5\text{——}4\Big)3\text{——}2$$
$$+\Big(\big(6\text{——}\square\ 5\text{——}\square+6\text{——}5\big)4\text{——}2+\big(6\text{——}2\ 5\text{——}\square+6\text{——}\square\ 5\text{——}2\big)4\text{——}\square$$
$$+6\text{——}4\ 5\text{——}2+6\text{——}2\ 5\text{——}4\Big)3\text{——}\square$$
$$+\big(6\text{——}2\ 5\text{——}\square+6\text{——}\square\ 5\text{——}2\big)4\text{——}3$$
$$+\big(6\text{——}3\ 5\text{——}\square+6\text{——}\square\ 5\text{——}3\big)4\text{——}2$$
$$+\big(6\text{——}3\ 5\text{——}2+6\text{——}2\ 5\text{——}3\big)4\text{——}\square\Big)\frac{Z_0[J]}{Z_0[0]}\Big]|_{J=0}=$$
$$\Big(6\text{——}5\ 4\text{——}3+6\text{——}4\ 5\text{——}3+6\text{——}3\ 5\text{——}4\Big)2\text{——}1+$$
$$\big(6\text{——}5\ 4\text{——}1+6\text{——}4\ 5\text{——}1+6\text{——}1\ 5\text{——}4\big)3\text{——}2+$$
$$\big(6\text{——}5\ 4\text{——}2+6\text{——}4\ 5\text{——}2+6\text{——}2\ 5\text{——}4\big)3\text{——}1+$$
$$\big(6\text{——}2\ 5\text{——}1+6\text{——}1\ 5\text{——}2\big)4\text{——}3+$$
$$\big(6\text{——}3\ 5\text{——}1+6\text{——}1\ 5\text{——}3\big)4\text{——}2+$$
$$\big(6\text{——}3\ 5\text{——}2+6\text{——}2\ 5\text{——}3\big)4\text{——}1$$

Consequence:

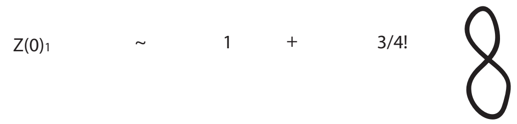
$G_0^{(4)}(x_1,\ldots,x_4)=:$



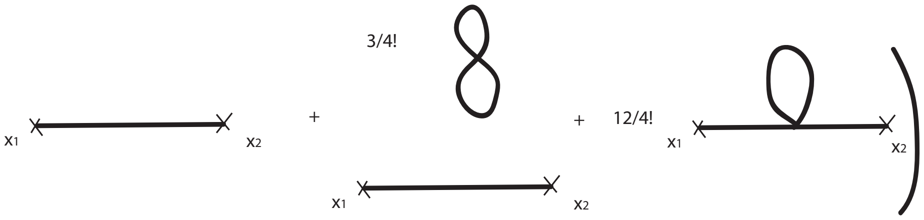
To order 1 in g,

$$
\begin{aligned}
Z[0]_1 &= \langle +\infty; 0|0; -\infty\rangle_1 && (2.93)\\
&= \mathcal{K}(1 - \frac{i}{\hbar}\frac{g}{4!}\int d^4x\,(\frac{\hbar}{i}\frac{\delta}{\delta J(x)})^4)\exp\frac{i}{2\hbar}J(\mathcal{D}^{-1})J]|_{J=0}\\
&= \mathcal{K}(1 - \frac{3ig}{\hbar 4!}\int d^4x\frac{\hbar}{i}\mathcal{D}^{-1}(x,x)\frac{\hbar}{i}\mathcal{D}^{-1}(x,x))\,. && (2.94)
\end{aligned}
$$

If an internal point (vertex) of a diagram corresponds to $-\frac{ig}{\hbar}\int d^4x$, one gets the following graphical representation

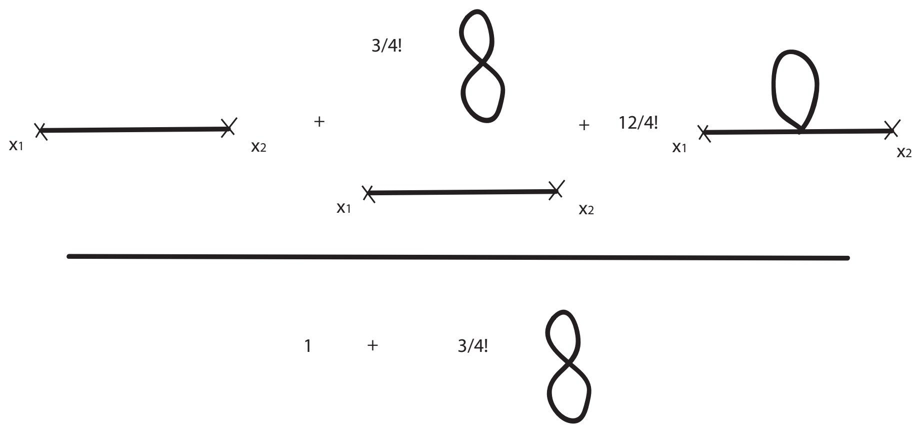


$$
\begin{aligned}
&\langle +\infty; 0|T\{\widehat{\phi}(x_1)\widehat{\phi}(x_2)\}|0; -\infty\rangle_1 =\\
&= \mathcal{K}(\frac{\hbar}{i})^2\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_1)}(1 - \frac{i}{\hbar}\frac{g}{4!}\int d^4x\,(\frac{\hbar}{i}\frac{\delta}{\delta J(x)})^4)\exp\frac{i}{2\hbar}J(\mathcal{D}^{-1})J]|_{J=0}\\
&= \mathcal{K}\Big(&& (2.95)
\end{aligned}
$$

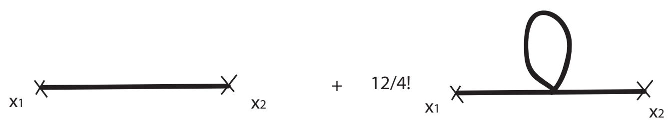


This follows from the expansion of $G_0^{(6)}(x_1,\ldots,x_6)$ by setting $x_3 = x_4 = x_5 = x_6 = x$. It follows that

$G_1^{(2)}(x_1,x_2) =$



$=$



Remarks: (i) The complete Feynman rules also include so-called symmetry factors that allow one to get the numerical factors correctly. They have not given here. The aim was rather to show how the rules follow from the fundamental formula (2.80). There are symbolic computer packages that can be used to perform such computations to high orders. (ii) When using expression (1.31) for $\mathcal{D}^{-1}(x, y)$, one sees that in the limit $x - y \to 0$, the integral over d^3k diverges. We will thus need to modify the theory to give a meaning to $\mathcal{D}^{-1}(x, x)$ (except in mechanics where no such mometa integrals are involved).

(ii) If $G^{(k)}(x_1, \dots, x_k) = \frac{1}{Z[0]}\langle +\infty; 0|T\{\widehat{\phi}(x_1)\dots\widehat{\phi}(x_k)|0; -\infty\rangle$, the effect of dividing by $Z[0]$ amounts to removing vacuum parts of the Green's functions at all orders. Here, a vacuum part is the analytic expression that corresponds to a disconnected part from the rest of the diagram that does not involve external points. Indeed,

$$
\begin{aligned}
G^{(k)}(x_k, \dots, x_1) &= \frac{e^{\frac{i}{\hbar}I_1[\frac{\hbar}{i}\frac{\delta}{\delta J}]}\frac{\hbar}{i}\frac{\delta}{\delta J(x_k)}\cdots\frac{\hbar}{i}\frac{\delta}{\delta J(x_1)}e^{\frac{i}{2\hbar}J\mathcal{D}^{-1}J}}{\exp\frac{i}{\hbar}I_1[\frac{\hbar}{i}\frac{\delta}{\delta J}]e^{\frac{i}{2\hbar}J\mathcal{D}^{-1}J}}|_{J=0} \\
&= \frac{e^{\frac{i}{\hbar}I_1[\frac{\hbar}{i}\frac{\delta}{\delta J}]}[\frac{\hbar}{i}\frac{\delta}{\delta J(x_k)}\cdots\frac{\hbar}{i}\frac{\delta}{\delta J(x_3)}(\frac{\hbar}{i}\mathcal{D}^{-1}(x_1, x_2) + \mathcal{D}^{-1}_{x_1}J\mathcal{D}^{-1}_{x_2}J)e^{\frac{i}{2\hbar}J\mathcal{D}^{-1}J}}{e^{\frac{i}{\hbar}I_1[\frac{\hbar}{i}\frac{\delta}{\delta J}]}e^{\frac{i}{2\hbar}J\mathcal{D}^{-1}J}}|_{J=0} \qquad (2.96)
\end{aligned}
$$

and one sees that the denominator precisely cancels the part of the numerator where $e^{\frac{i}{\hbar}I_1[\frac{\hbar}{i}\frac{\delta}{\delta J}]}$ acts entirely on the term $e^{\frac{i}{2\hbar}J\mathcal{D}^{-1}J}$.

2.4.4 Feynman rules in momentum space

to be added

2.4.5 Exercises

2.4.5.1 Propagator for massive vector field

Compute the propagator of the massive vector field with Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}m^2 A_\mu A^\mu \,. \tag{2.97}$$

Answer:

$$\widetilde{\Delta}_F^{\rho\sigma} = \frac{1}{p^2 + m^2 - i\epsilon}[\eta^{\rho\sigma} + \frac{p^\rho p^\sigma}{m^2}].$$

What happens for the propagator of the photons ($m = 0$) ? Show that a potential that is pure gauge, $A_\mu = \partial_\mu \lambda$, corresponds to an eigenvector with eigenvalue 0 of the quadratic kernel $\mathcal{D}_{\mu\nu}(x, x')$.

2.4.5.2 Divergent integrals through dimensional continuation

1. The Gamma function is defined by

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t}.$$

   Show that

   (a) $\Gamma(1) = 1$,
   (b) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,
   (c) $\Gamma(x+1) = x\Gamma(x)$,
   (d) $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

2. Show that the volume of the unit sphere in $\mathbb{R}^d$ is given by

$$\text{Vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

   Hint: compute the integral $\int d^d x\, e^{-|\vec{x}|^2}$ using two different methods.
   Specify the result for S^0, S^1, S^2 and S^3.

3. The beta function is defined by

$$B(x, y) = \int_0^1 dt\, t^{x-1}(1-t)^{y-1}.$$

   Show that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

4. Consider the integrals

$$i\Phi(m, d, A) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 + m^2 - i\epsilon)^A}, \tag{2.98}$$

   where $p^2 = -{p^0}^2 + p_i p^i$.

   (a) Where are the poles of the integrands ?

(b) Show that the position of these poles allow one to write

$$\Phi(m,d,A) = \int \frac{d^d p_E}{(2\pi)^d} \frac{1}{(p_E^2+m^2)^A}, \tag{2.99}$$

where $p_E^2 = (p_E^0)^2 + p_i^E p_E^i$.

(c) Go to spherical coordinates to show that

$$\Phi(m,d,A) = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty \frac{r^{d-1}dr}{(r^2+m^2)^A}. \tag{2.100}$$

(d) For a given value of d, for what values of A does the integral converge ?

(e) If the upper bound of the integral in the previous expression is Λ instead of $+\infty$, how does the integral behave when $\Lambda \to \infty$. Specify for $d = 4$, $A = 1,2$ and for $d = 6$, $A = 1,2,3$.

5. Show that

$$\Phi(m,d,A) = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(A-d/2)}{\Gamma(A)} (\frac{1}{m^2})^{A-d/2}. \tag{2.101}$$

6. Show that

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^{2B}}{(p^2+m^2-i\epsilon)^A} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(A-d/2-B)\Gamma(d/2+B)}{\Gamma(A)\Gamma(d/2)} (\frac{1}{m^2})^{A-d/2-B}.$$

7. The divergent behavior of Γ around zero is given by

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon),$$

where $\gamma = 0,5772\ldots$ is the Euler-Mascheroni constant. Show that

$$\begin{aligned}
a)\quad & \Gamma(-1+\epsilon) = -(\frac{1}{\epsilon} - \gamma + 1 + O(\epsilon)), \\
b)\quad & \Gamma(-2+\epsilon) = \tfrac{1}{2}(\frac{1}{\epsilon} - \gamma + \frac{3}{2} + O(\epsilon)), \\
c)\quad & \Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}(\frac{1}{\epsilon} - \gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{n} + O(\epsilon)), \quad n \in \mathbb{N}.
\end{aligned}$$

8. Compute $\Phi(m,d,A)$ for

(a) $d = 3 - 2\epsilon$ and $A = -\frac{1}{2}$. Answer: after introducing an arbitrary renormalization scale parameter μ of dimension of mass through $1 = \mu^{-2\epsilon}\mu^{2\epsilon}$, one finds

$$\Phi(m, 3-2\epsilon, -\frac{1}{2}) = \frac{-m^4\mu^{-2\epsilon}}{32\pi^2}\Big[\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2} + \ln 4\pi - \gamma + \frac{3}{2} + O(\epsilon)\Big]. \tag{2.102}$$

(b) $d = 4 - 2\epsilon$ and $A = 1,2$. Answer:

$$\Phi(m, 4-2\epsilon, 2) = \frac{1}{16\pi^2}\mu^{-2\epsilon}\Big[\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2} + \ln 4\pi - \gamma + O(\epsilon)\Big]. \tag{2.103}$$

$$\Phi(m, 4-2\epsilon, 1) = -\frac{1}{16\pi^2}m^2\mu^{-2\epsilon}\Big[\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2} + \ln 4\pi - \gamma + 1 + O(\epsilon)\Big]. \tag{2.104}$$

(c) $d = 6 - 2\epsilon$ and $A = 1, 2, 3$.

9. Show that
$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}.$$
In this formula, x is called the "Feynman parameter".

10. Show the generalization
$$\frac{1}{A_1 \dots A_n} = \int d^n x \, \delta(\sum_{i=1}^n x^i - 1) \frac{(n-1)!}{[x^i A_i]^n},$$
by first showing that
$$\frac{1}{AB^n} = \int dxdy \, \delta(x + y - 1) \frac{ny^{n-1}}{[xA + yB]^{n+1}},$$
using the previous exercise.

2.5 Holomorphic representation

cf. [9], [11] [1].

2.5.1 Coherent states

For a single degree of freedom (q, p), let us define

$$a = \frac{1}{\sqrt{2\hbar}}(\sqrt{\omega}q + i\sqrt{\omega}^{-1}p). \tag{2.105}$$

For the associated operator and its complex conjugate, the canonical commutation relations of $\widehat{q}, \widehat{p}$ translate into

$$[\widehat{a}, \widehat{a}^\dagger] = 1, \quad [\widehat{a}, \widehat{a}] = 0 = [\widehat{a}^\dagger, \widehat{a}^\dagger]. \tag{2.106}$$

The vacuum is defined by $\widehat{a}|0\rangle = 0$ and coherent states by

$$\boxed{|a\rangle = e^{a\widehat{a}^\dagger}|0\rangle, \quad \widehat{a}|a\rangle = a|a\rangle}, \tag{2.107}$$

where a is a complex number. The commutation relations imply that the coherent state is an eigenvector of $\widehat{a}$ with eigenvalue a. Similarily,

$$\langle a^*| = \langle 0|e^{\widehat{a}a^*}, \ \langle a^*|\widehat{a}^\dagger = \langle a^*|a^*. \tag{2.108}$$

The scalar product of such coherent states is given by

$$\langle b^*|a\rangle = e^{b^*a}. \tag{2.109}$$

Indeed, if $a = 0$, we find 1 on both sides. Differentiation with respect to a, one finds b^* times the expression itself, both on the left and the right hand side.

An orthonormal basis of the Hilbert space is provided by

$$|n\rangle = \frac{1}{\sqrt{n!}}(\widehat{a}^\dagger)^n|0\rangle, \ \langle m|n\rangle = \delta_{mn}, \tag{2.110}$$

and

$$\langle a^*|n\rangle = \langle 0|e^{a^*\widehat{a}}\frac{1}{\sqrt{n!}}(\widehat{a}^\dagger)^n|0\rangle = \frac{1}{\sqrt{n!}}(a^*)^n, \tag{2.111}$$

Indeed, this holds for $n = 0$, and assuming it to hold for $n - 1 \geqslant 0$, we have

$$\langle a^*|n\rangle = \langle 0|[e^{a^*\widehat{a}}, \frac{1}{\sqrt{n}}\widehat{a}^\dagger]\frac{1}{\sqrt{(n-1)!}}(\widehat{a}^\dagger)^{n-1}|0\rangle = \frac{1}{\sqrt{n}}a^*\langle a^*|n-1\rangle. \tag{2.112}$$

One then concludes by using the induction hypotheses.

In the same way, one shows that

$$\langle n|a\rangle = \frac{1}{\sqrt{n!}}a^n. \tag{2.113}$$

For any state $|\psi\rangle$ of the Hilbert space, we have

$$\langle a^*|\psi\rangle = \psi(a^*), \ \langle\psi|a\rangle = \bar{\psi}(a), \tag{2.114}$$

with $\psi(a^*)$ a series in a^* with complex coefficients and $\bar{\psi}(a)$ the series in a whose coefficients are complex conjugates of those of $\psi(a^*)$.

Let us show that

$$\langle\psi|\phi\rangle = \int \frac{da^*da}{2\pi i} e^{-a^*a}\langle\psi|a\rangle\langle a^*|\phi\rangle = \int \frac{da^*da}{2\pi i} e^{-a^*a}\bar{\psi}(a)\phi(a^*), \tag{2.115}$$

$$\boxed{\widehat{1} = \int \frac{da^*da}{2\pi i} e^{-a^*a}|a\rangle\langle a^*|}. \tag{2.116}$$

where the integral (by definition) is a real double integral defined by inverting (2.105) and its complex conjugate relation.

It is enough to show this relation for basis elements.

$$\begin{aligned}\langle n|m\rangle &= \int \frac{da^*da}{2\pi i} e^{-a^*a}\frac{1}{\sqrt{n!}}a^n\frac{1}{\sqrt{m!}}(a^*)^m \\ &= \int \frac{dqdp}{2\pi\hbar} e^{-\frac{1}{2\hbar}(\omega q^2+\omega^{-1}p^2)}\frac{1}{\sqrt{n!}}(\frac{\sqrt{\omega}q+i\sqrt{\omega}^{-1}p}{\sqrt{2\hbar}})^n\frac{1}{\sqrt{m!}}(\frac{\sqrt{\omega}q-i\sqrt{\omega}^{-1}p}{\sqrt{2\hbar}})^m, \end{aligned} \tag{2.117}$$

by using that the Jacobian of the change of variables $a = \frac{\sqrt{\omega}q+i\sqrt{\omega}^{-1}p}{\sqrt{2\hbar}}$, $a^* = \frac{\sqrt{\omega}q-i\sqrt{\omega}^{-1}p}{\sqrt{2\hbar}}$ is $\frac{i}{\hbar}$. Defining

$$I(\alpha,\alpha^*) = \int \frac{dqdp}{2\pi\hbar} e^{-\frac{1}{2\hbar}(\omega q^2+\omega^{-1}p^2)+\alpha^*\frac{\sqrt{\omega}q+i\sqrt{\omega}^{-1}p}{\sqrt{2\hbar}}+\alpha\frac{\sqrt{\omega}q-i\sqrt{\omega}^{-1}p}{\sqrt{2\hbar}}} \tag{2.118}$$

$$= \int \frac{dqdp}{2\pi\hbar} e^{-\frac{1}{2\hbar}(\omega q^2+\omega^{-1}p^2)+\frac{2\alpha_R}{\sqrt{2\hbar}}\sqrt{\omega}q+\frac{2\alpha_I}{\sqrt{2\hbar}}\sqrt{\omega}^{-1}p} \tag{2.119}$$

$$= e^{\alpha_R^2+\alpha_I^2} = e^{\alpha^*\alpha}, \tag{2.120}$$

by using exercice 2.2.6.1. This gives

$$\langle n|m\rangle = \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}(\frac{\partial}{\partial\alpha})^m(\frac{\partial}{\partial\alpha^*})^n I(\alpha,\alpha^*)|_{\alpha=0=\alpha^*} \tag{2.121}$$

$$= \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}(\frac{\partial}{\partial\alpha})^m(\alpha^n e^{\alpha^*\alpha})|_{\alpha=0=\alpha^*} = \begin{cases} 0 \text{ if } n\neq m \\ 1 \text{ if } n=m \end{cases} \tag{2.122}$$

NB: The considerations of this section could have been streamlined by using $\omega = 1 = \hbar$ since the end result does not depend on either of these parameters. We have have chosen to drag them along so that (2.105) is the change of variables appropriate to the harmonic oscillator.

For later use, note that we have also shown

$$I(\alpha,\alpha^*) = \int \frac{da^*da}{2\pi i} e^{-a^*a+\alpha^*a+\alpha a^*} = e^{\alpha^*\alpha}, \tag{2.123}$$

which is the value at the extremum, $a^* = \alpha^*, a = \alpha$, without any prefactor.

2.5.2 Kernel and normal symbol

Matrix elements of the operator $\widehat{O}$ in the orthornormal basis (2.110) are given by

$$O_{nm} = \langle n|\widehat{O}|m\rangle,\ \widehat{O} = \sum_{n,m}|n\rangle O_{nm}\langle m|. \tag{2.124}$$

The kernel of $\widehat{O}$ in the holomorphic representation is then given by

$$O(a^*,a) = \langle a^*|\widehat{O}|a\rangle = \sum_{n,m}\langle a^*|n\rangle O_{nm}\langle m|a\rangle = \sum_{n,m} O_{nm}\frac{(a^*)^n}{\sqrt{n!}}\frac{a^m}{\sqrt{m!}} \tag{2.125}$$

Using (2.115) yields

$$
\begin{aligned}
(\widehat{O}|\psi\rangle)(a^*) &\equiv \langle a^*|\widehat{O}|\psi\rangle = \int \frac{d\alpha^* d\alpha}{2\pi i} O(a^*,\alpha)\psi(\alpha^*)e^{-\alpha^*\alpha}, && (2.126)\\
(\widehat{O}_1\widehat{O}_2)(a^*,a) &\equiv \langle a^*|\widehat{O}_1\widehat{O}_2|a\rangle = \int \frac{d\alpha^* d\alpha}{2\pi i} O_1(a^*,\alpha)O_2(\alpha^*,a)e^{-\alpha^*\alpha}. && (2.127)
\end{aligned}
$$

Furthermore $\widehat{a}^\dagger$ and $\widehat{a}$ act respectively like multiplication and differentiation by a^*,

$$
\begin{aligned}
(\widehat{a}^\dagger|\psi\rangle)(a^*) &\equiv \langle a^*|\widehat{a}^\dagger|\psi\rangle = a^*\psi(a^*), && (2.128)\\
(\widehat{a}|\psi\rangle)(a^*) &\equiv \langle a^*|\widehat{a}|\psi\rangle = \frac{\partial}{\partial a^*}\psi(a^*). && (2.129)
\end{aligned}
$$

If $\widehat{O}(\widehat{a},\widehat{a}^\dagger)$ is an operator, its normal symbol $O^N(a^*,a)$ is by definition the function of a, a^* that one obtains by arranging in $\widehat{O}$ the $\widehat{a}^\dagger$ to the left and the $\widehat{a}$ to the right by using the commutation rules, and then replacing the operators $\widehat{a}^\dagger, \widehat{a}$ by the complex numbers a^*, a. Conversely, in $O^N(a^*,a)$ one may rearrange the a^* to the left and the a to the right without changing the expressions since we are dealing with c-numbers. If one then replaces the complex numbers a^*, a by operators $\widehat{a}^\dagger, \widehat{a}$, one obtains an operator in normal form that is equal to the starting point operator $\widehat{O}(\widehat{a},\widehat{a}^\dagger)$ after using the commutation relations.

This is not to be confused with the normal ordered operator $:\widehat{O}(\widehat{a},\widehat{a}^\dagger):$ which is the operator that one gets when rearranging in $\widehat{O}$ the $\widehat{a}^\dagger$ to the left and the $\widehat{a}$ to the right without using the commutation relations. In particular, the projector on the vacuum may be represented as

$$
|0\rangle\langle 0| =: e^{-\widehat{a}^\dagger\widehat{a}} : . \tag{2.130}
$$

It is enough to show that $: e^{-\widehat{a}^\dagger\widehat{a}} : |n\rangle = \delta_n^0|0\rangle$. This relation holds for $n=0$. For $n\neq 0$, it also holds if one shows that

$$
[: e^{-\widehat{a}^\dagger\widehat{a}} :, (\widehat{a}^\dagger)^n] = -(\widehat{a}^\dagger)^n : e^{-\widehat{a}^\dagger\widehat{a}} :, \tag{2.131}
$$

since then $e^{-\widehat{a}^\dagger\widehat{a}} : |n\rangle = [e^{-\widehat{a}^\dagger\widehat{a}} :, \frac{(\widehat{a}^\dagger)^n}{\sqrt{n!}}]|0\rangle + \frac{(\widehat{a}^\dagger)^n}{\sqrt{n!}} : e^{-\widehat{a}^\dagger\widehat{a}} : |0\rangle = 0$.
For $n=1$, $[\sum_{n=0}(-)^n\frac{(\widehat{a}^\dagger)^n\widehat{a}^n}{n!}, \widehat{a}^\dagger] = \sum_{n=1}(-)^n\frac{(\widehat{a}^\dagger)^n\widehat{a}^{n-1}}{(n-1)!} = -\widehat{a}^\dagger : e^{-\widehat{a}^\dagger\widehat{a}} :$. If (2.131) holds for $n-1\geqslant 0$, it follows that $[: e^{-\widehat{a}^\dagger\widehat{a}} :, (\widehat{a}^\dagger)^n] = \widehat{a}^\dagger[: e^{-\widehat{a}^\dagger\widehat{a}} :, (\widehat{a}^\dagger)^{n-1}] + [: e^{-\widehat{a}^\dagger\widehat{a}} :, \widehat{a}^\dagger](\widehat{a}^\dagger)^{n-1} = -(\widehat{a}^\dagger)^n : e^{-\widehat{a}^\dagger\widehat{a}} : -\widehat{a}^\dagger : e^{-\widehat{a}^\dagger\widehat{a}} : (\widehat{a}^\dagger)^{n-1} = -(\widehat{a}^\dagger)^n : e^{-\widehat{a}^\dagger\widehat{a}} : -\widehat{a}^\dagger[: e^{-\widehat{a}^\dagger\widehat{a}} :, (\widehat{a}^\dagger)^{n-1}] - (\widehat{a}^\dagger)^n : e^{-\widehat{a}^\dagger\widehat{a}} := -(\widehat{a}^\dagger)^n : e^{-\widehat{a}^\dagger\widehat{a}} :$.

The kernel and the normal symbol of an operator are related by:

$$
O(a^*,a) = e^{a^*a}O^N(a^*,a). \tag{2.132}
$$

Indeed, $\widehat{O} = \sum_{n,m} O_{nm}\frac{(\widehat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle\langle 0|\frac{\widehat{a}^m}{\sqrt{m!}} = \sum_{n,m} O_{nm}\frac{(\widehat{a}^\dagger)^n}{\sqrt{n!}} : e^{-\widehat{a}^\dagger\widehat{a}} : \frac{\widehat{a}^m}{\sqrt{m!}} = \sum_{n,m} O_{nm} : \frac{(\widehat{a}^\dagger)^n}{\sqrt{n!}} e^{-\widehat{a}^\dagger\widehat{a}} \frac{\widehat{a}^m}{\sqrt{m!}} :$
We thus have $O^N(a^*,a) = \sum_{n,m} O_{nm}\frac{(a^*)^n}{\sqrt{n!}}e^{-a^*a}\frac{a^m}{\sqrt{m!}} = O(a^*,a)e^{-a^*a}$ on account of (2.125).
For an alternative proof of (2.132), one first notes that it is enough to show this for operators $\widehat{O}_{n,m} = (\widehat{a}^\dagger)^n a^m$ for which $O^N_{n,m} = (a^*)^n a^m$. To compute the kernel, one uses that $(\widehat{O}_{n,m}|\psi\rangle)(a^*) = (a^*)^n(\frac{\partial}{\partial a^*})^m\psi(a^*) = (a^*)^n(\frac{\partial}{\partial a^*})^m\int\frac{d\alpha^*d\alpha}{2\pi i}e^{a^*\alpha-\alpha^*\alpha}\psi(\alpha^*) = \int\frac{d\alpha^*d\alpha}{2\pi i}(a^*)^n\alpha^m e^{a^*\alpha-\alpha^*\alpha}\psi(\alpha^*)$. Since $(\widehat{O}_{n,m}|\psi\rangle)(a^*) = \int\frac{d\alpha^*d\alpha}{2\pi i}O_{n,m}(a^*,\alpha)e^{-\alpha^*\alpha}\psi(\alpha^*)$, it follows that $O_{n,m}(a^*,\alpha) = (a^*)^n\alpha^m e^{a^*\alpha} = O^N_{n,m}(a^*,\alpha)e^{a^*\alpha}$.

In holomorphic representation, the trace is given by

$$
\mathrm{Tr}\,\widehat{O} = \sum_n\langle n|\widehat{O}|n\rangle = \int\frac{da^*da}{2\pi i}\sum_n\langle n|a\rangle\langle a^*|\widehat{O}|n\rangle e^{-a^*a} = \int\frac{da^*da}{2\pi i}O(a^*,a)e^{-a^*a}. \tag{2.133}
$$

2.5.3 Evolution operator

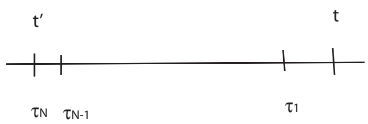
In this section, we start by using $\hbar = 1$. We want to compute the kernel of the evolution operator in holomorphic representation,

$$\langle a^*; t'|a; t\rangle = \langle a^*|e^{-i\widehat{H}(t'-t)}|a\rangle \equiv U(a^*, t'; a, t). \tag{2.134}$$

For $t' - t = \epsilon$ infinitesimal,

$$\langle a^*|1 - i\epsilon\widehat{H}|a\rangle = e^{a^*a} - i\epsilon\mathcal{H}(a^*, a) = e^{a^*a}[1 - i\epsilon h(a^*, a)] = e^{a^*a - i\epsilon h(a^*,a)}, \tag{2.135}$$

where $\mathcal{H}(a^*, a)$ is the kernel of the Hamiltonian $\widehat{H}$ while $h(a^*, a)$ is its normal symbol. In this case, a finite interval is decomposed in N pieces:



Inserting

$$\int \frac{da^*_{N-1}da_{N-1}}{2\pi i} e^{-a^*_{N-1}a_{N-1}}\ |a_{N-1}; \tau_{N-1}\rangle\langle a^*_{N-1}; \tau_{N-1}|\ \ldots\ \int \frac{da^*_1 da_1}{2\pi i} e^{-a^*_1 a_1}|a_1; \tau_1\rangle\langle a^*_1; \tau_1|, \tag{2.136}$$

at the appropriate places, one finds

$$U(a^*, t'; a, t) = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \frac{da^*_k da_k}{2\pi i} \quad e^{S'_D}, \tag{2.137}$$

where

$$S'_D = -i\epsilon[h(a^*_N, a_{N-1}) + \cdots + h(a^*_1, a_0)] + a^*_N a_{N-1} + \cdots + a^*_1 a_0 - a^*_{N-1}a_{N-1} - \cdots - a^*_1 a_1. \tag{2.138}$$

In the limit $N \to \infty$, this gives

$$\boxed{U(a^*, t'; a, t) = \int_{a(t)=a}^{a^*(t')=a^*} \prod_{\tau} \frac{da^*(\tau)da(\tau)}{2\pi i} \quad e^{iS'_H[a^*,a]}}, \tag{2.139}$$

$$\boxed{S'_H[a^*, a] = \int_t^{t'} d\tau \Big[\frac{1}{2i}(\dot{a}^* a - a^*\dot{a}) - h(a^*, a)\Big] + \frac{1}{2i}\Big(a^*(t')a(t') + a^*(t)a(t)\Big)}. \tag{2.140}$$

Indeed,when discretizing in the argument of the exponential all the terms besides the Hamiltonian, one finds

$$\frac{1}{2}\big[(a^*_N - a^*_{N-1})a_{N-1} + \cdots + (a^*_1 - a^*_0)a_0 - a^*_N(a_N - a_{N-1}) - \cdots - a^*_1(a_1 - a_0) + a^*_N a_N + a^*_0 a_0\big]$$
$$= a^*_N a^*_{N-1} + \cdots + a^*_1 a_0 - a^*_{N-1}a_{N-1} - \ldots a_1 a_1. \quad (2.141)$$

N.B.: The action to be used in the path integral is the one that has a true extremum when taking into account the boundary conditions $\delta a^*(t') = 0 = \delta a(t)$, and when the field equations hold:

$$\begin{aligned}
\delta S'_H &= \int_t^{t'} d\tau\Big(\delta a^*(-\frac{1}{i}\dot{a} - \frac{\partial h}{\partial a^*}) + \delta a(\frac{1}{i}\dot{a}^* - \frac{\partial h}{\partial a})\Big) \\
&\quad + \frac{1}{2i}[\delta a^* a]_t^{t'} - \frac{1}{2i}[a^*\delta a]_t^{t'} + \frac{1}{2i}[\delta a^*(t')a(t') + a^*(t')\delta a(t') + \delta a^*(t)a(t) + a^*(t)\delta a(t)] \\
&= \int_t^{t'} d\tau\Big(\delta a^*(-\frac{1}{i}\dot{a} - \frac{\partial h}{\partial a^*}) + \delta a(\frac{1}{i}\dot{a}^* - \frac{\partial h}{\partial a})\Big).
\end{aligned}$$

For the forced harmonic oscillator, the quantum Hamiltonian is

$$\widehat{H} = \omega(\widehat{a}^\dagger\widehat{a} + \frac{\alpha}{2}) - j(t)\widehat{a}^\dagger - j^*(t)\widehat{a}, \quad (2.142)$$

$$h(a^*, a) = \omega(a^* a + \frac{\alpha}{2}) - j(t)a^* - j^*(t)a. \quad (2.143)$$

NB: $\alpha = 1$ corresponds to using a symmetrical ordering during quantization, while $\alpha = 0$ corresponds to using the normal ordering.

To compute the path integral, since all discretized integrals are Gaussian, we admit the shortcut that the result will be given by the value of the exponential at the extremum, without prefactor. We have

$$\dot{a} = -i\frac{\partial h}{\partial a^*} = -i(\omega a - j),\ a(t) = a, \quad (2.144)$$

$$\dot{a}^* = i\frac{\partial h}{\partial a} = i(\omega a^* - j^*),\ a^*(t') = a^*. \quad (2.145)$$

N.B.: At the boundary, one gives a and a^* independently. The solution to these equations is

$$a(\tau) = e^{-i\omega(\tau - t)}a + i\int_t^\tau d\tau' j(\tau')e^{-i\omega(\tau-\tau')}, \quad (2.146)$$

$$a^*(\tau) = e^{-i\omega(t'-\tau)}a^* + i\int_\tau^{t'} d\tau' j^*(\tau')e^{-i\omega(\tau'-\tau)}. \quad (2.147)$$

Substituting this solution in the classical action gives

$$\boxed{\ln U(a^*, t'; a, t) = -i(t' - t)\frac{\alpha\omega}{2} + a^* e^{-i\omega(t'-t)}a + ia^*\int_t^{t'} d\tau j(\tau)e^{-i\omega(t'-\tau)} +}$$
$$+\boxed{ia\int_t^{t'} d\tau j^*(\tau)e^{-i\omega(\tau-t)} - \int_t^{t'} d\tau\int_t^{t'} d\tau' j^*(\tau)e^{-i\omega(\tau-\tau')}\theta(\tau-\tau')j(\tau')}. \quad (2.148)$$

Details of this lengthy computation follow. They show in particular that:

In the absence of sources and for the normal ordered theory $\alpha = 0$, the contributions from the first terms with time derivatives and the Hamiltonian cancel so that the whole result comes from the correct boundary term.

Factors of $\hbar$ can be restored as follows. In the first term of the Hamiltonian, there should be an $\hbar$. This amounts to changing ω by $\hbar\omega$. Furthermore, at the start of the computation, the Hamiltonian including the sources in the evolution operator in (2.134) should be divided by $\hbar$. None of the kinetic terms coming from holomorphic insertions of the completeness relation and the norm of coherent states carry an $\hbar$. The result of these two manipulations in the determination of the extremum in (2.144) amounts to dividing j, j^* by $\hbar$. This is what one has to do in the final result,

$$\ln U(a^*,t';a,t) = -i(t'-t)\frac{\alpha\omega}{2} + a^* e^{-i\omega(t'-t)}a + ia^* \int_t^{t'} d\tau \frac{j(\tau)}{\hbar} e^{-i\omega(t'-\tau)} + \\ + ia \int_t^{t'} d\tau \frac{j^*(\tau)}{\hbar} e^{-i\omega(\tau-t)} - \int_t^{t'} d\tau \int_t^{t'} d\tau' \frac{j^*(\tau)}{\hbar} e^{-i\omega(\tau-\tau')}\theta(\tau-\tau')\frac{j(\tau')}{\hbar}. \quad (2.149)$$

Forced harmonic oscillator : on-shell action

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$$\hat{H} = \omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) - j(t)\,\hat{a}^+ - j^*(t)\,\hat{a}$$

$$U(a^*, t'; a, t) = e^{i S_H|_{extr}[a^*, a]}$$

$$S_H|_{extr} = \int_t^{t'} d\tau \Big\{ \frac{1}{2i}\Big[i\Big(\omega\Big[e^{-i\omega(t'-\tau)}a^* + i\int_\tau^{t'} d\tau'\, j^*(\tau')\, e^{-i\omega(\tau'-\tau)}\Big] - j^*\Big)$$

$$\Big(e^{-i\omega(\tau-t)}a + i\int_t^{\tau} d\tau'\, j(\tau')\, e^{-i\omega(\tau-\tau')}\Big)$$

$$-\Big(e^{-i\omega(t'-\tau)}a^* + i\int_\tau^{t'} d\tau'\, j^*(\tau')\, e^{-i\omega(\tau'-\tau)}\Big)$$

$$(-i)\Big(\omega\Big[e^{-i\omega(\tau-t)}a + i\int_t^{\tau} d\tau'\, j(\tau')\, e^{-i\omega(\tau-\tau')}\Big] - j\Big)\Big]$$

$$-\omega\Big(\Big[e^{-i\omega(t'-\tau)}a^* + i\int_\tau^{t'} d\tau'\, j^*(\tau')\, e^{-i\omega(\tau'-\tau)}\Big]$$

$$\Big[e^{-i\omega(\tau-t)}a + i\int_t^{\tau} d\tau'\, j(\tau')\, e^{-i\omega(\tau-\tau')}\Big] + \frac{1}{2}\Big)$$

$$+ j(\tau)\Big(e^{-i\omega(t'-\tau)}a^* + i\int_\tau^{t'} d\tau'\, j^*(\tau')\, e^{-i\omega(\tau'-\tau)}\Big)$$

$$+ j^*(\tau)\Big[e^{-i\omega(\tau-t)}a + i\int_t^{\tau} d\tau'\, j(\tau')\, e^{-i\omega(\tau-\tau')}\Big]\Big\}$$

$$+\frac{1}{2i}\Big[a^*\Big(e^{-i\omega(t'-t)}a + i\int_t^{t'} d\tau'\, j(\tau')\, e^{-i\omega(t'-\tau')}\Big)$$

$$+\Big(e^{-i\omega(t'-t)}a^* + i\int_t^{t'} d\tau'\, j^*(\tau')\, e^{-i\omega(\tau'-t)}\Big)a\Big]$$

$$= -\frac{\alpha\omega}{2}(t'-t) \quad + \int_t^{t'} d\zeta \left\{ \frac{1}{2i}\left[2i\,\omega\, a^\dagger a\, \cancel{e^{-i\omega(t'-t)}}\right] - \omega\, a^\dagger a\, \cancel{e^{-i\omega(t'-t)}} \right\} + \boxed{\frac{1}{i}\, a^\dagger e^{-i\omega(t'-t)}\, a}$$

$$+ \int_t^{t'} d\zeta \left\{ \frac{1}{2}\left(i\omega \int_\zeta^{t'} j^*(\zeta')\, \cancel{e^{-i\omega(\zeta'-t)}}\, a - j^*(\zeta)\, \cancel{e^{-i\omega(\zeta-t)}}\, a \right.\right.$$

$$+ i\omega \int_\zeta^{t'} d\zeta'\, \cancel{j^*(\zeta')}\, e^{-i\omega(\zeta'-t)}\, a \Big)$$

$$- i\omega \int_\zeta^{t'} d\zeta'\, \cancel{j^*(\zeta')}\, e^{-i\omega(\zeta'-t)}\, a$$

$$\boxed{+ a\, j^*(\zeta)\, e^{-i\omega(\zeta-t)}} \quad \Big\} \qquad + \frac{1}{2\cancel{i}}\cancel{i} \int_t^{t'} d\zeta'\, j^*(\zeta')\, \cancel{e^{-i\omega(\zeta'-t)}}\, a$$

$$+ \int_t^{t'} d\zeta \Big\{ \frac{1}{2}\Big[\omega i\, a^\dagger \int_t^{\zeta} d\zeta'\, \cancel{j(\zeta')}\, e^{-i\omega(t'-\zeta')}$$

$$+ \omega i\, a^\dagger \int_t^{\zeta} \cancel{d\zeta'}\, j(\zeta')\, e^{-i\omega(t'-\zeta')} - a^\dagger j(\zeta)\, \cancel{e^{-i\omega(t'-\zeta)}} \Big]$$

$$- \omega i\, a^\dagger \int_t^{\zeta} d\zeta'\, \cancel{j(\zeta')}\, e^{-i\omega(t'-\zeta')}$$

$$\boxed{+ a^\dagger\, j(\zeta)\, e^{-i\omega(t'-\zeta)}} \quad \Big\} \qquad + \frac{1}{2\cancel{i}}\cancel{i} \int_t^{t'} d\zeta'\, j(\zeta')\, \cancel{e^{-i\omega(t'-\zeta')}}\, a^\dagger$$

$$+ \int_t^{t'} d\zeta\; \frac{1}{2}\left(-j^*(\zeta)\, i \int_t^{\zeta} d\zeta'\, \cancel{j(\zeta')}\, e^{-i\omega(\zeta-\zeta')} - j(\zeta)\, i \int_\zeta^{t'} d\zeta'\, \cancel{j^*(\zeta')}\, e^{-i\omega(\zeta'-\zeta)} \right)$$

$$+ \frac{i}{2} \int_t^{t'} d\zeta\; j(\zeta) \int_\zeta^{t'} d\zeta'\, j^*(\zeta')\, e^{-i\omega(\zeta'-\zeta)} + \frac{i}{2} \int_t^{t'} d\zeta\; j^*(\zeta) \int_t^{\zeta} d\zeta'\, j(\zeta')\, e^{-i\omega(\zeta-\zeta')}$$

$$\stackrel{?}{=} \boxed{i \int_t^{t'} d\tau \int_t^{t'} d\tau' \, j^*(\tau)\, e^{-i\omega(\tau-\tau')}\, \theta(\tau-\tau')\, j(\tau')}$$

$$\Big(= \frac{i}{2} \int_t^{t'} d\tau \int_t^{t'} d\tau' \left[j^*(\tau)\, e^{-i\omega(\tau-\tau')}\, \theta(\tau-\tau')\, j(\tau') + j^*(\tau')\, e^{-i\omega(\tau'-\tau)}\, \theta(\tau'-\tau)\, j(\tau) \right]$$

$$= \frac{i}{2} \int_t^{t'} d\tau' \int_{\tau'}^{t'} d\tau \quad j^*(\tau)\, e^{-i\omega(\tau-\tau')}\, j(\tau') + \frac{i}{2} \int_t^{t'} d\tau \int_\tau^{t'} d\tau' \, j^*(\tau')\, e^{-i\omega(\tau'-\tau)}\, j(\tau) \Big)$$

2.6 Reduction formulas

2.6.1 S-matrix of forced harmonic oscillator

The $\widehat{S}$ operator is given by

$$\widehat{S} = \lim_{\substack{t' \to +\infty \\ t \to -\infty}} \widehat{S}(t', t), \qquad \widehat{S}(t', t) = e^{i\widehat{H}_0 t'}\, \widehat{U}(t', t)\, e^{-i\widehat{H}_0 t}. \tag{2.150}$$

The kernel of $\widehat{S}(t', t)$ in holomorphic representation is

$$S(a^*, t'; a, t) = \int \frac{d\alpha^* d\alpha}{2\pi i} \frac{d\beta^* d\beta}{2\pi i} e^{-\alpha^*\alpha - \beta^*\beta} \langle a^* | e^{i\widehat{H}_0 t'} | \alpha \rangle U(\alpha^*, t'; \beta, t) \langle \beta^* | e^{-i\widehat{H}_0 t} | a \rangle. \tag{2.151}$$

Since $\langle a^* | e^{i\widehat{H}_0 t'} | \alpha \rangle$ is a particular case of (2.148) with $j = 0$ and $t' - t \to -t'$, and similarily for $\langle \beta^* | e^{-i\widehat{H}_0 t} | a \rangle$ with $t' - t \to t$, one finds

$$S(a^*, t'; a, t) = \int \frac{d\alpha^* d\alpha}{2\pi i} \frac{d\beta^* d\beta}{2\pi i} \quad e^A \tag{2.152}$$

with

$$\begin{aligned} A = -\alpha^*\alpha + a^* e^{i\omega t'}\alpha + \alpha^* e^{-i\omega(t'-t)}\beta + i\alpha^* \int_t^{t'} d\tau e^{-i\omega(t'-\tau)} j(\tau) + i \int_t^{t'} d\tau j^*(\tau) e^{-i\omega(\tau - t)}\beta \\ - \int_t^{t'} d\tau \int_t^{t'} d\tau' j^*(\tau)\theta(\tau - \tau') e^{-i\omega(\tau' - \tau)} j(\tau') - \beta^*\beta + \beta^* e^{-i\omega t} a. \end{aligned} \tag{2.153}$$

If $u^* = a^* e^{i\omega t'}$ and $u = e^{-i\omega(t'-t)}\beta + i\int_t^{t'} d\tau e^{-i\omega(t'-\tau)} j(\tau)$, the integral over $\alpha^*\alpha$ transforms the first 4 terms of the exponential into $u^* u$. Hence, the argument of the exponential becomes

$$\begin{aligned} a^* e^{i\omega t}\beta + ia^* \int_t^{t'} d\tau e^{i\omega\tau} j(\tau) + i \int_t^{t'} d\tau j^*(\tau) e^{-i\omega(\tau - t)}\beta \\ - \int_t^{t'} d\tau \int_t^{t'} d\tau' j^*(\tau)\theta(\tau - \tau') e^{-i\omega(\tau' - \tau)} j(\tau') - \beta^*\beta + \beta^* e^{-i\omega t} a. \end{aligned} \tag{2.154}$$

If $v = e^{-i\omega t} a$ et $v^* = a^* e^{i\omega t} + i\int_t^{t'} d\tau j^*(\tau) e^{-i\omega(\tau - t)}$, the integral over $\beta^*\beta$ gives $e^{v^* v}$ and one finds

$$\begin{aligned} \ln S(a^*, t'; a, t) = a^* a + ia^* \int_t^{t'} d\tau e^{i\omega\tau} j(\tau) + ia \int_t^{t'} d\tau j^*(\tau) e^{-i\omega\tau} \\ - \int_t^{t'} d\tau \int_t^{t'} d\tau' j^*(\tau) e^{-i\omega(\tau - \tau')}\theta(\tau - \tau') j(\tau')]. \end{aligned} \tag{2.155}$$

It follows that the normal symbol is

$$\begin{aligned} \ln S^N(a^*, t'; a, t) = ia^* \int_t^{t'} d\tau e^{i\omega\tau} j(\tau) + ia \int_t^{t'} d\tau j^*(\tau) e^{(-i\omega\tau)} \\ - \int_t^{t'} d\tau \int_t^{t'} d\tau' j^*(\tau) e^{-i\omega(\tau - \tau')}\theta(\tau - \tau') j(\tau'). \end{aligned} \tag{2.156}$$

2.6.2 S-matrix of interacting scalar field

In Fourier transform, the scalar field with real external source $j(x)$ is a superposition of decoupled harmonic oscillators:

$$\begin{aligned}
\widehat{H}_0^j &= \int d^3x\, [\frac{1}{2}\widehat{\pi}^2 + \frac{1}{2}\partial_k\widehat{\phi}\partial^k\widehat{\phi} + \frac{1}{2}m^2\widehat{\phi}^2 - j\widehat{\phi}], && (2.157)\\
j(\vec{x}) &= \frac{1}{(2\pi)^{3/2}}\int d^3k\,\sqrt{2\omega(\vec{k})}e^{i\vec{k}\cdot\vec{x}}\widetilde{j}(\vec{k}),\ \widetilde{j}^*(\vec{k}) = \widetilde{j}(-\vec{k}), && (2.158)\\
\widehat{\phi}(\vec{x}) &= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega(\vec{k})}}\,[\widehat{a}(\vec{k})e^{i\vec{k}\cdot\vec{x}} + \text{h.c.}], && (2.159)\\
\widehat{H}_0^j &= \int d^3k\,[\omega(\vec{k})\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}) - \widetilde{j}(\vec{k})\widehat{a}^\dagger(\vec{k}) - \widetilde{j}^*(\vec{k})\widehat{a}(\vec{k})]. && (2.160)
\end{aligned}$$

One thus finds for the normal symbol

$$\begin{aligned}
\ln S_0^N(a^*, +\infty; a, -\infty) = i\int_{-\infty}^{+\infty} d\tau \int d^3k\,\big[a^*(\vec{k})e^{i\omega(\vec{k})\tau}\widetilde{j}(\tau,\vec{k}) + \widetilde{j}^*(\tau,\vec{k})e^{-i\omega(\vec{k})\tau}a(\vec{k})\big]\\
-\int_{-\infty}^{+\infty} d\tau\int_{-\infty}^{+\infty} d\tau'\int d^3k\,\widetilde{j}^*(\tau,\vec{k})e^{(-i\omega(\vec{k})(\tau-\tau'))}\theta(\tau-\tau')\widetilde{j}(\tau',\vec{k}). \quad (2.161)
\end{aligned}$$

Defining

$$\phi_{as}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega(\vec{k})}}\,[a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}], \quad (2.162)$$

with $a(\vec{k})$ the initial condition in $t \to -\infty$ and $a^*(\vec{k})$ the final condition in $t \to +\infty$, the first 2 terms of the exponential combine into

$$i\int d^4x\,\phi_{as}(x)j(x). \quad (2.163)$$

The last term can be written as

$$\begin{aligned}
-\int d^4x\int d^4x'\int d^3k\frac{1}{(2\pi)^3 2\omega(\vec{k})}\,e^{-i\omega(\vec{k})(t-t')}\theta(t-t')e^{i\vec{k}\cdot\vec{x}}j(x)e^{-i\vec{k}\cdot\vec{x}'}j(x')\\
= -\frac{1}{2}\int d^4x\int d^4x'\,j(x)j(x')\int d^3k\frac{1}{(2\pi)^3 2\omega(\vec{k})}[e^{ik(x-x')}\theta(t-t') + e^{ik(x'-x)}\theta(t'-t)] \quad (2.164)
\end{aligned}$$

When taking (1.29) and (1.31) into account, one gets

$$\ln S_0^N(a^*, +\infty; a, -\infty) = i\int d^4x\,\phi_{as}(x)j(x) + \frac{i}{2}\int d^4x\int d^4x'\,j(x)\Delta_F(x-x')j(x'), \quad (2.165)$$

or, equivalently,

$$\boxed{S_0^N(a^*, +\infty; a, -\infty) = e^{i\int d^4x\,\phi_{as}(x)j(x)}\frac{Z_0[j]}{Z_0[0]}}. \quad (2.166)$$

Note that

$$\frac{\delta}{\delta j(y)}S_0^N = \big(i\phi_{as}(y) + i\int d^4x\Delta_F(y,x)j(x)\big)S_0^N. \quad (2.167)$$

For the operator $\widehat{S}_0$, one replaces ϕ_{as} by $\widehat{\phi}$ and takes normal order:

$$\widehat{S}_0 =: e^{i\int d^4x\, \widehat{\phi}(x)j(x)} : \frac{Z_0[j]}{Z_0[0]}. \tag{2.168}$$

Again, if there is an interaction $I_1[\phi]$, it is treated perturbatively in terms of the expression we have just computed:

$$U(a^*, t'; a, t) = e^{iI_1[\frac{1}{i}\frac{\delta}{\delta j}]} \int_{a(t)=a}^{a^*(t')=a^*} \prod_{\tau,\vec{k}} \frac{da(\vec{k},\tau)da(\vec{k},\tau)}{2\pi i}\, e^{iS'{}_{H_0}^{j}}, \tag{2.169}$$

$$\boxed{S^N(a^*, +\infty; a, -\infty) = e^{iI_1[\frac{1}{i}\frac{\delta}{\delta j}]} S_0^N}, \quad \widehat{S} = e^{iI_1[\frac{1}{i}\frac{\delta}{\delta j}]} : e^{i\int d^4x\, \widehat{\phi}(x)j(x)} : \frac{Z_0[j]}{Z_0[0]}. \tag{2.170}$$

Remarks:

(i) Equation (2.170) describes the generating functional for the S-matrix of an interacting theory, to be computed perturbatively in the interaction, in the presence of an external source. Usually one is only interested in S-matrix elements, computed to a given order, in the absence of a source. This is obtained from the above by putting the source to zero at the end of the computation.

(ii) As we have briefly discussed in (2.96), dividing by $Z[0]$ instead of $Z_0[0]$ removes the vacuum parts of the diagrams for Green's functions. In the same way, dividing $S^N(a^*, +\infty; a, -\infty)$ by $S^N(0, +\infty; 0, -\infty)$ removes the vacuum diagrams that contribute to various S-matrix elements.

(iii) One may consider a generalization of the generating functional for the S-matrix in the presence of the source, where the external states, contained in the Fourier coefficients of $\widetilde{\phi}(x)$ are off-the mass-shell,

$$\widetilde{S}^N(\widetilde{\phi}; j) = e^{iI_1[\frac{1}{i}\frac{\delta}{\delta j}]} e^{i\int d^4x \widetilde{\phi}(x)j(x)} \frac{Z_0[j]}{Z_0[0]}. \tag{2.171}$$

In particular, on the one hand, $Z[j] = \widetilde{S}^N(0; j)$: the generating functional for Green's function is obtained when keeping the source, but setting the field to zero; it is the generating functional for the vacuum to vacuum transition amplitudes (in the presence of the source). On the other hand, after having computed to a given order in perturbation theory, S-matrix elements (in the form of its normal symbol) are obtained by setting the source to zero and replacing $\widetilde{\phi}(x)$ by $\phi_{as}(x)$, $S^N = \widetilde{S}^N(\phi_{as}; 0)$.

2.6.3 S-matrix from Green's functions

We now want to show that it is enough to compute Green's functions. Indeed, there is a simple procedure that allows one to pass from Green's functions to S-matrix elements. It goes under the name of reduction formulas. We follow [17].

The normal symbol for the S-matrix is entirely determined by its Taylor coefficients,

$$S^N = \sum_n \frac{1}{n!} \int d^4x_1 \dots d^4x_n \phi_{as}(x_1) \dots \phi_{as}(x_n) S^N(x_1, \dots, x_n), \tag{2.172}$$

where

$$\begin{aligned} S^N(x_1, \dots, x_n) &= \frac{\delta}{i\delta\widetilde{\phi}(x_1)} \cdots \frac{\delta}{i\delta\widetilde{\phi}(x_n)} \widetilde{S}^N|_{j=0=\widetilde{\phi}} \\ &= [e^{iI_1[\frac{1}{i}\frac{\delta}{\delta j}]} j(x_1) \dots j(x_n) e^{i\int d^4x \widetilde{\phi}(x)j(x)} \frac{Z_0[j]}{Z_0[0]}]|_{j=0=\widetilde{\phi}}. \end{aligned} \tag{2.173}$$

But

$$(-\Box_{x_k} + m^2)\frac{\delta}{i\delta j(x_k)} Z[j] = e^{iI_1[\frac{1}{i}\frac{\delta}{\delta j}]} j(x_k)\frac{Z_0[j]}{Z_0[0]}, \tag{2.174}$$

since

$$(-\Box_{x_k} + m^2)\frac{1}{i}\frac{\delta}{\delta j(x_k)}\frac{Z_0[j]}{Z_0[0]} = (-\Box_{x_k} + m^2)\int d^4x'\, \Delta_F(x_k - x')j(x')\frac{Z_0[j]}{Z_0[0]} = j(x_k)\frac{Z_0[j]}{Z_0[0]}. \tag{2.175}$$

This then implies that

$$\int d^4x_1 \dots d^4x_n \phi_{as}(x_1)\dots\phi_{as}(x_n)\Big(S^N(x_1,\dots,x_n) - [(-\Box_{x_1} + m^2)\frac{\delta}{i\delta j(x_1)}\dots(-\Box_{x_n} + m^2)\frac{\delta}{i\delta j(x_n)}]Z[j]|_{j=0}\Big) = 0. \tag{2.176}$$

Indeed, in this expression, all additional terms that appear in the second term when subsequent derivatives with respect to j act on the explicit $j(x_k)$ in (2.174) vanish because the result is contracted with ϕ_{as}, which satisfies $(-\Box_x + m^2)\phi_{as}(x) = 0$. Note that in terms of Feynman rules, applying $(-\Box_{x_k} + m^2)\frac{\delta}{i\delta j(x_k)}$ to $Z[j]$ amputates the external propagator at x_k of the Green's function.

The reduction formulas can then be described as follows.

The normal symbol of the S-matrix is obtained by taking Green's function of order n, amputating the external propagators, multiplying instead by ϕ_{as} at the external points, dividing by $n!$ and summing over n.

This procedure can also be summarized as follows [9, 1]:

$$\boxed{S^N = e^{\int d^4x\, \phi_{as}(x)(-\Box+m^2)\frac{\delta}{\delta j(x)}}\frac{Z[j]}{Z[0]}}, \tag{2.177}$$

in the case vacuum diagrams have been removed, and with the understanding that $(-\Box + m^2)\dfrac{\delta}{\delta j(x)}$ only acts on external propagators.

We find for the kernel

$$S(a^*, +\infty; a, -\infty) = \langle a^*, +\infty|\widehat{S}|a, -\infty\rangle = e^{\int d^3k\, a^*(\vec{k})a(\vec{k})} e^{\int d^4x\, \phi_{as}(x)(-\Box+m^2)\frac{\delta}{\delta j(x)}}\frac{Z[j]}{Z[0]}. \tag{2.178}$$

For the free theory, $I_1[\phi] = 0 = j$, and the result reduces to

$$S_0(a^*, +\infty; a, -\infty) = e^{\int d^3k\, a^*(\vec{k})a(\vec{k})}, \tag{2.179}$$

Since

$$\langle a^*, +\infty| = \langle +\infty; 0|e^{\int d^3k\, \widehat{a}(\vec{k},+\infty)a^*(\vec{k})} \tag{2.180}$$

$$|a, -\infty\rangle = e^{\int d^3k\, a(\vec{k})\widehat{a}^*(\vec{k},-\infty)}|0; -\infty\rangle \tag{2.181}$$

and $\widehat{a}(\vec{k}, t) = \widehat{a}(\vec{k})e^{-i\omega(\vec{k})t}$, $|0, t\rangle = e^{i\omega(\vec{k})t}|0\rangle$ for the free theory, one may check this result by evaluating $\langle a^*, +\infty|a, -\infty\rangle$ on the one hand and $e^{\int d^3k\, a^*(\vec{k})a(\vec{k})}$ on the other, for instance,

$$\langle 0|0\rangle = 1, \tag{2.182}$$

$$\langle \vec{q}|\vec{p}\rangle = \frac{\delta}{\delta a^*(\vec{q})}\frac{\delta}{\delta a(\vec{p})} S_0(a^*, +\infty; a, -\infty)|_{a^*=0=a} = \delta^3(\vec{q} - \vec{p}). \tag{2.183}$$

Decomposing $\widehat{S} = \widehat{\mathbf{1}} + i\widehat{T}$, the factor $e^{\int d^3k\, a^*(\vec{k})a(\vec{k})}$ corresponds to $\widehat{\mathbf{1}}$. The transition matrix $\widehat{T}$ contributes as soon as there are particles interacting:

$$\langle a^*, +\infty | i\widehat{T} | a, -\infty\rangle = e^{\int d^4x\, \phi_{as}(x)(-\Box+m^2)\frac{\delta}{\delta j(x)}} \frac{Z[j]}{Z[0]}. \tag{2.184}$$

This gives

$$\begin{aligned}
&\langle +\infty; 0| \prod_i \widehat{a}(\vec{q}_i, +\infty)(i\widehat{T}) \prod_j \widehat{a}^\dagger(\vec{p}_j, -\infty)|0; -\infty\rangle = \\
&\prod_i \frac{\delta}{\delta a^*(\vec{q}_i)} \prod_j \frac{\delta}{\delta a(\vec{p}_j)} e^{\int d^4x\, \phi_{as}(x)(-\Box+m^2)\frac{\delta}{\delta j(x)}} \frac{Z[j]}{Z[0]}|_{a^*=0=a=j} = \\
&\prod_i \int d^4y_i \frac{e^{-iq_iy_i}}{\sqrt{2\omega(\vec{q}_i)}(2\pi)^{3/2}}(-\Box_{y_i} + m^2)(\frac{i}{\hbar}) \prod_j \int d^4x_j \frac{e^{ip_jx_j}}{\sqrt{2\omega(\vec{p}_j)}(2\pi)^{3/2}}(-\Box_{x_j} + m^2)(\frac{i}{\hbar}) \\
&\frac{\langle +\infty; 0|T\{\prod_i \widehat{\phi}(y_i) \prod_j \widehat{\phi}(x_j)\}|0; -\infty\rangle}{\langle +\infty; 0|0; -\infty\rangle}.
\end{aligned} \tag{2.185}$$

Since

$$\widehat{a}^\dagger(p_j, -\infty)|0; -\infty\rangle = Z^{\frac{1}{2}}\widehat{a}^\dagger(p_j, in)|0; -\infty\rangle, \tag{2.186}$$

$$\langle +\infty; 0|\widehat{a}(q_i, +\infty) = Z^{\frac{1}{2}}\langle +\infty; 0|\widehat{a}(q_i, out), \tag{2.187}$$

one finds

$$\begin{aligned}
&\boxed{\langle +\infty; 0| \prod_i \widehat{a}(\vec{q}_i, out)(i\widehat{T}) \prod_j \widehat{a}^\dagger(\vec{p}_j, in)|0; -\infty\rangle =} \\
&\boxed{\prod_i \int d^4y_i \frac{e^{-iq_iy_i}}{\sqrt{2\omega(\vec{q}_i)}(2\pi)^{3/2}Z^{\frac{1}{2}}}(-\Box_{y_i} + m^2)(\frac{i}{\hbar}) \prod_j \int d^4x_j \frac{e^{ip_jx_j}}{\sqrt{2\omega(\vec{p}_j)}(2\pi)^{3/2}Z^{\frac{1}{2}}}(-\Box_{x_j} + m^2)(\frac{i}{\hbar})} \\
&\boxed{\frac{\langle +\infty; 0|T\{\prod_i \widehat{\phi}(y_i) \prod_j \widehat{\phi}(x_j)\}|0; -\infty\rangle}{\langle +\infty; 0|0; -\infty\rangle}.}
\end{aligned} \tag{2.188}$$

As said before, the effect of $(-\Box_{x_j} + m^2)$ is to amputate the external propagators of the Feynman diagrams since $(-\Box_{x_j} + m^2)\Delta_F(x, z) = \delta^4(z)$.

2.7 Fermions

cf. [2], [9], [11], [16].

2.7.1 Grassmann variables

When canonically quantizing fermions, considerations of stability and the desire for a standard interpretation of the energy-momentum tensor force one to quantize fermions using anticommutators instead of commutators,

$$\begin{aligned}
\widehat{\psi}^l(x) &= \frac{1}{(2\pi)^{3/2}}\int d^3p\, u^l_\alpha(\vec{p})\widehat{b}^\alpha(\vec{p})e^{ipx} + v^l_\alpha(\vec{p})\widehat{d}^{\dagger\alpha}(\vec{p})e^{-ipx}, \\
[\widehat{b}^\alpha(\vec{p}), \widehat{b}^{\dagger\beta}(\vec{q})]_+ &= \hbar\delta^{\alpha\beta}\delta^3(\vec{p}-\vec{q}), \\
[\widehat{d}^\alpha(\vec{p}), \widehat{d}^{\dagger\beta}(\vec{q})]_+ &= \hbar\delta^{\alpha\beta}\delta^3(\vec{p}-\vec{q}), \\
[\widehat{\psi}^l(x), \widehat{\psi}^{\dagger m}(y)]_+ &= \big((-\gamma^\mu\partial_\mu + m)\beta\big)^{lm}\Delta(x-y), \\
\Delta(x) &= \int \frac{d^3p}{2p^0}(2\pi)^3(e^{ipx} - e^{-ipx}). \qquad (2.189)
\end{aligned}$$

In order to use a path integral approach, one would like to represent the basic fermionic operators $[\widehat{b}, \widehat{b}^\dagger]_+ = 1$ as analytic functions, similar to the path integral representation in holomorphic representation. In order to do so, one needs unusual classical variables, called Grassmann variables.

Consider complex series in 2 variables η, η^* and declare

$$\eta\eta^* + \eta^*\eta = 0, \quad \eta^2 = 0, \quad (\eta^*)^2 = 0. \qquad (2.190)$$

A general series has the form

$$f(\eta, \eta^*) = f_0 + f_1\eta + f_2\eta^* + f_3\eta\eta^*, \quad f_i \in \mathbb{C}. \qquad (2.191)$$

More generally, one may consider n anticommuting variables η^α:

$$\eta^\alpha\eta^\beta + \eta^\beta\eta^\alpha = 0. \qquad (2.192)$$

Parity $|f|$ of a monomial f in η^α is the number of anticommuting variables modulo 2. Differentiation is defined by

$$\frac{\partial^L fg}{\partial\eta^\alpha} = \frac{\partial^L f}{\partial\eta^\alpha}g + (-)^{|f|}f\frac{\partial^L g}{\partial\eta^\alpha}, \quad \frac{\partial^L\eta^\beta}{\partial\eta^\alpha} = \delta^\alpha_\beta, \quad \frac{\partial^L 1}{\partial\eta^\alpha} = 0, \qquad (2.193)$$

on monomials and extended by linearity.

For 2 complex conjugated variables, $\frac{\partial^L f}{\partial\eta} = 0$ implies $f = f_0 + f_2\eta^*$. These are "analytic" functions, to be compared to the functions $f(a^*)$ of the holomorphic representation. The Fock space associated to a fermionic oscillator $[\widehat{b}, \widehat{b}^\dagger]_+ = 1$ is defined by

$$|f\rangle = f_0|0\rangle + f_1\widehat{b}^\dagger|0\rangle, \quad \widehat{b}|0\rangle = 0, \quad |1\rangle = \widehat{b}^\dagger|0\rangle, \qquad (2.194)$$

$$\widehat{b}^\dagger|1\rangle = 0, \quad \langle 0|0\rangle = 1 = \langle 1|1\rangle, \quad \langle f|g\rangle = f_0^*g_0 + f_1^*g_1. \qquad (2.195)$$

One may represent this scalar product through an integral provided one defines

$$\int d\eta^*\eta^* = 1 = \int d\eta\eta, \quad \int d\eta 1 = 0 = \int d\eta^* 1, \qquad (2.196)$$

and extends by linearity. Furthermore, $d\eta d\eta^* = -d\eta^* d\eta$. The integral acts like differentiation,

$$\int d\eta^* d\eta \eta\eta^* = 1. \tag{2.197}$$

For the function $f(\eta^*) = f_0 + f_1\eta^*$, let us define $f^*(\eta) = f_0^* + f_1^*\eta$ and

$$\begin{aligned} \langle f|g\rangle &= \int d\eta^* d\eta f^*(\eta) g(\eta^*) e^{-\eta^*\eta} && (2.198)\\ &= \int d\eta^* d\eta (f_0^* + f_1^*\eta)(g_0 + g_1\eta^*)(1-\eta^*\eta) && (2.199)\\ &= \int d\eta^* d\eta(-\eta^*\eta f_0^* g_0 + \eta\eta^* f_1^* g_1) = f_0^* g_0 + f_1^* g_1. && (2.200)\end{aligned}$$

If $|\eta\rangle = e^{\widehat{b}^\dagger\eta}|0\rangle = |0\rangle - \eta\widehat{b}^\dagger|0\rangle$ and $\langle\eta^*| = \langle 0|e^{\eta^*\widehat{b}} = \langle 0| - \langle 0|\widehat{b}\eta^*$, it follows that $\widehat{b}|\eta\rangle = \eta|\eta\rangle$, $\langle\eta^*|\widehat{b}^\dagger = \langle\eta^*|\eta^*$, $\langle\eta^*|f\rangle = f(\eta^*) = f_0 + f_1\eta^*$, $\langle f|\eta\rangle = f^*(\eta) = f_0^* + f_1^*\eta$. We thus have

$$\langle f|g\rangle = \int d\eta^* d\eta e^{-\eta^*\eta}\langle f|\eta\rangle\langle\eta^*|g\rangle, \tag{2.201}$$

$$\langle\eta^*|\widehat{b}^\dagger|f\rangle = \langle 0|(1+\eta^*\widehat{b})\widehat{b}^\dagger(f_0|0\rangle + f_1\widehat{b}^\dagger)|0\rangle = \eta^* f_0 = \eta^* f(\eta^*), \tag{2.202}$$

$$\langle\eta^*|\widehat{b}|f\rangle = \langle 0|(1+\eta^*\widehat{b})\widehat{b}(f_0|0\rangle + f_1\widehat{b}^\dagger)|0\rangle = f_1 = \frac{\partial}{\partial\eta^*} f(\eta^*). \tag{2.203}$$

It follows that $\widehat{b}^\dagger$ acts by multiplication by η^*, while $\widehat{b}$ acts like differentiation with respect to η^*.

The vacuum projector is given by

$$|0\rangle\langle 0| =: e^{-\widehat{b}^\dagger\widehat{b}} := 1 - \widehat{b}^\dagger\widehat{b}. \tag{2.204}$$

Indeed,

$$|0\rangle\langle 0|(f_0|0\rangle + f_1\widehat{b}^\dagger)|0\rangle = f_0|0\rangle, \tag{2.205}$$

$$(1-\widehat{b}^\dagger\widehat{b})(f_0|0\rangle + f_1\widehat{b}^\dagger)|0\rangle = f_0|0\rangle. \tag{2.206}$$

We have $\widehat{O} = \sum_{n,m=0,1}|n\rangle O_{nm}\langle m|$, $\langle\eta^*|0\rangle = 1$, $\langle\eta^*|1\rangle = \eta^*$, $\langle 0|\eta\rangle = 1$, $\langle 1|\eta\rangle = \eta$. The kernel of $\widehat{O}$ in holomorphic representation is

$$O(\eta^*,\eta) = \langle\eta^*|\widehat{O}|\eta\rangle = \sum_{n,m}\langle\eta^*|n\rangle O_{nm}\langle m|\eta\rangle = \sum_{n,m}(\eta^*)^n O_{nm}\eta^m. \tag{2.207}$$

Furthermore,

$$\langle\eta^*|\widehat{O}|f\rangle = \int d\xi^* d\xi e^{-\xi^*\xi} O(\eta^*,\xi) f(\xi^*), \tag{2.208}$$

$$\langle\eta^*|\widehat{O}_1\widehat{O}_2|\eta\rangle = \int d\xi^* d\xi e^{-\xi^*\xi} O_1(\eta^*,\xi) O_2(\xi^*,\eta). \tag{2.209}$$

The normal symbol is

$$O^N(\eta^*,\eta) = e^{-\eta^*\eta} O(\eta^*,\eta). \tag{2.210}$$

Indeed,

$$\widehat{O} = \sum_{n,m}(\widehat{b}^\dagger)^n|0\rangle O_{nm}\langle 0|\widehat{b}^m = \sum_{n,m}(\widehat{b}^\dagger)^n : e^{-\widehat{b}^\dagger\widehat{b}} : \widehat{b}^m O_{nm} = \sum_{n,m} : (\widehat{b}^\dagger)^n e^{-\widehat{b}^\dagger\widehat{b}}\widehat{b}^m : O_{nm}, \tag{2.211}$$

which gives the result.

Up to a factor of $2\pi i$ in the scalar product, all formulas look like those of the bosonic case. But if one preforms a change of integration variables,

$$\begin{pmatrix}\eta\\ \eta^*\end{pmatrix} = A\begin{pmatrix}\xi\\ \xi^*\end{pmatrix}, \tag{2.212}$$

and considers the polynomial $P(\eta,\eta^*) = Q(\xi,\xi^*)$ that one gets by substitution, the term in $\xi\xi^*$ comes from $P_{12}\eta\eta^* = P_{12}(A^{11}\xi + A^{12}\xi^*)(A^{21}\xi + A^{22}\xi^*)$ and is given by $P_{12}(detA)\xi\xi^* \equiv Q_{12}\xi\xi^*$. This implies

$$\int d\eta^* d\eta P(\eta,\eta^*) = P_{12} = (detA)^{-1}Q_{12} = (detA)^{-1}\int d\xi^* d\xi Q(\xi,\xi^*). \tag{2.213}$$

with

$$detA = |\frac{\partial(\eta\eta^*)}{\partial(\xi\xi^*)}|. \tag{2.214}$$

For ordinary variables, it is the Jacobian of the change of variables that occurs,

$$\int\prod_i dx^i f(x) = \int\prod_j dy^j \left|\frac{\partial x}{\partial y}\right| f(x(y)), \tag{2.215}$$

but in the fermionic case, one sees that the rules for integration imply that it is now the inverse Jacobian that occurs.

The trace of fermionic operator like $\widehat{b}$ or $\widehat{b}^\dagger$ vanishes. For the trace of a bosonic operator $\widehat{O}$, we now find

$$\begin{aligned}\mathrm{Tr}\,\widehat{O} = \sum_n \langle n|\widehat{O}|n\rangle = \int d\eta^* d\eta \sum_n \langle n|\eta\rangle\langle\eta^*|\widehat{O}|n\rangle e^{-\eta^*\eta} = \int d\eta^* d\eta \sum_n \langle\eta^*|\widehat{O}|n\rangle\langle n|-\eta\rangle e^{-\eta^*\eta} \\ = \int d\eta^* d\eta\, O(\eta^*,-\eta)e^{-\eta^*\eta} = \int d\eta d\eta^*\, O(\eta^*,\eta)e^{\eta^*\eta}.\end{aligned} \tag{2.216}$$

2.7.2 Evolution operator

By repeating the arguments of the bosonic case for the fermionic functional integral, one finds for the kernel of the evolution operator:

$$U(\eta^*,t';\eta,t) = \int_{\eta(t)=\eta}^{\eta^*(t')=\eta^*} \mathcal{D}\eta^*\mathcal{D}\eta\; e^{iS'_H[\eta^*,\eta]}, \tag{2.217}$$

$$S'_H[\eta^*,\eta] = \int_t^{t'} d\tau\Big[\frac{1}{2i}(\dot{\eta}^*\eta - \eta^*\dot{\eta}) - h(\eta^*,\eta)\Big] + \frac{1}{2i}\big(\eta^*(t')\eta(t') + \eta^*(t)\eta(t)\big). \tag{2.218}$$

2.7.3 Exercises

2.7.3.1 Fermionic Gaussian integration

Show that

$$\int d\xi^n \dots d\xi^1 e^{\xi^m M_{mn} \xi^n} = 2^{\frac{n}{2}} \sqrt{\det M}. \tag{2.219}$$

Hint: A skew-symmetric matrix may be put into the form

$$M' = \begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & & \\ & & 0 & m_2 & \\ & & -m_2 & 0 & \\ & & & & \dots \end{pmatrix} \tag{2.220}$$

with $M' = O^T M O$ et $\det O = 1$.

If

$$Z(\eta, \eta^*) = \int \prod_k d\xi^{*k} d\xi^k e^{-\xi^{*k} A_{kl} \xi^l + \xi^{*k} \eta_k + \eta^*_k \xi^k} \tag{2.221}$$

show that

$$Z(\eta, \eta^*) = \det A e^{\eta^*_k (A^{-1})^{kl} \eta_l}. \tag{2.222}$$

Hint: perform the change of variables: $\xi^k = \xi'^k + (A^{-1})^{kl} \eta_l$, $\xi^{*k} = \xi^{*\prime k} + \eta^*_l (A^{-1})^{lk}$.

2.7.3.2 Fermionic Wick theorem

If

$$\det A \langle \xi^{*i_1} \xi^{j_1} \dots \xi^{*i_n} \xi^{j_n} \rangle \equiv \int \prod_i d\xi^{*i} d\xi^i \xi^{*i_1} \xi^{j_1} \dots \xi^{*i_n} \xi^{j_n} e^{-\xi^{*i} A_{ij} \xi^j}, \tag{2.223}$$

show that

$$\langle \xi^{*i_1} \xi^{j_1} \dots \xi^{*i_n} \xi^{j_n} \rangle = \sum_{\sigma \in \{1, \dots, n\}} \epsilon(\sigma) (A^{-1})^{i_1 j_{\sigma(1)}} \dots (A^{-1})^{i_n j_{\sigma(n)}}, \tag{2.224}$$

where $\epsilon(\sigma)$ is the signature of the permutation.

Hint:

$$\det A \langle \xi^{*i_1} \xi^{j_1} \dots \xi^{*i_n} \xi^{j_n} \rangle = \frac{\partial^L}{\partial \eta^{i_1}} \frac{\partial^L}{\partial \eta^{*j_1}} \dots \frac{\partial^L}{\partial \eta^{i_n}} \frac{\partial^L}{\partial \eta^{*j_n}} Z(\eta, \eta^*)\big|_{\eta = 0 = \eta^*}. \tag{2.225}$$

2.7.3.3 Path integral representation for the trace in holomorphic representation

Show that in the bosonic case, the trace of the evolution operator admits the path integral representation

$$\text{Tr}\, \widehat{U}(t', t) = \int \prod_\tau \frac{da^*(\tau) da(\tau)}{2\pi i} \quad e^{i S^P_H [a^*, a]}, \quad S^P_H[a^*, a] = \int_0^{t'-t} d\tau \Big[\frac{1}{i} \dot{a}^* a - h(a^*, a)\Big] \tag{2.226}$$

and periodic boundary conditions on $[0, t' - t]$

Hint: start from (2.138). The trace in the holomorphic introduces an additional integral over $\frac{da_N^* da_0}{2\pi i} e^{-a_N^* a_0}$. Since there are no a_0^*, a_N involved in the integrand, we are free to set $a_0^* = a_N^*, a_N = a_0$. With this, all terms are reproduced by the discretization of S_H^P.

Show that in the fermionic case,

$$\mathrm{Tr}\,\widehat{U}(t',t) = \int \prod_\tau d\eta^*(\tau) d\eta(\tau)\, e^{iS_H^{AP}[\eta^*,\eta]},\ S_H^{AP}[\eta^*,\eta] = \int_0^{t'-t} d\tau \Big[\frac{1}{i}\dot{\eta}^*\eta - h(\eta^*,\eta)\Big] \tag{2.227}$$

and anti-periodic boundary conditions on $[0, t'-t]$.

Hint: In this case, the additional integral from the trace (2.216) is $d\eta_0 d\eta_N^* e^{\eta_N^* \eta_0}$ so that now one needs to set $\eta_0^* = -\eta_N^*$, $\eta_N = -\eta_0$.

2.7.4 Fermionic propagator

For Dirac fermions,

$$Z(\eta,\bar{\eta}) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{\frac{i}{\hbar}\int d^4x(-\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi + \bar{\psi}\eta + \bar{\eta}\psi)}. \tag{2.228}$$

We then have $A(x,y) = \frac{i}{\hbar}(\gamma^\mu \partial_\mu^x + m)\delta^4(x,y)$ and thus

$$A(x) = \frac{i}{\hbar}\frac{1}{(2\pi)^4}\int d^4p(i\gamma^\mu p_\mu + m)e^{ipx}, \tag{2.229}$$

$$A^{-1}(x) = \boxed{\frac{\hbar}{i}\frac{1}{(2\pi)^4}\int d^4p \frac{-i\gamma^\mu p_\mu + m}{p^2+m^2-i\epsilon} e^{ip\cdot x} \equiv \frac{\hbar}{i}S_F(x)}. \tag{2.230}$$

Furthermore,

$$\frac{Z(\eta,\bar{\eta})}{Z(0,0)} = e^{\frac{i}{\hbar}\int d^4x \int d^4y\ \bar{\eta}(x) S_F(x,y)\eta(y)}, \tag{2.231}$$

$$\boxed{\frac{\hbar}{i}S_F(x,y) =} (\frac{\hbar}{i})^2 \frac{\delta^L}{\delta\bar{\eta}(x)}\frac{\delta^R}{\delta\eta(y)}\frac{Z(\eta,\bar{\eta})}{Z(0,0)} = \boxed{\frac{\langle +\infty; 0|T\{\widehat{\psi}(x)\widehat{\bar{\psi}}(y)\}|0;-\infty\rangle}{\langle +\infty;0|0;-\infty\rangle}}. \tag{2.232}$$

2.8 Finite temperature results

The reference for the subsection on thermal correlation function is for instance this script here, equation 8.44 or the associated book [18].

2.8.1 Harmonic oscillator: Partition function, thermal 2-point function

We already saw that $Z(\beta) = \text{Tr } e^{-\beta \widehat{H}} = \text{Tr } \widehat{U}(t', t)$ with $t' - t = -i\hbar\beta$. It then follows from (2.149) that

$$U(a^*, a; -i\hbar\beta) = e^{-\frac{\alpha\hbar\omega\beta}{2} + a^* e^{-\hbar\beta\omega} a}. \tag{2.233}$$

Using formula (2.133) for the trace, this yields for the partition function

$$Z(\beta) = e^{-\frac{\alpha\hbar\omega\beta}{2}} \int \frac{da^* da}{2\pi i} e^{(-a^* a + a^* e^{-\hbar\beta\omega} a)} = e^{-\frac{\alpha\hbar\omega\beta}{2}} \frac{1}{1 - e^{-\hbar\beta\omega}}, \tag{2.234}$$

where the last equality follows from the redefinitions $a = \widetilde{a}/\sqrt{1 - e^{-\hbar\beta\omega}}$, $a^* = \widetilde{a}^*/\sqrt{1 - e^{-\hbar\beta\omega}}$. In particular, for $\alpha = 1$, one finds

$$\boxed{Z(\beta) = \frac{1}{2\sinh\frac{\hbar\beta\omega}{2}}}. \tag{2.235}$$

From (2.234), it follows that the free energy is

$$F = -\beta^{-1} \ln Z(\beta) = \frac{\alpha\hbar\omega}{2} + \beta^{-1} \ln(1 - e^{-\hbar\beta\omega}) \approx \begin{cases} \frac{\alpha\hbar\omega}{2}, & \beta \gg \hbar\omega \\ \beta^{-1} \ln \hbar\beta\omega, & \beta \ll \hbar\omega \end{cases}, \tag{2.236}$$

the internal energy is

$$E = -\frac{\partial \ln Z(\beta)}{\partial\beta} = \hbar\omega\left(\frac{\alpha}{2} + \frac{1}{e^{\hbar\beta\omega} - 1}\right) \approx \begin{cases} \frac{\alpha\hbar\omega}{2}, & \beta \gg \hbar\omega \\ \beta^{-1}, & \beta \ll \hbar\omega \end{cases}. \tag{2.237}$$

This means in particular that the internal energy is the ground state energy in low-temperature limit and rises linearly with temperature in the high-temperature limit. The entropy is

$$S(\beta) = \left(1 - \beta\frac{\partial}{\partial\beta}\right) \ln Z(\beta) = -\ln(1 - e^{-\hbar\beta\omega}) + \frac{\hbar\beta\omega}{e^{\hbar\beta\omega} - 1} \approx \begin{cases} \hbar\beta\omega e^{-\hbar\beta\omega}, & \hbar\beta\omega \gg 1 \\ 1 - \ln \hbar\beta\omega, & \hbar\beta\omega \ll 1 \end{cases}. \tag{2.238}$$

More generally, putting $t' \to t' - t$, $t \to 0$ and then using $t' - t = -i\hbar\beta$, $\tau = -i\lambda$, $\tau' = -i\lambda'$ in (2.149), we get

$$\begin{aligned}\ln U(a^*, a; -i\hbar\beta) = &-\frac{\alpha\hbar\beta\omega}{2} + a^* e^{-\hbar\beta\omega} a + a^* \int_0^{\hbar\beta} d\lambda \frac{j(\lambda)}{\hbar} e^{-\omega(\hbar\beta - \lambda)} + \\ &+ a \int_0^{\hbar\beta} d\lambda \frac{j^*(\lambda)}{\hbar} e^{-\omega\lambda} + \int_0^{\hbar\beta} d\lambda \int_0^{\beta} d\lambda' \frac{j^*(\lambda)}{\hbar} e^{-\omega(\lambda - \lambda')} \theta(\lambda - \lambda') \frac{j(\lambda')}{\hbar}.\end{aligned} \tag{2.239}$$

Taking the trace now gives

$$\ln Z(\beta; j) = \ln Z(\beta; 0) + \int_0^{\hbar\beta} d\lambda \int_0^{\hbar\beta} d\lambda' \frac{j^*(\lambda)}{\hbar} \frac{j(\lambda')}{\hbar} e^{-\omega(\lambda - \lambda')} \left[\theta(\lambda - \lambda') + \frac{e^{-\hbar\beta\omega}}{1 - e^{-\hbar\beta\omega}}\right]. \tag{2.240}$$

For the harmonic oscillator,

$$a = \frac{\sqrt{\omega} q + i\sqrt{\omega}^{-1} p}{\sqrt{2\hbar}}, \tag{2.241}$$

while the source term in the path integral representation of the partition function (i.e., in the Euclidean version of (2.139)) is the exponential of

$$\frac{1}{\hbar}\int_0^{\hbar\beta} d\lambda[j^*(\lambda)a(\lambda)+j(\lambda)a^*(\lambda)]. \tag{2.242}$$

It follows that, if $j=\sqrt{\frac{\hbar}{2\omega}}J_R$, with J_R real, this source term becomes

$$\frac{1}{\hbar}\int_0^{\hbar\beta} d\lambda J_R(\lambda)q(\lambda), \tag{2.243}$$

while (2.240) can be written as

$$\begin{aligned}\ln Z(\beta;J_R) &= \ln Z(\beta;0)+\frac{1}{2\hbar\omega}\int_0^{\hbar\beta}d\lambda\int_0^{\hbar\beta}d\lambda' J_R(\lambda)J_R(\lambda')e^{-\omega(\lambda-\lambda')}[\theta(\lambda-\lambda')+\frac{e^{-\hbar\beta\omega}}{1-e^{-\hbar\beta\omega}}]\\ &= \ln Z(\beta;0)+\frac{1}{2\hbar\omega}\int_0^{\hbar\beta}d\lambda\int_0^{\hbar\beta}d\lambda' J_R(\lambda)J_R(\lambda')\frac{\cosh\omega(|\lambda-\lambda'|-\frac{\hbar\beta}{2})}{2\sinh\frac{\hbar\beta\omega}{2}}\end{aligned} \tag{2.244}$$

This implies that the thermal 2-point function is given by

$$\begin{aligned}G(\lambda,0) &= \boxed{\frac{1}{Z(\beta)}\mathrm{Tr}\,\widehat{q}(\lambda)\,\widehat{q}(0)\,e^{-\beta\widehat{H}}} = \left[\hbar\frac{\delta}{\delta J_R(\lambda)}\hbar\frac{\delta}{\delta J_R(0)}\ln Z(\beta;J_R)\right]|_{J_R=0}\\ &\boxed{=\frac{\hbar}{2\omega}\frac{\cosh\omega(\lambda-\frac{\hbar\beta}{2})}{\sinh\frac{\hbar\beta\omega}{2}}},\end{aligned} \tag{2.245}$$

with $\hbar\beta\geqslant\lambda\geqslant 0$.

Details on intermediate computations (with $\hbar=1$) follow:

replace $t' \to t'-t$ and $t \to 0$

$$\ln U(a^*, t'-t, a, 0; j) = -i(t'-t)\frac{\alpha\omega}{2} + a^* e^{-i\omega(t'-t)} a + i a^* \int_0^{t'-t} d\tau\, j(\tau)\, e^{-i\omega(t'-t-\tau)}$$

$$+ i a \int_0^{t'-t} d\tau\, j^*(\tau)\, e^{-i\omega\tau} - \frac{1}{2}\int_0^{t'-t} d\tau\, j(\tau) \int_\tau^{t'-t} d\tau'\, j^*(\tau')\, e^{-i\omega(\tau'-\tau)}$$

$$- \frac{1}{2}\int_0^{t'-t} d\tau\, j^*(\tau) \int_0^{\tau} d\tau'\, j(\tau')\, e^{-i\omega(\tau-\tau')}$$

$t'-t = -i\beta$, $\tau = -i\lambda$, $\tau' = -i\lambda'$

$$\ln U(a^*, -i\beta, a, 0; j) = \left(-\frac{\alpha\beta\omega}{2}\right) + a^* e^{-\beta\omega} a + a^* \int_0^\beta d\lambda'\, j(\lambda')\, e^{-\omega(\beta-\lambda')}$$

$$+ a \int_0^\beta d\lambda\, j^*(\lambda)\, e^{-\omega\lambda} + \frac{1}{2}\int_0^\beta d\lambda\, j(\lambda) \int_{\cancel{\lambda}}^\beta \cancel{d\lambda'}\, j^*(\lambda')\, e^{-\omega(\lambda'-\lambda)}$$

$$+ \frac{1}{2}\int_0^\beta d\lambda\, j^*(\lambda) \int_{\cancel{0}}^{\lambda} \cancel{d\lambda'}\, j(\lambda')\, e^{-\omega(\lambda-\lambda')}$$

$$\left[\begin{array}{l} \frac{1}{2}\, j^*(\lambda)\, \theta(\lambda-\lambda')\, e^{-\omega(\lambda-\lambda')}\, j(\lambda') \\ + \frac{1}{2}\, j^*(\lambda')\, \theta(\lambda'-\lambda)\, e^{-\omega(\lambda'-\lambda)}\, j(\lambda) \end{array}\right.$$

$$+ \int_0^\beta d\lambda \int_0^\beta d\lambda'\, j^*(\lambda)\, j(\lambda')\, \theta(\lambda-\lambda')\, e^{-\omega(\lambda-\lambda')}$$

Taking the trace:

exponent: $-a^* a + a^* e^{-\beta\omega} a + a^* b + a c^* + d$

$d = \bigcirc$, $b = \int_0^\beta d\lambda'\, j(\lambda')\, e^{-\omega(\beta-\lambda')}$, $c^* = \int_0^\beta d\lambda\, j^*(\lambda)\, e^{-\omega\lambda}$

redefinition: $a^* = \dfrac{\tilde{a}^*}{\sqrt{1-e^{-\beta\omega}}}$ $\quad a = \dfrac{\tilde{a}}{\sqrt{1-e^{-\beta\omega}}}$

$$Z(\beta; j) = \frac{e^{\alpha}}{1-e^{-\beta\omega}} \int \frac{d\tilde{a}^* \, d\tilde{a}}{2\pi i} \, e^{-\tilde{a}^*\tilde{a} + \tilde{a}^* \frac{b}{\sqrt{1-e^{-\beta\omega}}} + \tilde{a} \frac{c^*}{\sqrt{1-e^{-\beta\omega}}}}$$

extremum : $\tilde{a}^* = \frac{c^*}{\sqrt{1-e^{-\beta\omega}}}$, $a = \frac{b}{\sqrt{1-e^{-\beta\omega}}}$

$$= \frac{e^{\alpha}}{1-e^{-\beta\omega}} \, e^{\frac{1}{1-e^{-\beta\omega}}\left(- \cancel{c^* b} + \cancel{c^* b} + c^* b \right)}$$

$$\ln Z(\beta, j) = \underbrace{- \alpha \frac{\beta\omega}{2} - \ln\left(1-e^{-\beta\omega}\right)}_{= \ln Z(\beta, 0)}$$

$$\underbrace{+ \int_0^\beta d\lambda \int_0^\beta d\lambda' \, j^*(\lambda) \, j(\lambda) \, e^{-\omega(\lambda-\lambda')} \left[\theta(\lambda-\lambda') + \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \right]}_{= (X)}$$

$$\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} = \frac{e^{-\beta\frac{\omega}{2}}}{2 \operatorname{sh} \frac{\beta\omega}{2}}$$

$$j = \sqrt{\frac{1}{2\omega}} \, j_R$$

$$(X) = \int_0^\beta d\lambda \int_0^\beta d\lambda' \, j_R(\lambda) \, j_R(\lambda') \, \frac{1}{4\omega} \underbrace{\Big[e^{-\omega(\lambda-\lambda')} \theta(\lambda-\lambda') + e^{-\omega(\lambda-\lambda')} \frac{e^{-\beta\frac{\omega}{2}}}{2 \operatorname{sh} \frac{\beta\omega}{2}} + e^{-\omega(\lambda'-\lambda)} \theta(\lambda'-\lambda) + e^{-\omega(\lambda'-\lambda)} \frac{e^{-\beta\frac{\omega}{2}}}{2 \operatorname{sh} \beta\frac{\omega}{2}} \Big]}_{(Y)}$$

$$G(\tau) = e^{-\omega|\tau-\tau'|} + \frac{e^{-\frac{\beta\omega}{2}}}{2\,\mathrm{sh}\,\frac{\beta\omega}{2}}\left[e^{-\omega(\tau-\tau')} + e^{-\omega(\tau'-\tau)}\right]$$

$$e^{\frac{\beta\omega}{2}} = e^{-\frac{\beta\omega}{2}} + 2\,\mathrm{sh}\,\frac{\beta\omega}{2}$$

$$\stackrel{?}{=} \frac{\mathrm{ch}\;\omega\left(|\tau-\tau'| - \frac{\beta\omega}{2}\right)}{\mathrm{sh}\;\frac{\beta\omega}{2}} = \frac{1}{2\,\mathrm{sh}\,\frac{\beta\omega}{2}}\Big[\left(e^{\omega(\tau-\tau')}e^{-\frac{\beta\omega}{2}} + e^{-\omega(\tau-\tau')}\underline{e^{\frac{\beta\omega}{2}}}\right)\theta(\tau-\tau')$$
$$+ \left(e^{+\omega(\tau'-\tau)}e^{-\frac{\beta\omega}{2}} + e^{-\omega(\tau'-\tau)}\underline{e^{\frac{\beta\omega}{2}}}\right)\theta(\tau'-\tau)\Big]$$

$$= \frac{e^{-\frac{\beta\omega}{2}}}{2\,\mathrm{sh}\,\frac{\beta\omega}{2}}\left[e^{\omega(\tau-\tau')}\theta(\tau-\tau') + e^{-\omega(\tau-\tau')}\theta(\tau-\tau') + e^{\omega(\tau'-\tau)}\theta(\tau'-\tau) + e^{-\omega(\tau'-\tau)}\theta(\tau'-\tau)\right]$$
$$+ e^{-\omega(\tau-\tau')}\theta(\tau-\tau') + e^{-\omega(\tau'-\tau)}\theta(\tau'-\tau)$$

$$= \frac{e^{-\frac{\beta\omega}{2}}}{2\,\mathrm{sh}\,\frac{\beta\omega}{2}}\left[e^{\omega(\tau-\tau')} + e^{-\omega(\tau-\tau')}\right] + e^{-\omega|\tau-\tau'|}$$

✓

2.8.2 Thermal correlation functions

Thermal correlation functions

Thursday, September 26, 2019 6:59 PM

(Reference: Wipf, Path integrals or statistical approach to QFT)

Objective: derive the energy & wave function of the first excited state from thermal correlation functions

$$\hat{q}_E(\tau) = e^{\tau \hat{H}/\hbar}\, \hat{q}\, e^{-\tau \hat{H}/\hbar}, \quad \hat{q}_E(0) = \hat{q}(0)$$

$$\langle \hat{q}_E(\tau_n) \dots \hat{q}_E(\tau_1)\rangle_\beta = \frac{1}{Z(\beta)} \operatorname{Tr} e^{-\beta H} \hat{q}_E(\tau_n) \dots \hat{q}_E(\tau_1)$$

2 pt function: $\langle \hat{q}_E(\tau_2)\, \hat{q}_E(\tau_1)\rangle_\beta = \frac{1}{Z(\beta)} \operatorname{Tr} e^{-(\hbar\beta - \tau_2)\hat{H}/\hbar}\, \hat{q}\, e^{-(\tau_2-\tau_1)\hat{H}/\hbar}\, \hat{q}\, e^{-\tau_1 \hat{H}/\hbar}$

Suppose that energy eigenstates are non-degenerate, like in the harmonic oscillator,

take the trace $\sum_n \langle n| \quad |n\rangle$ and insert $\mathbb{1} = \sum_m |m\rangle\langle m|$

$$\langle \hat{q}_E(\tau_2)\, \hat{q}_E(\tau_1)\rangle_\beta = \frac{1}{Z(\beta)} \sum_{n,m} e^{-(\hbar\beta-\tau_2)E_n/\hbar} \langle n|\hat{q}|m\rangle e^{-(\tau_2-\tau_1)E_m/\hbar} \langle m|\hat{q}|n\rangle e^{-\tau_1 E_n/\hbar}$$

$$= \frac{1}{Z(\beta)} \sum_{n,m} e^{-(\hbar\beta - \tau_2+\tau_1)E_n/\hbar}\, |\langle n|\hat{q}|m\rangle|^2\, e^{-(\tau_2-\tau_1)E_m/\hbar}$$

low temperature limit: $\beta \to \infty$

$\Rightarrow$ E_n, $n \neq 0$ are exponentially suppressed

$$Z(\beta) \xrightarrow[\beta\to\infty]{} e^{-\beta E_0}$$

$$\lim_{\beta\to\infty} \langle \hat{q}_E(\tau_2)\, \hat{q}_E(\tau_1)\rangle_\beta = \sum_{m\geq 0} |\langle 0|\hat{q}|m\rangle|^2\, e^{-(\tau_2-\tau_1)(E_m-E_0)/\hbar} = \langle 0|\, \hat{q}_E(\tau_2)\, \hat{q}_E(\tau_1)|0\rangle$$

$\beta \to \infty$ ' ' ' $m > 0$

Similarly $\lim_{\beta \to \infty} \langle \hat{q}_E(\tau) \rangle_\beta = \langle 0 | \hat{q} | 0 \rangle$

$\Rightarrow$

$$\lim_{\beta \to \infty} \Big(\langle \hat{q}_E(\tau_2)\, \hat{q}_E(\tau_1) \rangle_\beta - \langle \hat{q}_E(\tau_2) \rangle_\beta \langle \hat{q}_E(\tau_1) \rangle_\beta \Big) = \sum_{m > 0} e^{-(\tau_2 - \tau_1)(E_m - E_0)/\hbar} \, | \langle 0 | \hat{q} | m \rangle |^2$$

$$\lim_{\beta \to 0} \langle \underbrace{\{ \hat{q}_E(\tau_2) - \langle q_E(\tau_2) \rangle_\beta \} \{ \hat{q}_E(\tau_1) - \langle \hat{q}_E(\tau_1) \rangle_\beta \}} \rangle_\beta$$

connected 2-point function $\langle \hat{q}_E(\tau_2)\, \hat{q}_E(\tau_1) \rangle_\beta^c$

if $\tau_2 \gg \tau_1$, large Euclidean time differences,

$$\lim_{\substack{\beta \to \infty \\ \tau_2 - \tau_1 \to \infty}} \langle \hat{q}_E(\tau_2)\, \hat{q}_E(\tau_1) \rangle_\beta^c \to e^{-(E_1 - E_0)(\tau_2 - \tau_1)/\hbar} \, | \langle 0 | \hat{q} | 1 \rangle |^2$$

$\Rightarrow E_1 - E_0$, $| \langle 0 | q | 1 \rangle |^2$ are determined by large time behavior of thermal 2pt function.

—

NB: Path integral representation of thermal correlation functions:

$$\langle T \hat{q}_E(\tau_n) \dots \hat{q}_E(\tau_1) \rangle := \frac{1}{Z(\beta)} \operatorname{Tr} T \hat{q}_E(\tau_n) \dots \hat{q}_E(\tau_1)\, e^{-\beta \hat{H}}$$

$$= \frac{1}{Z(\beta)} \int_{\text{periodic}} \mathcal{D}q \; q(\tau_n) \dots q(\tau_1)\, e^{-\frac{1}{\hbar} S_E^c}$$

$$= (\hbar)^n \frac{\delta}{\delta J(\tau_n)} \dots \frac{\delta}{\delta J_n(\tau_1)} \ln Z[\beta; J]$$

$$= (\hbar)^n \frac{\delta}{\delta j_R(\tau_n)} \cdots \frac{\delta}{\delta j_R(\tau_1)} \ln Z[\beta; j]$$

$$Z[\beta; j] = \int_{\text{periodic}} \mathcal{D}q \; e^{-\frac{1}{\hbar}\left(S_E^L - \int_0^{\hbar\beta} d\tau \, j_R(\tau)\, q(\tau)\right)}$$

to be computed perturbatively using Feynman rules for non-quadratic interactions.

2.8.3 Fermionic oscillator partition function

The partition function of a fermionic harmonic oscillator with $\widehat{H} = \hbar\omega(\widehat{b}^\dagger\widehat{b} - \frac{\alpha}{2})$ is given by

$$Z[\beta] = e^{\frac{\hbar\omega\beta\alpha}{2}}(1 + e^{-\hbar\beta\omega}). \tag{2.246}$$

When $\alpha = 1$, this can be written as

$$\boxed{Z[\beta] = 2\cosh\frac{\hbar\beta\omega}{2}}. \tag{2.247}$$

This is trivial to show in the operator formalism in the basis $|n\rangle$ since one only needs to sum over $n = 0$ and $n = 1$. From the path integral point of view, following the derivation of the bosonic result, we get

$$\text{Tr}\, e^{-\beta\widehat{H}} = \int d\eta d\eta^* e^{\frac{\hbar\beta\omega\alpha}{2} + \eta^* e^{-\hbar\beta\omega}\eta + \eta^*\eta} = e^{\frac{\hbar\beta\omega\alpha}{2}} \int d\eta d\eta^* e^{-\eta\eta^*(1+e^{-\hbar\beta\omega})}. \tag{2.248}$$

The result then follows from the basic Gaussian fermionic integral after the change of variables $\eta = \widetilde{\eta}/\sqrt{1+e^{-\hbar\beta\omega}}$, $\eta^* = \widetilde{\eta}^*/\sqrt{1+e^{-\hbar\beta\omega}}$, when taking into acount that it is the inverse Jacobian that has to be used.

2.8.4 Massive scalar field

We follow [19], sections 2.2 and 2.3. Note also that this reference contains an alternative derivation of (2.251) based on an Euclidean path integral in position space in the Lagrangian formalism and involving a sum over Matsubara frequencies.

A massive scalar field is a superposition of decoupled harmonic oscillators with frequencies $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$. If we consider the field in a box with periodic boundary conditions, the frequencies are quantized $k^i = \frac{2\pi n^i}{L_{(i)}}$ with $n^i \in \mathbb{Z}^d$ and L_i the lengths of the different sides of the box. (cf. sections 1.2 and 1.5.3).

We then find from the partition function of a single harmonic oscillator in (2.234) (with $\hbar = 1$) that the partition function of a massive scalar field in d dimensions is given by

$$Z(\beta) = \prod_{\vec{k}} (e^{-\frac{\alpha\omega(\vec{k})\beta}{2}} \frac{1}{1 - e^{-\beta\omega(\vec{k})}}), \tag{2.249}$$

For the free energy, defined through $Z(\beta) = e^{-\beta F(\beta)}$, this gives

$$F(\beta) = \sum_{\vec{k}} \Big[\frac{\alpha\omega(\vec{k})}{2} + \beta^{-1}\ln(1 - e^{-\beta\omega(\vec{k})})\Big]. \tag{2.250}$$

In the limit of a large box we replace, as before, the sum by an integral,

$$\frac{F(\beta)}{V} = \int \frac{d^d k}{(2\pi)^d} \Big[\frac{\alpha\omega(\vec{k})}{2} + \beta^{-1}\ln(1 - e^{-\beta\omega(\vec{k})})\Big]. \tag{2.251}$$

In a low temperature expansion, one separates the piece at zero temperature ($\beta \to \infty$) from the rest. It is given by

$$(\frac{F(\beta)}{V})_0 = \int \frac{d^d k}{(2\pi)^d} \frac{\alpha\omega(\vec{k})}{2}. \tag{2.252}$$

For $d = 3$, it is an ultraviolet (large $|\vec{k}|$) divergent integral that is absent if one chooses normal ordering ($\alpha = 0$). For symmetric ordering ($\alpha = 1$), it can be treated through dimensional regularization: according to (2.99), (2.101) and (2.102), the result for $d = 3 - 2\epsilon$ is

$$(\frac{F(\beta)}{V})_0 = \frac{1}{2}\Phi(m, 3-2\epsilon, -\frac{1}{2}) = \frac{-m^4\mu^{-2\epsilon}}{64\pi^2}\Big[\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2} + \ln 4\pi - \gamma_E + \frac{3}{2} + O(\epsilon)\Big]. \tag{2.253}$$

How to deal with such ultraviolet divergences in a systematic way in terms of renormalized parameters will be discussed later.

The thermal part of the free energy density is convergent for $d = 3$ and given by

$$(\frac{F(\beta)}{V})_T = \beta^{-1} \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta\omega(\vec{k})}), \tag{2.254}$$

Doing the integral over the angles and setting $k = \frac{x}{\beta}$, $y = m\beta$, we find

$$(\frac{F(\beta)}{V})_T = \beta^{-4} \frac{1}{2\pi^2} \int_0^\infty dx x^2 \ln(1 - e^{-\sqrt{x^2+y^2}}). \tag{2.255}$$

In a low temperature expansion, $y \gg 1$, and using $\ln(1-\epsilon) = -\epsilon + O(\epsilon^2)$, $e^{-\sqrt{x^2+y^2}} = e^{-y} e^{\sqrt{1+\frac{x^2}{y^2}}}$,

$$\begin{aligned}
\int_0^\infty dx x^2 \ln(1 - e^{-\sqrt{x^2+y^2}}) &= -\int_0^\infty dx x^2 e^{-\sqrt{x^2+y^2}} + O(e^{-2y}) \\
&= -\int_y^\infty dw w \sqrt{w^2 - y^2} e^{-w} + O(e^{-2y}) \\
&= -e^{-y} \int_0^\infty dv (v+y) \sqrt{v^2 + 2vy} e^{-v} + O(e^{-2y}) \\
&= -\sqrt{2} y^{\frac{3}{2}} e^{-y} \int_0^\infty dv v^{\frac{1}{2}} (1 + \frac{v}{y})(1 + \frac{v}{2y})^{\frac{1}{2}} e^{-v} + O(e^{-2y}) \\
&= -\sqrt{2} y^{\frac{3}{2}} e^{-y} \Gamma(\frac{3}{2}) \big[1 + O(\frac{1}{y})\big] + O(e^{-2y}),
\end{aligned} \tag{2.256}$$

where we have made the changes of variables $w = \sqrt{x^2 + y^2}$ and $v = w - y$. Using $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, we finally have

$$(\frac{F(\beta)}{V})_T = -\beta^{-4} (\frac{y}{2\pi})^{\frac{3}{2}} e^{-y} \big[1 + O(\frac{1}{y}) + O(e^{-y})\big]. \tag{2.257}$$

At low temperature, finite temperature effects in massive, (relativistic) scalar field theory are exponentially suppressed.

At high temperature, $y \ll 1$, we find from (2.255) that, to lowest order,

$$(\frac{F(\beta)}{V})_T (y = 0) = \frac{\beta^{-4}}{2\pi^2} \int_0^\infty dx x^2 \ln(1 - e^{-x}) = -\beta^{-4} \frac{\pi^2}{90}. \tag{2.258}$$

cf. (1.104) for the evaluation of the integral. Since $y = 0$ corresponds to the massless theory, it is not surprising that the partition function for a single massless scalar field gives half the black body result.

2.8.5 Functional approach to massless scalar partition function in large volume limit

The functional evaluation of partition functions [20] (see also e.g. [21, 22] for earlier connected work, [23] for the inclusion of chemical potentials and [24] for a review) starts from the Hamiltonian path integral representation

$$Z_d(\beta, \mu^j) = \int \prod_{x^a} \mathrm{d}\phi(x^a) \prod_{x^a} \frac{\mathrm{d}\pi(x^a)}{2\pi} e^{-S^E_H}, \tag{2.259}$$

where $x^a = (x^1, \dots, x^{d+1})$ and the first order Euclidean action is

$$S_H^E[\phi, \pi] = \int_0^\beta dx^{d+1} \big[\int_{V_d} d^d x (-i\pi \partial_{d+1}\phi) + (H - i\mu^j P_j) \big]. \tag{2.260}$$

The sum is over periodic phase space paths of period $\beta = L_{d+1}$ in Euclidean time x^{d+1},

$$\phi(x^i, x^{d+1} + \beta) = \phi(x^i, x^{d+1}), \quad \pi(x^i, x^{d+1} + \beta) = \pi(x^i, x^{d+1}). \tag{2.261}$$

After integration over the momenta, this leads to

$$Z_d(\beta, \mu^j) = \big(\det[2\pi\delta^{d+1}(x', x)]\big)^{-\frac{1}{2}} \int \prod_{x^a} d\phi(x^a)\, e^{-S_L^E} \tag{2.262}$$

with

$$S_L^E[\phi] = \frac{1}{2} \int_{V_{d+1}} d^{d+1}x [(\partial_{d+1}\phi - \mu^j \partial_j \phi)^2 + \partial_j \phi \partial^j \phi], \tag{2.263}$$

where the integration is now over a $(d+1)$-dimensional hyperrectangle in Euclidean space of volume $V_{d+1} = V_d \beta = \prod_{a=1}^{d+1} L_a$. Except for the replacement $\partial_{d+1} \to \partial_{d+1} - \mu^j \partial_j$, the operator in the action is minus the Laplacian in $d+1$ dimensions with periodic boundary conditions in all dimensions. Since the eigenfunctions are $\frac{1}{\sqrt{V_{d+1}}} e^{ik_a x^a}$ with $k_{d+1} = \frac{2\pi n_{d+1}}{\beta}$, the eigenvalues are

$$\lambda_{n_a} = \Big[\Big(\frac{2\pi n_{d+1}}{\beta} - \sum_j \mu^j \frac{2\pi n_j}{L_j} \Big)^2 + \sum_i \Big(\frac{2\pi n_i}{L_i} \Big)^2 \Big]. \tag{2.264}$$

In order to define the partition function, formally given by

$$Z_d(\beta, \mu^j) = \prod_{n_a \in \mathbb{Z}^{d+1}}{}' \sqrt{\frac{2\pi}{\lambda_{n_a}}} = \Big[\det(\frac{-\Delta_\mu}{2\pi}) \Big]^{-\frac{1}{2}}, \tag{2.265}$$

one introduces a parameter ν of dimension inverse length and considers the dimensionless operators $A_\mu = -\Delta_\mu / 2\pi\nu^2$. The associated zeta function is defined by

$$\zeta_{A_\mu}(s) = \sum_{n_a \in \mathbb{Z}^{d+1}}{}' \lambda_{n_a}^{-s} (2\pi\nu^2)^s. \tag{2.266}$$

In these terms, the partition function is given by

$$\ln Z_d(\beta, \mu^j) = \frac{1}{2}\zeta'_{A_\mu}(0) = \frac{1}{2}\zeta'_{-\Delta_\mu}(0) + \frac{1}{2}\ln(2\pi\nu^2)\zeta_{-\Delta_\mu}(0). \tag{2.267}$$

Let us recover the canonical result of section 1.4.2 through the functional approach. In this case, in the absence of chemical potentials, $\mu^j = 0$, $A_0 = -\frac{\Delta_0}{2\pi\nu^2}$ we again turn the sums over the n_i into integrals,

$$\zeta_{A_0}(s) = (2\pi\nu^2)^s \frac{V_d}{(2\pi)^d} \int d^d k \sum_{n_{d+1} \in \mathbb{Z}} [(\frac{2\pi n_{d+1}}{\beta})^2 + k_i k^i]^{-s}, \tag{2.268}$$

From the Euclidean spacetime point of view, there is thus only one small dimension x^{d+1} of length β. After performing the integrals over the angles in hyperspherical coordinates, we have

$$\zeta_{A_0}(s) = (2\pi\nu^2)^s \frac{V_d 2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})} \int_0^\infty dk\, k^{d-1} \sum_{n_{d+1} \in \mathbb{Z}} [(\frac{2\pi n_{d+1}}{\beta})^2 + k^2]^{-s} \tag{2.269}$$

The divergent integral contained in the term at $n_{d+1} = 0$ is regulated through an infrared cut-off ϵ,

$$\int_\epsilon^\infty dk\; k^{d-1-2s} = -\frac{\epsilon^{d-2s}}{d-2s}, \quad \mathfrak{Re}(s) > \frac{d}{2}. \tag{2.270}$$

This expression and its derivative with respect to s both vanish at $s = 0$ in the limit when $\epsilon \to 0$ and can thus be discarded. In the remaining terms, the integrals converge for $\mathfrak{Re}(s) > \frac{d}{2}$. After the change of variables, $k = \frac{2\pi}{\beta} y$, we are left with

$$\zeta_{A_0}(s) = (2\pi\nu^2)^s \frac{V_d 2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})} (\frac{2\pi}{\beta})^{d-2s} \int_0^\infty dy\; y^{d-1} \sum_{n_{d+1}\in\mathbb{Z}}{}' [(n_{d+1})^2 + y^2]^{-s}. \tag{2.271}$$

The additional change of variables $y = n_{d+1} \sinh x$ together with

$$\int_0^\infty \mathrm{d}x \frac{\sinh^{d-1}(x)}{\cosh^{2s-1}(x)} = \frac{1}{2} \frac{\Gamma(\frac{d}{2})\Gamma(s-\frac{d}{2})}{\Gamma(s)}, \tag{2.272}$$

then yields

$$\zeta_{A_0}(s) = (2\pi\nu^2)^s \frac{V_d 2\pi^{\frac{d}{2}}}{(2\pi)^d} (\frac{2\pi}{\beta})^{d-2s} \frac{\Gamma(s-\frac{d}{2})\zeta(2s-d)}{\Gamma(s)}, \tag{2.273}$$

which can be written more compactly as

$$\zeta_{A_0}(s) = \frac{\nu^{2s} V_d}{\beta^{d-2s}} 2^{1-s} \frac{\xi(2s-d)}{\Gamma(s)}. \tag{2.274}$$

Since $\frac{1}{\Gamma(s)} = s + O(s^2)$, it follows that $\zeta_{A_0}(0) = 0$ and

$$\ln Z_d(\beta) = \xi(-d) \frac{V_d}{\beta^d}. \tag{2.275}$$

When using the reflection formula

$$\pi^{-\frac{z}{2}} \Gamma(\frac{z}{2}) \zeta(z) = \pi^{\frac{z-1}{2}} \Gamma(\frac{1-z}{2}) \zeta(1-z) \iff \xi(z) = \xi(1-z). \tag{2.276}$$

at $z = -d$, this agrees with the result derived by canonical methods in section 1.4.2.

2.8.6 Exercises

2.8.6.1 Thermal 2-point function in the operator formalism

Derive directly equation (2.245) in the operator formalism.
(Solution: [19], Chapter 1, appendix A, page 9.)

Chapter 3

Symmetries and Ward identities

cf. [12].

3.1 Finite transformations

3.1.1 General case

Let us consider transformations that can involve both a change of spacetime coordinates and a redefinition of the fields,

$$\begin{cases} x'^{\mu} = x'^{\mu}(x) \\ \phi'^{i}(x') = F^{i}[\phi(x)]. \end{cases} \tag{3.1}$$

3.1.2 Examples: Poincaré transformations and dilatations

For instance, for Poincaré transformations, we consider

$$\begin{cases} x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu} \\ \phi'^{i}(x') = L^{-1\,i}{}_{j}(\Lambda)\phi^{j}(x), \end{cases} \tag{3.2}$$

with $L^{i}{}_{j}(\Lambda)$ a matrix representation of the Lorentz group. This means that if $g = (\Lambda, a)$ denotes an element of the Poincaré group, with group law and inverse given by

$$g_1 g_2 = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1), \quad g^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a), \tag{3.3}$$

then a Poincaré transformation at x acts on the fields as a linear transformation of the fields at $g^{-1}x = \Lambda^{-1}(x-a)$ through

$$g \circ \phi^{i}(x) = L^{-1\,i}{}_{j}(\Lambda)\phi^{j}(g^{-1}x) \quad \text{with } L(\Lambda_1)L(\Lambda_2) = L(\Lambda_1\Lambda_2). \tag{3.4}$$

These transformations form a (reducible) representation of the Poincaré group:

$$\begin{aligned} g_1 \circ (g_2 \circ \phi^{i}(x)) &= g_1 \circ (L^{-1\,i}{}_{j}(\Lambda_2)\phi^{j}(g_2^{-1}x)) = \\ &= L^{-1\,i}{}_{j}(\Lambda_2)L^{-1\,j}{}_{k}(\Lambda_1)\phi^{k}(g_2^{-1}g_1^{-1}x) = (g_1g_2) \circ \phi^{i}(x). \end{aligned} \tag{3.5}$$

In particular, for a scalar, vector or spinor field, we have

$$\phi'(x') = \phi(x), \tag{3.6}$$

$$A'_{\mu}(x') = \Lambda_{\mu}{}^{\nu}A_{\nu}(x), \tag{3.7}$$

$$\Psi'(x') = S^{-1}(\Lambda)\Psi(x). \tag{3.8}$$

Another example of a transformation that we are interested in is a spacetime dilatation,

$$x' = \lambda x, \quad \phi'^i(x') = \lambda^{-\Delta_{(i)}}\phi^i(x), \tag{3.9}$$

with $\Delta_{(i)}$ the canonical dimension of the field (and parenthesis meaning that the summation convention is suspended).

3.2 Finite symmetries

3.2.1 General case

Consider then an action of the type

$$S[\phi] = \int d^n x\, \mathcal{L}(x, \phi^i, \frac{\partial\phi^i}{\partial x^\mu}). \tag{3.10}$$

A transformation is a symmetry if the action is invariant,

$$S[\phi'] = S[\phi]. \tag{3.11}$$

We have

$$S[\phi'] = \int d^n x\, \mathcal{L}(x, \phi'^i(x), \frac{\partial}{\partial x^\mu}\phi'^i(x)) = \int d^n x'\, \mathcal{L}(x', \phi'^i(x'), \frac{\partial}{\partial x'^\mu}\phi'^i(x')) =$$
$$= \int d^n x\, |\frac{\partial x'}{\partial x}|\mathcal{L}\Big(x'(x), F^i[\phi(x)], \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}F^i[\phi(x)]\Big).$$

The symmetry condition is satisfied if [1]

$$|\frac{\partial x'}{\partial x}|\mathcal{L}\Big(x'(x), F^i[\phi(x)], \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}F^i[\phi(x)]) = \mathcal{L}(x, \phi^i, \frac{\partial}{\partial x^\mu}\phi^i\Big). \tag{3.12}$$

3.2.2 Examples: Poincaré transformations and dilatations

For instance, for Poincaré transformations with $\Lambda \in L^\uparrow_+$, we have $|\frac{\partial x'}{\partial x}| = 1$. It follows in particular that spacetime translations are symmetries of the action if the Lagrangian does not depend explicitly on x^μ. The standard Lagrangians

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{k!}\phi^k, && (3.13)\\
\mathcal{L} &= -\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi, && (3.14)\\
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, && (3.15)
\end{aligned}$$

do not depend explicitly on time. They are also Lorentz and thus Poincaré invariant since $S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = \Lambda^{-1\mu}{}_\nu\gamma^\nu$, $\beta S(\Lambda)^\dagger\beta = S^{-1}(\Lambda)$ and $\Lambda^T\eta\Lambda = \eta$.

For a dilatation, $|\frac{\partial x'}{\partial x}| = \lambda^n$. An Lagrangian that does not depend explicitly on x^μ is invariant under a dilatation if all terms have dimension n. Since $\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \lambda^{-1}\frac{\partial}{\partial x^\mu}$, the canonical dimension of a derivative is 1. If $m = 0$, the kinetic term of the scalar field Lagrangian is invariant if $\Delta_\phi = \frac{n-2}{2}(= 1$ for $n = 4)$. In 4 dimensions, the ϕ^4 interaction preserves invariance, while for $n = 6$, the Lagrangian is invariant for a ϕ^3 interaction.

The Dirac Lagrangian is invariant if $m = 0$ and $\Delta_\psi = \frac{n-1}{2}(= \frac{3}{2}$ for $n = 4)$, while the electromagnetic Lagrangian is invariant for $\Delta_{A_\mu} = \frac{n-2}{2}(= 1$ for $n = 4)$.

[1]Note that one could also admit that the transformed and the old Lagrangian be equal only up to a total derivative. Unlike for the Galilean boosts in classical mechanics, we will however not need this more general condition here in the discussion of Poincaré and dilatation invariance.

3.3 Infinitesimal transformations

3.3.1 General case

For infinitesimal transformations, we have

$$\begin{cases} x'^\mu = x^\mu + \epsilon\xi^\mu(x), \\ \phi'^i(x') = \phi^i(x) + \epsilon f^i(x) \end{cases}, \tag{3.16}$$

which gives

$$\delta_\epsilon\phi^i(x) \equiv \phi'^i(x) - \phi^i(x) = \epsilon(f^i(x) - \xi^\mu\partial_\mu\phi^i(x)), \tag{3.17}$$

$$\delta_\epsilon\partial_\nu\phi^i = \partial_\nu\delta_\epsilon\phi^i. \tag{3.18}$$

The theory is said to be invariant under infinitesimal transformations if

$$\delta_\epsilon S = 0. \tag{3.19}$$

When the fields decrease sufficiently fast at infinity, this is equivalent to the requirement that

$$\boxed{\delta_\epsilon\mathcal{L} = \partial_\mu k^\mu_\epsilon}, \tag{3.20}$$

for some k^μ_ϵ. More generally, we will take this as the condition which $\delta_\epsilon\phi^i$ has to satisfy in order to define an (infinitesimal) symmetry.

Let us now show that the infinitesimal transformations associated to finite symmetries define an infinitesimal symmetry. If the finite symmetry depends continuously on parameters θ^a,

$$x' = X(x,\theta), \quad X(x,0) = x, \tag{3.21}$$

$$\phi'^i(x') = F^i[\phi(x),\theta] \quad F^i[\phi(x),0] = \phi^i(x), \tag{3.22}$$

and ϵ^a denotes an infinitesimal variation of θ^a at $\theta^a = 0$, the infinitesimal transformations associated to a finite transformation are given by

$$\epsilon\xi^\mu = \epsilon^a\frac{\partial X^\mu}{\partial\theta^a}|_{\theta=0}, \quad \epsilon f^i = \epsilon^a\frac{\partial F^i}{\partial\theta^a}|_{\theta=0}, \tag{3.23}$$

$$\delta_\epsilon\phi^i = \epsilon^a(\frac{\partial F^i}{\partial\theta^a}|_{\theta=0} - \frac{\partial X^\mu}{\partial\theta^a}|_{\theta=0}\partial_\mu\phi^i). \tag{3.24}$$

Since

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \frac{\partial}{\partial x^\nu}(\epsilon\frac{\partial X^\mu}{\partial\theta}|), \tag{3.25}$$

the LHS of (3.12) can be written as

$$(1+\partial_\mu(\epsilon\frac{\partial X^\mu}{\partial\theta}|))\mathcal{L}\big(x+\epsilon\frac{\partial X}{\partial\theta}|, \phi+\epsilon\frac{\partial\mathcal{F}}{\partial\theta}|, (\delta^\nu_\mu - \frac{\partial}{\partial x^\mu}(\epsilon\frac{\partial X^\nu}{\partial\theta}|))\partial_\nu(\phi+\epsilon\frac{\partial\mathcal{F}}{\partial\theta}|)\big) + O(\epsilon^2). \tag{3.26}$$

Invariance of the action for finite transformation in the form of (3.12) then implies in particular that the terms linear in ϵ in the above expression vanish.

More explicitly, this becomes

$$\partial_\mu(\frac{\partial X^\mu}{\partial\theta}|)\mathcal{L} + (\frac{\partial X^\mu}{\partial\theta}|)\frac{\partial\mathcal{L}}{\partial x^\mu} + \frac{\partial\mathcal{F}}{\partial\theta}|\frac{\partial\mathcal{L}}{\partial\phi} + (\partial_\mu\frac{\partial\mathcal{F}}{\partial\theta}| - \partial_\mu\frac{\partial X^\nu}{\partial\theta}|\partial_\nu\phi)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0.$$

When integrating the first term by parts, this gives

$$\partial_\mu(\frac{\partial X^\mu}{\partial\theta}|\mathcal{L}) - \frac{\partial X^\mu}{\partial\theta}|(\partial_\mu\phi\frac{\partial\mathcal{L}}{\partial\phi} + \partial_\mu\partial_\nu\phi\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}) + \frac{\partial\mathcal{F}}{\partial\theta}|\frac{\partial\mathcal{L}}{\partial\phi} + (\partial_\mu\frac{\partial\mathcal{F}}{\partial\theta}| - \partial_\mu\frac{\partial X^\nu}{\partial\theta}|\partial_\nu\phi)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0.$$

Regrouping the various terms, we have shown:

Under the associated infinitesimal transformation, the Lagrangian is invariant up to a total derivative,

$$\delta_\epsilon\phi\frac{\partial\mathcal{L}}{\partial\phi} + \partial_\mu(\delta_\epsilon\phi)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = -\partial_\mu(\epsilon\frac{\partial X^\mu}{\partial\theta}|\mathcal{L}). \tag{3.27}$$

Note that the total derivative is determined by the transformations of the spacetime coordinates as $k^\mu_\epsilon = -\epsilon\frac{\partial X^\mu}{\partial\theta}|\mathcal{L}$.

3.3.2 Examples: Infinitesimal Poincaré transformations and dilatations

For an infinitesimal translation for instance, $a^\mu = \epsilon^\mu$, the associated field variation is simply

$$\delta_\epsilon\phi^i = -\epsilon^\mu\partial_\mu\phi^i. \tag{3.28}$$

For an infinitesimal Lorentz transformation, $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and $L^i{}_j(\Lambda) = \delta^i_j + \frac{1}{2}\omega_{\mu\nu}L^{\mu\nu}$ where $L^{\mu\nu}$ is a matrix representation of the Lie algebra associated to the Lorentz group, we have $x'^\alpha = x^\alpha + \omega^\alpha{}_\nu x^\nu$ and thus $\epsilon\xi^\alpha = \frac{1}{2}\omega_{\mu\nu}(\eta^{\alpha\mu}x^\nu - \eta^{\alpha\nu}x^\mu)$, which gives

$$\delta_\omega\phi^i \;=\; -\frac{1}{2}\omega_{\mu\nu}\big(L^{\mu\nu i}{}_j\phi^j + (x^\nu\partial^\mu - x^\mu\partial^\nu)\phi^i\big). \tag{3.29}$$

These definitions imply that the fields form a representation of the Poincaré algebra,

$$[\delta_{(\omega_1,\epsilon_1)}, \delta_{(\omega_2,\epsilon_2)}]\phi^i = \delta_{([\omega_1,\omega_2],\omega_1\epsilon_2-\omega_2\epsilon_1)}\phi^i. \tag{3.30}$$

For an infinitesimal dilatation $\lambda = 1 + \alpha$, one gets

$$\delta_\alpha\phi^i(x) = -\alpha(x^\nu\partial_\nu + \Delta_{(i)})\phi^i. \tag{3.31}$$

3.4 Noether theorem

3.4.1 General case

The first Noether theorem can be obtained from the previous equation, or more generally from (3.20) by an integrations by parts of the second term on the LHS that makes appear the Euler-Lagrange equations of motions,

$$\boxed{\delta_\epsilon\phi^i\frac{\delta\mathcal{L}}{\delta\phi^i} = \partial_\mu j^\mu_\epsilon}, \tag{3.32}$$

$$\boxed{j^\mu_\epsilon = k^\mu_\epsilon - \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\delta_\epsilon\phi^i}. \tag{3.33}$$

This means that the current j^μ_ϵ is conserved when the equations of motion are satisfied, which is sometimes denoted by

$$\partial_\mu j^\mu_\epsilon \approx 0. \tag{3.34}$$

As an alternative to the algebraic approach followed above, this result is often presented as follows: invariance of the action implies that the variation $\delta_\epsilon S = S[\phi'] - S[\phi]$ vanishes if ϵ is a constant parameter. If we now make the parameter "local", i.e., we consider instead a parameter that depends on the spacetime point, $\epsilon = \epsilon(x)$, one finds instead that

$$\delta_\epsilon S = -\int d^n x\, j^\mu\partial_\mu\epsilon(x) = \int d^n x\,\partial_\mu j^\mu\epsilon(x). \tag{3.35}$$

The second follows from an integrations by parts and the assumption that $\epsilon(x)$ is choosen so as to vanish at the boundary of spacetime. Since an arbitrary variation of the fields fixed at the end points leaves the action invariant at an extremum, i.e., for a solution to the Euler-Lagrange field equations, the LHS vanishes on solutions, and, since this needs to be true for arbitrary $\epsilon(x)$ in the inside, we recover (3.34).

For all solutions ϕ_s, the Noether charge

$$Q_\epsilon[\phi_s] = \int_\Sigma d^3x\, j^0_\epsilon|_{\phi_s}, \tag{3.36}$$

is time-independent whenever the spatial components j^i vanish at the boundary.

Indeed,

$$\frac{dQ}{dt}[\phi_s] = -\int_\Sigma d^3x\, \partial_i j^i_\epsilon|_{\phi_s} = \int_{\partial\Sigma} d\sigma_i j^i_\epsilon|_{\phi_s} = 0.$$

3.4.2 Examples: Noether currents for Poincaré transformations and dilatations

For a translation, we find from (3.28) that

$$j^\mu_\epsilon = -\epsilon^\mu \mathcal{L} + \epsilon^\nu \partial_\nu \phi^i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^i} = -\epsilon_\nu T^{\mu\nu}, \tag{3.37}$$

$$T^{\mu\nu} = -\partial^\nu \phi^i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^i} + \eta^{\mu\nu}\mathcal{L}. \tag{3.38}$$

The associated charges

$$P^\nu = \int d^3x\, T^{0\nu}, \tag{3.39}$$

form the energy-momentum of the field theory.

For Lorentz transformations, we find instead that

$$j^\alpha_\omega = \frac{1}{2}\omega_{\mu\nu}\Big[\big(L^{\mu\nu i}{}_j \phi^j + (x^\nu\partial^\mu - x^\mu\partial^\nu)\phi^i\big)\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^i} - (\eta^{\alpha\mu}x^\nu - \eta^{\alpha\nu}x^\mu)\mathcal{L}\Big] \tag{3.40}$$

$$= \frac{1}{2}\omega_{\mu\nu}\Big[L^{\mu\nu i}{}_j \phi^j \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^i} + T^{\alpha\nu}x^\mu - T^{\alpha\mu}x^\nu\Big]. \tag{3.41}$$

Finally, for dilatations, the Nother current is

$$j^\mu_\alpha = \alpha\big(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^i}(x^\nu \partial_\nu + \Delta_{(i)})\phi^i - x^\mu \mathcal{L}\big) \tag{3.42}$$

$$= \alpha\big(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^i}\Delta_{(i)}\phi^i - T^\mu{}_\nu x^\nu\big). \tag{3.43}$$

One may show that the Noether charges associated to Poincaré transformations and evaluated on solutions,

$$Q_{\omega,\epsilon} = -\epsilon_\mu P^\mu + \frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}, \tag{3.44}$$

with $\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} = \int d^3x\, j^0_\omega$ form a representation of the Poincaré algebra,

$$[Q_{\omega_1,\epsilon_1}, Q_{\omega_2,\epsilon_2}] \equiv \delta_{\omega_1,\epsilon_1} Q_{\omega_2,\epsilon_2} = Q_{([\omega_1,\omega_2],\omega_1\epsilon_2-\omega_2\epsilon_1)}. \tag{3.45}$$

A general proof, along the lines of the result in the Hamiltonian formalism, goes as follows.
Let $\delta_a\phi^i = Q^i_a$, be linearly independent infinitesimal field transformations. If these infinitesimal transformations define symmetries, then so does the commutator $\delta_{[a,b]}\phi^i := [\delta_a,\delta_b]\phi^i = \delta_a Q^i_b - \delta_b Q^i_a$ because $[\delta_a,\delta_b]\mathcal{L} = \partial_\mu k^\mu_{[a,b]}$ with $k^\mu_{[a,b]} := \delta_a k^\mu_b - \delta_b k^\mu_a$. Since the commutator satisfies the Jacobi identity, it follows that infinitesimal symmetries form a Lie algebra. Note also that the Noether current associated to the commutator is $j^\mu_{[a,b]} := \delta_a k^\mu_b - \delta_b k^\mu_a - \dfrac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\,\delta_{[a,b]}\phi^i$,

$$\delta_{[a,b]}\phi^i\frac{\delta\mathcal{L}}{\delta\phi^i} = \partial_\mu j^\mu_{[a,b]}. \tag{3.46}$$

When acting with a symmetry $\delta_{e_a}\phi^i$ on the definition of the Noether current associated to another symmetry $\delta_b\phi^i$, one finds

$$\delta_a(Q^i_b)\frac{\delta\mathcal{L}}{\delta\phi^i} + Q^i_b\delta_a(\frac{\delta\mathcal{L}}{\delta\phi^i}) = \partial_\mu(\delta_a j^\mu_b), \tag{3.47}$$

If $\mathcal{L}$ and Q_a depend at most on first order derivatives, one can show by direct computation that

$$\delta_a\frac{\delta\mathcal{L}}{\delta\phi^i} = \frac{\delta(\delta_a\mathcal{L})}{\delta\phi^i} - \frac{\partial Q^j_a}{\partial\phi^i}\frac{\delta\mathcal{L}}{\delta\phi^j} + \partial_\mu(\frac{\partial Q^j_a}{\partial\partial_\mu\phi^i}\frac{\delta\mathcal{L}}{\delta\phi^j}).$$

The first term on the RHS vanishes because $\delta_a\phi^i$ is a symmetry and Euler-Lagrange derivatives annihilate total divergences. When subsituting into (3.47) and integrating by parts in the last term we get

$$\delta_{[a,b]}\phi^i\frac{\delta\mathcal{L}}{\delta\phi^i} = \partial_\mu(\delta_a j^\mu_b - K^i_b\frac{\partial K^j_a}{\partial\partial_\mu\phi^i}\frac{\delta\mathcal{L}}{\delta\phi^j}). \tag{3.48}$$

Substracting from (3.46) and using a (suitably adapted) version of the Poincaré lemma, one finds

$$\delta_a j^\mu_b = j^\mu_{[a,b]} + K^i_b\frac{\partial K^j_a}{\partial\partial_\mu\phi^i}\frac{\delta\mathcal{L}}{\delta\phi^j} + \partial_\nu k^{[\mu\nu]}_{a,b}, \tag{3.49}$$

with $k^{[\mu\nu]}_{a,b} = -k^{[\nu\mu]}_{a,b}$. For the Noether charge, the last two terms are irrelevant,

$$Q_{[a,b]} := \delta_a Q_b, \tag{3.50}$$

because the first term vanishes on solutions of the Euler-Lagrange equations of motion, while the second term vanishes when using Stokes' theorem and the assumptions that the fields vanish at the boundary.
A sub-Lie algebra of the Lie algebra of symmetries forms a realization of a Lie algebra with generators e_a and structure constants f^c_{ab} with $[e_a,e_b] = f^c_{ab}e_c$, if there exists a basis such that $[\delta_a,\delta_b]\phi^i = f^c_{ab}\delta_c\phi^i$. What we have shown is that the associated Noether charges, if there is no contribution from boundary terms, satisfy

$$\delta_a Q_b = f^c_{ab}Q_c. \tag{3.51}$$

NB: This holds if the (algebraic) Poincaré lemma can be used and boundary terms can be neglected. If $d = dx^\mu\partial_\mu$ has non trivial cohomology, such as the constants in classical mechanics, this relation might be violated.

3.5 Ward identities

3.5.1 Finite transformations

We have

$$Z[J] = \int \mathcal{D}\phi\, e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)}$$
$$= \sum_k (\frac{i}{\hbar})^k \frac{1}{k!} \int d^dx_1 J_{i_1}(x_1)\cdots \int d^dx_k J_{i_k}(x_k)\langle +\infty, 0|T\{\widehat{\phi}^{i_1}(x_1)\dots\widehat{\phi}^{i_k}(x_k)\}|0,-\infty\rangle \quad (3.52)$$

Invariance of the action implies that, under the transformations $\phi'^i(x') = \phi^i(x)$, the action is invariant, $S[\phi'] = S[\phi]$. If we assume in addition that the measure is invariant, $\mathcal{D}\phi' = \mathcal{D}\phi$, that is to say that the Jacobian is the identity, we get

$$\boxed{\langle +\infty, 0|T\{\widehat{\phi}^{i_1}(x_1')\dots\widehat{\phi}^{i_n}(x_n')\}|0,-\infty\rangle = \langle +\infty, 0|T\{\widehat{F}^{i_1}[\phi(x_1)]\dots\widehat{F}^{i_n}[\phi(x_n)]\}|0,-\infty\rangle}. \quad (3.53)$$

Indeed,

$$Z[J] = \int \mathcal{D}\phi'\, e^{\frac{i}{\hbar}(S[\phi']+\int d^dx\, J_i(x)\phi'^i(x))} \quad (3.54)$$
$$= \int \mathcal{D}\phi\, e^{\frac{i}{\hbar}(S[\phi]+\int d^dx'\, J_i(x')\phi'^i(x'))} \quad (3.55)$$
$$= \int \mathcal{D}\phi\, e^{\frac{i}{\hbar}(S[\phi]+\int d^dx'\, J_i(x')F^i[\phi(x)])}. \quad (3.56)$$

The result follows by differentiating (3.52) and thus last expression with respect to $J_{i_k}(x_k')$.

In particular, for theories that are invariant under translations, Lorentz transformations or dilatations, this gives,

$$\langle +\infty, 0|T\prod_k \widehat{\phi}^{i_k}(x_k+a)|0,-\infty\rangle = \langle +\infty, 0|T\prod_k \widehat{\phi}^{i_k}(x_k)|0,-\infty\rangle, \quad (3.57)$$

$$\langle +\infty, 0|T\prod_k \widehat{\phi}^{i_k}(\Lambda x_k)|0,-\infty\rangle = \langle +\infty, 0|T\prod_k {L^{-1}}^{i_k}{}_{j_k}(\Lambda)\widehat{\phi}^{j_k}(x_k)|0,-\infty\rangle, \quad (3.58)$$

$$\langle +\infty, 0|T\prod_k \widehat{\phi}^{i_k}(\lambda x_k)|0,-\infty\rangle = \lambda^{-\sum_k \Delta_{(i_k)}}\langle +\infty, 0|T\prod_k \widehat{\phi}^{i_k}(x_k)|0,-\infty\rangle. \quad (3.59)$$

For dilatations, we will see later that the absorption of divergences leads to corrections of order $\hbar$ to this relation (that has been derived formally under the additional assumption of invariance of the measure).

3.5.2 Infinitesimal transformations

For an infinitesimal transformation, since ϕ is a dummy integration variable in the path integral, we have

$$Z[J] = \int \mathcal{D}(\phi+\delta\phi)\, e^{\frac{i}{\hbar}(S[\phi+\delta\phi]+J_A(\phi^A+\delta\phi^A))} =$$
$$= \int \mathcal{D}\phi\, (1+\frac{i}{\hbar}(\delta S + J_B\delta\phi^B))e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)}. \quad (3.60)$$

Again, if the measure is invariant, the terms linear in $\delta\phi$ have to vanish,

$$\int \mathcal{D}\phi\, (\delta S + J_B\delta\phi^B)e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)} = 0. \quad (3.61)$$

If $\delta\phi^i(x) = K^i[\phi]$ is a symmetry,

$$\int d^dx\, J_i(x) K^i[\frac{\hbar}{i}\frac{\delta}{\delta J}]\, Z[J] = 0. \tag{3.62}$$

Furthermore, if $K^i_j[\phi^j(x)] = K^i_j\phi^j(x) + K^{i\mu}_j\partial_\mu\phi^j(x) + \dots$ is linear, multiple derivations with respect to $J_i(x)$ give

$$\sum_{k=1}^{m}\langle +\infty, 0|T\{\widehat{\phi}^{i_1}(x_1)\dots K^{i_k}_j[\widehat{\phi}^j(x_k)]\dots\widehat{\phi}^{i_m}(x_m)\}|0,-\infty\rangle = 0. \tag{3.63}$$

3.5.3 Schwinger-Dyson equations

Invariance of the path integral under translations implies that

$$0 = \int \mathcal{D}\phi\, \frac{\delta}{\delta\phi^B} e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)} = \int \mathcal{D}\phi\, \frac{i}{\hbar}(\frac{\delta S}{\delta\phi^B} + J_B) e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)}. \tag{3.64}$$

In turn, this gives the equations of motion satisfied by the Green's functions,

$$(\frac{\delta S}{\delta\phi^B}[\frac{\hbar}{i}\frac{\delta}{\delta J}] + J_B)\, Z[J] = 0. \tag{3.65}$$

For the scalar field for instance,

$$\Big[\int d^ny\, \mathcal{D}(x,y)\frac{\delta}{\delta J(y)} + \frac{\partial V_I}{\partial\phi}[\frac{\hbar}{i}\frac{\delta}{\delta J(x)}] - J(x)\Big]\, Z[J] = 0. \tag{3.66}$$

3.5.4 Local version

The quantum analog of the Noether relation $\partial_\mu j^\mu_K = K^i\frac{\delta\mathcal{L}}{\delta\phi^i}$ is

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\int \mathcal{D}\phi\, j^\mu_K(x) e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)} &= \int \mathcal{D}\phi\, K^i_j[\phi^j(x)]\frac{\delta S}{\delta\phi^i(x)} e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)} = \\ &= \int \mathcal{D}\phi\, K^i_j[\phi^j(x)]\big(\frac{\hbar}{i}\frac{\delta}{\delta\phi^i(x)} - J_i(x)\big) e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)}.\end{aligned} \tag{3.67}$$

Assuming that $\delta(0) = 0$ and that the path integral is translation invariant, the first term vanishes, and one finds

$$\frac{\partial}{\partial x^\mu}\int \mathcal{D}\phi\, j^\mu_K e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)} = -J_i(x)K^i_j[\frac{\hbar}{i}\frac{\delta}{\delta J_j(x)}]Z[J]. \tag{3.68}$$

Derivation with respect to the sources yields

$$\begin{aligned}&\frac{\partial}{\partial x^\mu}\langle +\infty, 0|T\{\widehat{j}^\mu_K(x)\prod_{k=1}^{m}\widehat{\phi}^{i_k}(x_k)\}|0,-\infty\rangle = \\ &= -\frac{\hbar}{i}\sum_{k=1}^{m}\delta(x-x_k)\langle +\infty, 0|T\{\widehat{\phi}^{i_1}(x_1)\dots \widehat{K}^{i_k}_j[\phi^j(x_k)]\dots\widehat{\phi}^{i_m}(x_m)\}|0,-\infty\rangle.\end{aligned} \tag{3.69}$$

In the same manner, for insertions of Noether currents, one finds

$$\frac{\partial}{\partial x^\mu}\langle +\infty, 0|T\{\widehat{j}^\mu_a(x)\prod_{k=1}^m \widehat{j}^{\nu_k}_{b_k}(x_k)\}|0,-\infty\rangle =$$

$$= -\frac{\hbar}{i}\sum_{k=1}^m \delta(x-x_k)\langle +\infty, 0|T\prod_{k=1}^m \widehat{j}^{\nu_k}_{[a,b]}(x_k)|0,-\infty\rangle. \quad (3.70)$$

in the case where (3.49) simplifies to $\delta_a j^\mu_b = j^\mu_{[a,b]}$.

These derivations are based on the assumptions that the measure is invariant and/or that that $\delta(0) = 0$. In the renormalized theory that we are going to construct later on, this is not always justified. The above Ward identities are then only valid to lowest order in $\hbar$. In other words, there may be quantum corrections of higher order in $\hbar$ to the above relations.

Chapter 4

Functional methods

This chapter is based on [25], [9], [10], [16], [26], [27], [28].

4.1 Generating functional for connected Green's functions

4.1.1 Normalized generating functional and logarithm

The generating function for Green's function may be written as

$$Z[J] = \int \mathcal{D}\phi \; e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)}, \tag{4.1}$$

while the generating functional for normalized Green's functions is

$$\begin{aligned} \frac{Z[J]}{Z[0]} &= \sum_k \frac{(i/\hbar)^k}{k!} J_{A_1}\dots J_{A_k}\langle\widehat{\phi}^{A_1}\dots\widehat{\phi}^{A_k}\rangle, &(4.2)\\ \langle\widehat{\phi}^{A_1}\dots\widehat{\phi}^{A_k}\rangle &= \frac{\langle+\infty,0|T\prod_k\widehat{\phi}^{A_k}|0,-\infty\rangle}{\langle+\infty,0|0,-\infty\rangle}. &(4.3)\end{aligned}$$

We have already seen that this corresponds to summing over Feynman diagrams without vacuum parts.

In the following, it will be useful to consider Green's functions in the presence of the external source,

$$\langle\widehat{\phi}^{A_1}\dots\widehat{\phi}^{A_k}\rangle^J = \frac{1}{Z[J]}(\frac{\hbar}{i})^k \frac{\delta^k}{\delta J_{A_1}\dots\delta J_{A_k}} Z[J]. \tag{4.4}$$

Let

$$W[J] = \frac{\hbar}{i}\ln(\frac{Z[J]}{Z[0]}) \iff \boxed{\frac{Z[J]}{Z[0]} = e^{\frac{i}{\hbar}W[J]}}, \quad W[0]=0. \tag{4.5}$$

For an arbitrary polynomial $X[\phi]$, one has

$$\langle X[\widehat{\phi}]\rangle^J = X[\frac{\delta W}{\delta J} + \frac{\hbar}{i}\frac{\delta}{\delta J}]\,1. \tag{4.6}$$

Indeed, $\langle\widehat{\phi}^{A_1}\rangle^J = \frac{1}{Z[J]}\frac{\hbar}{i}\frac{\delta Z[J]}{\delta J_{A_1}} = \frac{\delta W}{\delta J_{A_1}}$. Suppose this relation is true for a monomial of order m,

$$\begin{aligned}\langle\widehat{\phi}^{A_{m+1}}\widehat{\phi}^{A_m}\dots\widehat{\phi}^{A_1}\rangle^J &= \frac{1}{Z[J]}(\frac{\hbar}{i})^{m+1}\frac{\delta}{\delta J_{A_m+1}}\big(Z[J]\frac{1}{Z[J]}\frac{\delta^m Z[J]}{\delta J_{A_m}\dots\delta J_{A_1}}\big)\\ &= \frac{1}{Z[J]}\frac{\hbar}{i}\frac{\delta}{\delta J_{A_m+1}}\big(Z[J](\frac{\delta W}{\delta J_{A_m}}+\frac{\hbar}{i}\frac{\delta}{\delta J_{A_m}})\dots(\frac{\delta W}{\delta J_{A_1}}+\frac{\hbar}{i}\frac{\delta}{\delta J_{A_1}})\,1\big)\\ &= (\frac{\delta W}{\delta J_{A_{m+1}}}+\frac{\hbar}{i}\frac{\delta}{\delta J_{A_{m+1}}})\dots(\frac{\delta W}{\delta J_{A_1}}+\frac{\hbar}{i}\frac{\delta}{\delta J_{A_1}})\,1 \quad \square\end{aligned}$$

In particular,

$$\langle\widehat{\phi}^{A_{m+1}}\dots\widehat{\phi}^{A_1}\rangle^J = (\frac{\delta W}{\delta J_{A_{m+1}}}+\frac{\hbar}{i}\frac{\delta}{\delta J_{A_{m+1}}})\langle\widehat{\phi}^{A_m}\dots\widehat{\phi}^{A_1}\rangle^J. \tag{4.7}$$

4.1.2 Connected Green's functions

Connected Green's functions in the presence of sources are defined recursively through the relation

$$\langle\widehat{\phi}^{A_1}\dots\widehat{\phi}^{A_m}\rangle^J_c = \langle\widehat{\phi}^{A_1}\dots\widehat{\phi}^{A_m}\rangle^J - \sum_{\text{partitions}\,\{1\dots k\}}\Big(\prod\langle\widehat{\phi}^{A_{i_1}}\dots\widehat{\phi}^{A_{i_k}}\rangle^J_c \quad \text{for } k<m\Big), \tag{4.8}$$

together with $\langle\widehat{\phi}^{A_1}\rangle^J_c = \langle\widehat{\phi}^{A_1}\rangle^J$. In particular, this implies for instance that

$$\langle\widehat{\phi}^{A_1}\widehat{\phi}^{A_2}\rangle^J_c = \langle\widehat{\phi}^{A_1}\widehat{\phi}^{A_2}\rangle^J - \langle\widehat{\phi}^{A_1}\rangle^J\langle\widehat{\phi}^{A_2}\rangle^J. \tag{4.9}$$

As a consequence of the correspondance between the analytic expression and Feynman diagrams, it follows that connected Green's fonctions only involve the summation over connected Feynman diagrams.

explain this better

Proposition 1. $\frac{i}{\hbar}W[J]$ *is the generating functional for connected Green's functions.*

$$\langle\widehat{\phi}^{A_1}\dots\widehat{\phi}^{A_m}\rangle^J_c = (\frac{\hbar}{i})^m\frac{\delta^m(\frac{i}{\hbar}W[J])}{\delta J_{A_1}\dots\delta J_{A_m}} \iff \boxed{\frac{i}{\hbar}W[J] = \sum_{k=1}(\frac{\hbar}{i})^k\frac{1}{k!}J_{A_k}\dots J_{A_1}\langle\widehat{\phi}^{A_1}\dots\widehat{\phi}^{A_k}\rangle_c}. \tag{4.10}$$

Indeed, $\langle\widehat{\phi}^{A_1}\rangle^J_c = \langle\widehat{\phi}^{A_1}\rangle^J = \frac{\delta W}{\delta J_{A_1}}$. Suppose the relation true for a monomial of order m. When using (4.7),

$$\begin{aligned}&\langle\widehat{\phi}^{A_{m+1}}\widehat{\phi}^{A_m}\dots\widehat{\phi}^{A_1}\rangle^J = (\frac{\delta W}{\delta J_{A_{m+1}}}+\frac{\hbar}{i}\frac{\delta}{\delta J_{A_{m+1}}})\langle\widehat{\phi}^{A_m}\dots\widehat{\phi}^{A_1}\rangle^J\\ &= (\frac{\delta W}{\delta J_{A_{m+1}}}+\frac{\hbar}{i}\frac{\delta}{\delta J_{A_{m+1}}})\Big[\langle\widehat{\phi}^{A_m}\dots\widehat{\phi}^{A_1}\rangle^J_c + \sum_{\text{part}\,\{1\dots k\}}\Big(\prod\langle\widehat{\phi}^{A_{i_1}}\dots\widehat{\phi}^{A_{i_k}}\rangle^J_c,\ k<m\Big)\Big]\\ &= (\frac{\delta W}{\delta J_{A_{m+1}}}+\frac{\hbar}{i}\frac{\delta}{\delta J_{A_{m+1}}})\Big[(\frac{\hbar}{i})^{m-1}\frac{\delta^m W[J]}{\delta J_{A_m}\dots\delta J_{A_1}} + \sum_{\text{part}\,\{1\dots k\}}\Big(\prod(\frac{\hbar}{i})^{k-1}\frac{\delta^k W[J]}{\delta J_{A_{i_1}}\dots\delta J_{A_{i_k}}},\ k<m\Big)\Big]\\ &= (\frac{\hbar}{i})^m\frac{\delta^{m+1}W[J]}{\delta J_{A_{m+1}}\dots\delta J_{A_1}} + \frac{\hbar}{i}\frac{\delta}{\delta J_{A_{m+1}}}\sum_{\text{part}\,\{1\dots k\}}\Big(\prod(\frac{\hbar}{i})^{k-1}\frac{\delta^k W[J]}{\delta J_{A_{i_1}}\dots\delta J_{A_{i_k}}},\ k<m\Big)\\ &\qquad + \frac{\delta W}{\delta J_{A_{m+1}}}\sum_{\text{part}\,\{1\dots k\}}\Big(\prod(\frac{\hbar}{i})^{k-1}\frac{\delta^k W[J]}{\delta J_{A_{i_1}}\dots\delta J_{A_{i_k}}},\ k<m\Big) + \frac{\delta W}{\delta J_{A_{m+1}}}(\frac{\hbar}{i})^{m-1}\frac{\delta^m W[J]}{\delta J_{A_m}\dots\delta J_{A_1}}\\ &= (\frac{\hbar}{i})^m\frac{\delta^{m+1}W[J]}{\delta J_{A_{m+1}}\dots\delta J_{A_1}} + \sum_{\text{partitions}\,\{1\dots k\}}\Big(\prod\langle\widehat{\phi}^{A_{i_1}}\dots\widehat{\phi}^{A_{i_k}}\rangle^J_c \quad \text{pour } k<m+1\Big),\end{aligned}$$

where the recursion assumption has been used at the beginning and at the end.

When comparing with the definition, we thus find $(\frac{\hbar}{i})^m \dfrac{\delta^{m+1} W[J]}{\delta J_{A_{m+1}} \ldots \delta J_{A_1}} = \langle \widehat{\phi}^{A_{m+1}} \ldots \widehat{\phi}^{A_1} \rangle^J_c$.The last line uses the fact that in order to go from the the sum of all partitions of m objects to the partitions of $m+1$ objects, the object $m+1$

- can either be a singleton multliplying all already existing partitions,
- or can enter in the already existing partitions.

  For instance, partitions of 3 objects are given by $1\ 2\ 3$, $(12)\ 3$, $(13)\ 2$, $(23)\ 1$, (123). To go to the partitions of 4 objects, the singleton 4 may multiply all partitions of 3 objects, (which corresponds to the last two terms (line before last) of the above equation, $1\ 2\ 3\ 4$, $(12)\ 3\ 4$, $(13)\ 2\ 4$, $(23)\ 1\ 4$, $(123)\ 4$ or 4 may enter in all existing partitions, $(14)\ 2\ 3$, $1\ (24)\ 3$, $1\ 2\ (34)$, $(124)\ 3$, $(12)\ (34)$, $(134)\ 2$, $(13)\ (24)$, $(234)\ 1$, $(23)\ (14)$, (1234), which gives indeed all partitions of 4 objects.

Once proposition 1 is proved, it follows that equation (4.6) allows one to explicitly work out the decomposition of any Green's function in terms of connected Green's functions.

4.1.3 Classical field and invertibility

We introduce the notation

$$\phi^A_J = \langle \widehat{\phi}^A \rangle^J = \frac{\delta^L W}{\delta J^A}. \tag{4.11}$$

One assumes that this relation can be inverted to give J_A as a function of ϕ^A,

$$\forall \phi^A\ \exists !\ J^\phi_A \text{ such that } \phi^A = \frac{\delta^L W}{\delta J^A}\bigg|_{J=J^\phi}. \tag{4.12}$$

The field ϕ^A, which is a source replacing J_A in the Legendre transform to be discussed next, is sometimes called the "classical field". Inversion is possible perturbatively in J_A, if $\dfrac{\delta^L W}{\delta J^A}\bigg|_{J=0} \equiv \langle \widehat{\phi}^A \rangle^0 = 0$ and if $\dfrac{\delta^{L2} W}{\delta J^B \delta J^A}\bigg|_0$ is invertible.

Indeed, in this case

$$\phi^A_J = J_B\, \frac{\delta^{L2} W}{\delta J^B \delta J^A}\bigg|_0 + O(J^2) \tag{4.13}$$

and one may invert such a series perturbatively for which the first term if the matrix multiplying the linear term is invertible.

In particular, under these assumptions,

$$J_A = 0 \iff \phi^A = 0. \tag{4.14}$$

In the following, we assume that $\langle \widehat{\phi}^A \rangle^0 = 0$. Note that this relation holds in the free theory, $\langle 0 | T \widehat{\phi}^A | 0 \rangle = 0$ since the creation operators annihilate the vacuum on the left and destruction operators annihilate the vacuum on the right.

4.2 Effective action

4.2.1 Legendre transform

The effective action is defined as the Legendre transform of $W[J]$ with respect to J,

$$\phi^A = \frac{\delta^L W}{\delta J_A} \Longleftrightarrow J_A = J^\phi_A, \qquad \Gamma[\phi] = (W[J] - J_A\phi^A)\big|_{J^\phi}. \tag{4.15}$$

This implies

$$\frac{\delta^R \Gamma}{\delta \phi^A} = \frac{\delta^R W}{\delta J_B}\bigg|_{J^\phi} \frac{\delta^R J^\phi_B}{\delta\phi^A} - (-)^{|A||B|}\frac{\delta^R J^\phi_B}{\delta\phi^A}\phi^B - J^\phi_A = -J^\phi_A. \tag{4.16}$$

The name effective action is justified because the equations of motion of the classical action with sources $S + J_A\phi^A$ are very similar, $\frac{\delta^R S}{\delta\phi^A} = -J_A$. The sign factor $(-)^{|A||B|}$ is only relevant if there are fermionic fields, for which $|A| = 1$. For bosonic fields, $|A| = 0$. When taking into account that $\delta\phi^A\frac{\delta^L F}{\delta\phi^A} = \frac{\delta^R F}{\delta\phi^A}\delta\phi^A$, it follows that $\frac{\delta^L F}{\delta\phi^A} = (-)^{(|F|+|A|)|A|}\frac{\delta^R F}{\delta\phi^A}$, with $|F|$ the parity of the (homogeneous) function F. In the following, we often omit the bar around A or F in the sign factors.

The expansion in terms of classical fields of $\Gamma[\phi]$ is

$$\Gamma[\phi] = \frac{1}{2}\phi^{A_2}\Gamma_{A_1A_2}\phi^{A_1} + \widehat{\Gamma}[\phi], \qquad \widehat{\Gamma}[\phi] = \sum_{k\geqslant 3}\frac{1}{k!}\phi^{A_k}\ldots\phi^{A_2}\Gamma_{A_1\ldots A_k}\phi^{A_1}. \tag{4.17}$$

Indeed, the constant and linear terms are absent because of $W[0] = 0$ and of (4.14).

We also have

$$\Gamma''_{AB} = \frac{\delta^L}{\delta\phi^B}\frac{\delta^R\Gamma}{\delta\phi^A}, \qquad (\Gamma''_{AB})^{-1} = -\frac{\delta^{L2}W}{\delta J_A\delta J_B}\bigg|_{J=J^\phi} = -\frac{i}{\hbar}\langle\widehat{\phi}^A\widehat{\phi}^B\rangle^{J^\phi}_c, \tag{4.18}$$

$$\langle X[\widehat{\phi}]\rangle^{J^\phi} = X[\phi - \frac{\hbar}{i}(\Gamma'')^{-1}\frac{\delta^L}{\delta\phi}]1. \tag{4.19}$$

Indeed, $\frac{\delta^L}{\delta J_A}\Big|_{J^\phi} = \frac{\delta^L\phi^B_J}{\delta J_A}\Big|_{J^\phi}\frac{\delta^L}{\delta\phi^B}$. Starting from $J^\phi_A = -\frac{\delta^R\Gamma[\phi]}{\delta\phi^A}$, one finds $\frac{\delta^L J^\phi_A}{\delta\phi^B} = -\frac{\delta^L}{\delta\phi^B}\frac{\delta^R\Gamma[\phi]}{\delta\phi^A}$ and (4.18) is a consequence of

$$\frac{\delta^L\phi^B_J}{\delta J_C}\bigg|_{J^\phi}\frac{\delta^L J^\phi_A}{\delta\phi^B} = \delta^A_C, \qquad \frac{\delta^{L2}W}{\delta J_B\delta J_C}\bigg|_{J^\phi}(-\frac{\delta^L}{\delta\phi^B}\frac{\delta^R\Gamma[\phi]}{\delta\phi^A}) = \delta^A_C. \tag{4.20}$$

When expressing (4.6) in terms of classical fields, one finds

$$\langle X[\widehat{\phi}]\rangle^{J^\phi} = X[\phi + \frac{\hbar}{i}\frac{\delta^L}{\delta J}\bigg|_{J^\phi}]1. \tag{4.21}$$

When using the expression of the derivative with respect to J in terms of a derivative with respect to ϕ, one gets (4.19).

4.2.2 Connected diagrams and topological relations

Definitions:

- An amputated diagram is a diagram from which one has removed the external propagators (both in the diagram and in the corresponding analytic expression).

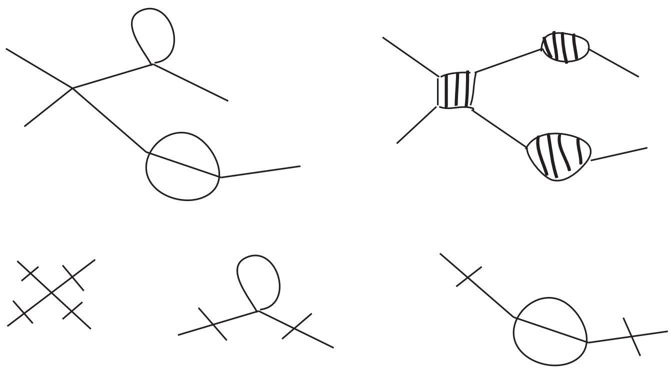
- A one-particle-irreducible diagram (1PI) is an amputated diagram that remains connected if one cuts an internal line.
- A tree diagram is a diagram without loops.
- The order of a diagram is the number of external points.
- A proper vertex of order n, $\Sigma_{A_1\ldots A_n}$, corresponds to the sum over all 1PI diagrams of order n.

Note that there may be poprer vertices of order n (e.g. $n = 2$) without having vertices of order n in the Lagrangian.

As a consequence of these definitions, it follows that

Every connected diagram may be decomposed into a tree diagram in which propagators connect 1PI diagrams.

Indeed, if one cuts an internal line, either the diagram becomes disconnected, and one has cut a propagator, or the diagram remains connected and one has cut in a 1PI diagram. The figure provides an example of a connected diagram in $\frac{g}{4!}\phi^4$ theory, and gives its decomposition into a connected tree diagram, relating the associated 1PI diagrams, 2 of orders 2 and 1 of order 4.



When summing these relation (and re-organizing the sum), one finds

$\sum$ (connected diagrams) = $\sum$ (connected tree diagrams) with (vertices of order $n \geqslant 2$ replaced by (proper vertex of order $n \geqslant 2$).

A connected diagram with L loops, V vertices and I internal lines satisfies

$$L = I - V + 1, \tag{4.22}$$

or, if there are C connected components,

$$\sum L = \sum I - \sum V + C. \tag{4.23}$$

Indeed, for a diagram with 1 vertex, $V = 1$, $L = 0$, $I = 0$, $C = 1$, and the relation holds. If one adds $V - 1$ vertices with just enough internal lines to keep the diagram connected, $I = V - 1$, $C = 1$, $L = 0$, and the relation still holds. In the case of the figure, $V = 5$, $I = 4$.

Every additional line gives a loop. For C connected components, one just sums the result for each connected component.

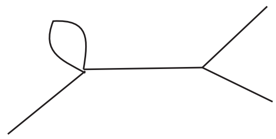
For a connected Feynman diagram, the expansion in $\hbar$ is related to an expansion in the number of loops. Indeed, a propagator brings $\frac{\hbar}{i}$ and a vertex brings $\frac{i}{\hbar}$. For a fixed number of external lines E, we thus have

$$\text{order in } \hbar \text{ of diagram} = \hbar^{E+I-V} = \hbar^{E-1+L}. \tag{4.24}$$

If there are N_i vertices of type i that involve n_i fields, and if there are E external lines, we have

$$2I + E = \sum_i N_i n_i. \tag{4.25}$$

Indeed, an external lines is attached to only one leg of a vertex, while an internal lines is attached to 2 legs of the same or of different vertices.



4.2.3 Complete propagator and proper vertex of order 2

Combining the previous two relations for a connected diagram by eliminating the number I of internal lines and using that $\sum_i N_i = V$, we get

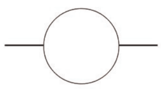
$$E - 2 = \sum_i N_i(n_i - 2) - 2L. \tag{4.26}$$

As a consequence, the sum over connected tree diagrams ($L = 0$) with 2 external legs ($E = 2$) can only contain the proper vertex of order 2 ($n_i = 2$) when there are no proper vertices with $n_i = 1$ in the theory. The latter holds because we assume $\langle \widehat{\phi}^A \rangle^0 = 0$.

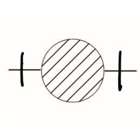
Denoting the 2-function

$$\langle \widehat{\phi}^A \widehat{\phi}^B \rangle^0_{(c)} = \frac{\hbar}{i} \frac{\delta^2 W[J]}{\delta J_A \delta J_B}|_{J=0}, \tag{4.27}$$

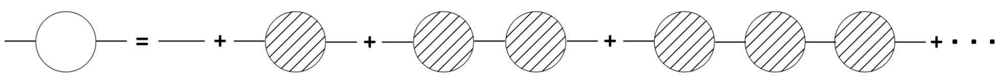
by



and the proper vertex of order 2 by



we thus have



so that

$$\begin{aligned}
\langle \widehat{\phi}^A \widehat{\phi}^B \rangle^0_{(c)} &= \frac{\hbar}{i}(\mathcal{D}^{-1})^{AB} + \frac{\hbar}{i}(\mathcal{D}^{-1})^{AC_1}\Sigma_{C_1C_2}\frac{\hbar}{i}(\mathcal{D}^{-1})^{C_2B} \qquad (4.28)\\
&\quad + \frac{\hbar}{i}(\mathcal{D}^{-1})^{AC_1}\Sigma_{C_1C_2}\frac{\hbar}{i}(\mathcal{D}^{-1})^{C_2C_3}\Sigma_{C_3C_4}\frac{\hbar}{i}(\mathcal{D}^{-1})^{C_4B} + \dots\\
&= \frac{\hbar}{i}(\mathcal{D}^{-1})^{AC_1}\Big(\delta^B_{C_1} - \Sigma_{C_1C_2}\frac{\hbar}{i}(\mathcal{D}^{-1})^{C_2B}\Big)^{-1}, \qquad (4.29)
\end{aligned}$$

by summing the (matrix) geometric series $1 + X + X^2 + \cdots = \frac{1}{1-X}$. Inverting (4.18) for $J_A = 0 = \phi^A$, we have

$$\boxed{\Gamma_{AB} = -\mathcal{D}_{AB} + \frac{\hbar}{i}\Sigma_{AB}.} \tag{4.30}$$

The second derivative of the effective action in $\phi^A = 0$ is thus equal to the second derivative of the classical action in $\phi^A = 0$ plus $\frac{\hbar}{i}$ times the proper vertex of order 2.

As a corollary of the expression of the complete propagator in terms of a sum of tree diagram involving the proper vertex of order 2, and the decomposition of the sum over connected diagrams in terms of tree graphs, we then have:

$\sum$ (connected diagrams) = $\sum$ (connected tree diagrams) with (propagators replaced by complete propagators) and (vertices of order $n \geqslant 3$ replaced by (proper vertex of order $n \geqslant 3$).

4.2.4 Semi-classical expansion of the effective action

We now use the path integral representation for Green's functions and we expand around the classical solution in the presence of a source. The expansion parameter is now $\hbar$, and thus the number of loops, instead of an expansion in terms of the coupling constant. We have

$$e^{\frac{i}{\hbar}W[J]} = \mathcal{N}^{-1}\int \mathcal{D}\phi e^{\frac{i}{\hbar}(S[\phi]+J_A\phi^A)}, \qquad \mathcal{N} = \int \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]}. \tag{4.31}$$

Denoting by ϕ_0^{AJ} the unique classical solution in the presence of a source, i.e., the solution to $\dfrac{\delta S}{\delta\phi^A} + J_A = 0$.

If the classical action is of the type $S[\phi] = -\frac{1}{2}\phi^A\mathcal{D}_{AB}\phi^B - V[\phi]$ where the potential $V[\phi]$ starts at cubic order in ϕ, this solution is indeed unique perturbatively in J. Let us for instance take a scalar field with $V[\phi] = \frac{g}{3!}\phi^3$. In this case, $\phi_0^J(x)$ is determined by $\int d^n\mathcal{D}(x,y)\phi(y) + \frac{g}{2}\phi^2(x) = J(x)$. The quadratic part $\mathcal{D}(x,y)$ is invertible because of the $i\epsilon$ terms, and one finds

$$\begin{aligned}
\phi(x) &= \int d^n y\mathcal{D}^{-1}(x,y)J(y) - \int d^n y\mathcal{D}^{-1}(x,y)\frac{g}{2}\phi^2(y) \\
&= \int d^n y\mathcal{D}^{-1}(x,y)J(y) - \int d^n x\mathcal{D}^{-1}(x,y)\frac{g}{2}\Big(\int d^n z\mathcal{D}^{-1}(y,z)J(z) - \int d^n z\mathcal{D}^{-1}(y,z)\frac{g}{2}\phi^2(z)\Big)^2 \\
&\qquad = \int d^n y\mathcal{D}^{-1}(x,y)J(y) - \int d^n x\mathcal{D}^{-1}(x,y)\frac{g}{2}\Big(\int d^n z\mathcal{D}^{-1}(y,z)J(z)\Big)^2 + O(J^3),
\end{aligned}$$

and so on. The solution $\phi_0^J(x)$ is thus unique as a series in J.

When translating the integration variables $\phi = \phi_0^J + \varphi$ with unit Jacobian, one finds

$$e^{\frac{i}{\hbar}W[J]} = \mathcal{N}^{-1}e^{\frac{i}{\hbar}(S[\phi_0^J]+J_A\phi_0^{AJ})}\int \mathcal{D}\varphi e^{\frac{i}{\hbar}(\frac{1}{2}\frac{\delta S}{\delta\phi^A\phi^B}|_{\phi_0^J}\varphi^A\varphi^B + O(\varphi^3))}, \tag{4.32}$$

because the term linear in φ^A vanishes on account of the definition of ϕ_0^J, $\dfrac{\delta S}{\delta\phi^A}|_{\phi_0^J} + J_A = 0$. If

$$S = \int d^d x\,\Big[-\frac{1}{2}\partial_\mu\phi^A\partial^\mu\phi_A - \frac{1}{2}m^2\phi^A\phi_A - V(\phi)\Big],$$

the new quadratic part in φ^A is

$$S^{(2)} = \int d^d x\,\Big[-\frac{1}{2}\partial_\mu\varphi^A\partial^\mu\varphi_A - \frac{1}{2}m^2\varphi^A\varphi_A - \frac{\partial^2 V}{\partial\phi^A\phi^B}|_{\phi_0^J}\varphi^A\varphi^B\Big],$$

which means $\frac{\delta^2 S}{\delta\phi^A\phi^B}|_{\phi_0^J} = -\mathcal{D}_{AB} - V''_{AB}[\phi_0^J]$ with $V''_{AB}[\phi_0^J] = \frac{\delta^2 \int d^dx V}{\delta\phi^A\phi^B}|_{\phi_0^J}$. One then considers the redefinition $\varphi^A \to \sqrt{\hbar}\varphi^A$ to find

$$e^{\frac{i}{\hbar}W[J]} = \mathcal{N}'^{-1} e^{\frac{i}{\hbar}(S[\phi_0^J]+J_A\phi_0^{AJ})} \int \mathcal{D}\varphi e^{i(\frac{1}{2}\frac{\delta S}{\delta\phi^A\phi^B}|_{\phi_0^J}\varphi^A\varphi^B + \sum_{k\geqslant 3}\frac{\hbar^{k/2-1}}{k!}\frac{\delta^k S}{\delta\phi^{A_1}\ldots\delta\phi^{A_k}}|_{\phi_0^J}\varphi^{A_1}\ldots\varphi^{A_k})} \quad (4.33)$$

When applying Wick's theorem to the perturbative expansion that one gets from this integral, the result is different from zero only for even polynomials in φ^A, so only integers powers of $\hbar$ appear.

We can then perform the Gaussian integration which gives

$$e^{\frac{i}{\hbar}W[J]} = \mathcal{N}''^{-1} e^{\frac{i}{\hbar}(S[\phi_0^J]+J_A\phi_0^{AJ})}\Big([\mathrm{Det}(-i\frac{\delta^2 S}{\delta\phi^A\delta\phi^B}[\phi_0^J])]^{-\frac{1}{2}} + O(\hbar)\Big). \quad (4.34)$$

The normalization follows by using $W[0] = 0 = \phi_0^0$,

$$e^{\frac{i}{\hbar}W[J]} = e^{\frac{i}{\hbar}(S[\phi_0^J]+J_A\phi_0^{AJ})}\Big(\frac{[\mathrm{Det}(-i\frac{\delta^2 S}{\delta\phi^A\delta\phi^B}[\phi_0^J])]^{-\frac{1}{2}}}{[\mathrm{Det}(-i\frac{\delta^2 S}{\delta\phi^A\delta\phi^B}[0])]^{-\frac{1}{2}}} + O(\hbar)\Big). \quad (4.35)$$

Since $\frac{\delta^2 S}{\delta\phi^A\delta\phi^B}[0] = -\mathcal{D}_{AB}$, $\frac{[\mathrm{Det}(-i\frac{\delta^2 S}{\delta\phi^A\delta\phi^B}[\phi_0^J])]^{-\frac{1}{2}}}{[\mathrm{Det}(-i\frac{\delta^2 S}{\delta\phi^A\delta\phi^B}[0])]^{-\frac{1}{2}}} = \big[\mathrm{Det}(-(\mathcal{D}^{-1})^{AB}\frac{\delta^2 S}{\delta\phi^B\delta\phi^C}[\phi_0^J])\big]^{-\frac{1}{2}}$ and $\mathrm{Det}\,A = e^{\mathrm{Tr}\ln A}$, we get

$$e^{\frac{i}{\hbar}W[J]} = e^{\frac{i}{\hbar}(S[\phi_0^J]+J_A\phi_0^{AJ})}\Big(e^{-\frac{1}{2}\mathrm{Tr}\ln(-(\mathcal{D}^{-1})^{AB}\frac{\delta^2 S}{\delta\phi^B\delta\phi^C}[\phi_0^J])} + O(\hbar)\Big). \quad (4.36)$$

and thus, using that $\frac{\delta^2 S}{\delta\phi^B\delta\phi^C}[\phi_0^J] = -\mathcal{D}_{BC} - V''_{BC}[\phi_0^J]$,

$$W[J] = S[\phi_0^J] + J_A\phi_0^{AJ} - \frac{\hbar}{2i}\mathrm{Tr}\ln\big(\delta^A_C + (\mathcal{D}^{-1})^{AB}V''_{BC}[\phi_0^J]\big) + O(\hbar^2). \quad (4.37)$$

To compute the effective action, one still needs to perform the Legendre transform

$$\phi^{AJ} = \frac{\delta W}{\delta J_A} = \frac{\delta S}{\delta\phi^B}|_{\phi_0^J}\frac{\delta\phi_0^{BJ}}{\delta J_A} + \phi_0^{AJ} + J_B\frac{\delta\phi_0^{BJ}}{\delta J_A} + O(\hbar). \quad (4.38)$$

By definition of ϕ_0^{AJ}, the first and third terms of the RHS cancel and

$$\phi^{AJ} = \phi_0^{AJ} + O(\hbar) \iff \phi^A = \frac{\delta W}{\delta J_A}|_{J^\phi} = \phi_0^{AJ^\phi} + O(\hbar). \quad (4.39)$$

We then find

$$\boxed{\Gamma[\phi] = S[\phi] - \frac{\hbar}{2i}\mathrm{Tr}\ln\big(\delta^A_C + (\mathcal{D}^{-1})^{AB}V''_{BC}[\phi]\big) + O(\hbar^2)}. \quad (4.40)$$

Indeed,

$$\Gamma[\phi] = W[J^\phi] - J^\phi_A\phi^A = S[\phi_0^{J^\phi}] + J^\phi_A\phi_0^{AJ^\phi} - J^\phi_A\phi^A - \frac{\hbar}{2i}\mathrm{Tr}\ln(\delta^A_C + (\mathcal{D}^{-1})^{AB}V''_{BC}[\phi_0^{J^\phi}]) + O(\hbar^2)$$
$$= S[\phi + (\phi_0^{J^\phi} - \phi)] + J^\phi_A(\phi_0^{AJ^\phi} - \phi^A) - \frac{\hbar}{2i}\mathrm{Tr}\ln(\delta^A_C + (\mathcal{D}^{-1})^{AB}V''_{BC}[\phi]) + O(\hbar^2).$$

Since $\phi_0^{AJ^\phi} - \phi^A = O(\hbar)$, one finds

$$S[\phi + (\phi_0^{J^\phi} - \phi)] + J^\phi_A(\phi_0^{AJ^\phi} - \phi^A) = S[\phi] + (\frac{\delta S}{\delta\phi^A}[\phi] + J^\phi_A)(\phi_0^{AJ^\phi} - \phi^A) + O(\hbar^2)$$
$$= S[\phi] + (\frac{\delta S}{\delta\phi^A}[\phi_0^{J^\phi}] + J^\phi_A)(\phi_0^{AJ^\phi} - \phi^A) + O(\hbar^2) = S[\phi] + O(\hbar^2),$$

when using the definition of $\phi_0^{AJ^\phi}$.

In particular, to order zero in $\hbar$, the effective action reduces to the starting point classical action.

NB: In the un-normalized case, where one does not keep track of $\mathcal{N}$ in (4.31) and one does not impose $W[0] = 0$, equation (4.35) contains an additional term given by

$$[\mathrm{Det}(-i\frac{\delta^2 S}{\delta\phi^A\delta\phi^B}[0])]^{-\frac{1}{2}} = e^{-\frac{1}{2}\mathrm{Tr}\,\ln i\mathcal{D}_{AB}}, \tag{4.41}$$

while equations (4.37) and (4.40) contain in addition the term

$$-\frac{\hbar}{2i}\mathrm{Tr}\,\ln i\mathcal{D}_{AB}. \tag{4.42}$$

check consistency with discussion (factor -i or not) at the end of section 7.7

4.2.5 Effective action as generating functional for proper vertices

We can now prove the following proposition:

a) Connected Green's functions may be computed by using $\Gamma[\phi] = \Gamma^{(2)} + \widehat{\Gamma}[\phi]$ instead of $S[\phi]$ in order to derive the Feynman rules and by summing only over connected tree diagrams.
b) $\frac{i}{\hbar}\widehat{\Gamma}[\phi]$ is the generating functional for proper vertices, i.e., the sum of all 1PI diagrams of order bigger than 3,

$$\frac{i}{\hbar}\Gamma_{A_1\dots A_k} = \Sigma_{A_1\dots A_k}, \quad k \geqslant 3. \tag{4.43}$$

Indeed, let $W_\Gamma[J;g]$ be the generating functional for connected Green's functions computed with $\Gamma[\phi]$ instead of $S[\phi]$ and $g\hbar$ instead of $\hbar$,

$$e^{\frac{i}{\hbar g}W_\Gamma[J;g]} = N^{-1}\int \mathcal{D}\phi e^{\frac{i}{\hbar g}(\Gamma[\phi]+J_A\phi^A)}. \tag{4.44}$$

Repeating the previous reasoning which gave (4.37), we have $W_\Gamma[J;g] = \Gamma[\phi^J_\Gamma] + J_A\phi^{AJ}_\Gamma + O(g)$ with $\frac{\delta\Gamma}{\delta\phi}[\phi^J_\Gamma] = -J_A$ and thus $W_\Gamma[J;0] = \Gamma[\phi^J_\Gamma] + J_A\phi^{AJ}_\Gamma$. But $\Gamma[\phi]$ is defined as the Legendre transform of $W[J]$. Since the Legendre transform is invertible (and thus unique), this means that $W[J] = (\Gamma[\phi] + J_A\phi^A)|_{\phi=\phi^J}$ with $-J_A = \frac{\delta\Gamma[\phi]}{\delta\phi^A}[\phi^J]$. These are the same relations satisfied by $W_\Gamma[J;0]$ which implies that $W[J] = W_\Gamma[J;0]$.
We have

$$\langle\widehat{\phi}^{A_1}\dots\widehat{\phi}^{A_k}\rangle_{c,\Gamma} = g^k(\frac{\hbar}{i})^k\frac{\delta^k\frac{i}{g\hbar}W_\Gamma[J;g]}{\delta J_{A_1}\dots J_{A_k}}|_{J=0} = g^{k-1}(\frac{\hbar}{i})^k\frac{\delta^k\frac{i}{\hbar}W_\Gamma[J;g]}{\delta J_{A_1}\dots J_{A_k}}|_{J=0} \tag{4.45}$$

A diagram $\langle\widehat{\phi}^{A_1}\dots\widehat{\phi}^{A_k}\rangle_{c,\Gamma}$ with k external propagators is of order $\sim g^{k-1+L}$. This means that

$$g^{1-k}\langle\widehat{\phi}^{A_1}\dots\widehat{\phi}^{A_k}\rangle_{c,\Gamma} = \sum_{L=0} g^L(\text{connected diagrams with L loops computed with }\Gamma[\phi]) = (\frac{\hbar}{i})^k\frac{\delta^k\frac{i}{\hbar}W_\Gamma[J;g]}{\delta J_{A_1}\dots J_{A_k}}|_{J=0}.$$

Putting $g=0$ in this relation proves item a).
On the one hand, the corollary of section **4.2.3** says that
$\sum$ (connected diagrams) = $\sum$ (connected trees) with (propagators replaced by complete propagators) and (vertices of order $n \geqslant 3$ replaced by proper vertex or order $n \geqslant 3$).
On the other hand,
$\sum$ (connected diagrams) =$\sum$ (connected trees)computed with $\Gamma[\phi]$.
In this latter representation, propagators are determined by $-(\frac{\delta^2\Gamma}{\delta\phi^A\delta\phi^B})^{-1}$, but we have already shown that this is equal to $\frac{\delta^2 W}{\delta J_A\delta J_B} = \frac{\hbar}{i}\langle\widehat{\phi}^A\widehat{\phi}^B\rangle_c$, which is the complete propagator. Since vertices are determined by $\frac{i}{\hbar}\widehat{\Gamma}[\phi]$, we thus have also shown item b).

4.2.6 Symmetries of the effective action

Supposons $\delta_Q\phi^A = Q^A[\phi]$, avec Q un polynôme en x,ϕ et ses dérivées, est une symétrie de l'action,

$$\delta_Q S = Q^A\frac{\delta S}{\delta\phi^A} = 0. \tag{4.46}$$

Si la mesure est invariante, on a dérivé les identités de Ward,

$$J_A Q^A[\frac{\hbar}{i}\frac{\delta}{\delta J}]Z[J] = 0 \iff J_A\langle Q^A[\widehat{\phi}]\rangle^J = 0,$$

où on a divisé par $Z[J]$ dans la deuxième relation. En effectuant la transformée de Legendre, on trouve

$$\frac{\delta^R\Gamma}{\delta\phi^A}\langle Q^A[\widehat{\phi}]\rangle^{J^\phi} = 0.$$

Si la symétrie de départ est linéaire, on a $Q^A_L[\phi] = Q^A_B[x]\phi^B$,

$$\langle Q^A_L[\widehat{\phi}]\rangle^J = Q^A_L[\phi_J], \quad \langle Q^A[\widehat{\phi}]\rangle^{J^\phi} = Q^A_L[\phi],$$

par définition du champs classique.

En termes de l'action effective les identités de Ward deviennent donc

$$Q^A_L[\phi]\frac{\delta^L\Gamma}{\delta\phi^A} = 0 \iff \delta_{Q_L}\Gamma = 0. \tag{4.47}$$

Pour les symétries linéaires, ces identités prennent la même forme que l'invariance de l'action classique de départ.

4.2.7 Background field method

Considérons une source additionnelle, le champ de fonds $\widetilde{\phi}^A$, et la fonctionnelle génératrice que l'on obtient en translatant le champ quantique ϕ^A dans l'action,

$$\widetilde{Z}[J,\widetilde{\phi}] = \int \mathcal{D}\phi e^{\frac{i}{\hbar}(S[\phi+\widetilde{\phi}]+J_A\phi^A)}. \tag{4.48}$$

La transformée de Legendre n'est pas affectée par cette source additionnelle,

$$\widetilde{W}[J,\widetilde{\phi}] = \frac{\hbar}{i}\ln\frac{\widetilde{Z}[J,\widetilde{\phi}]}{\widetilde{Z}[0,0]}, \qquad \phi^A_{J,\widetilde{\phi}} = \frac{\delta^L \widetilde{W}}{\delta J_A}, \tag{4.49}$$

et on suppose que cette dernière relation est inversible pour donner $J = J^{\phi,\widetilde{\phi}}$, et la transformée de Legendre est définie par

$$\widetilde{\Gamma}[\phi,\widetilde{\phi}] = (\widetilde{W}[J,\widetilde{\phi}] - J_A\phi^A)\Big|_{J=J^{\phi,\widetilde{\phi}}}. \tag{4.50}$$

Ces définitions impliquent que l'action effective en présence du champ de fonds coïncide avec l'ancienne action effective dans laquelle on translate le champ classique par le champ de fond,

$$\boxed{\widetilde{\Gamma}[\phi,\widetilde{\phi}] = \Gamma[\phi+\widetilde{\phi}].} \tag{4.51}$$

En particulier, l'action effective peut donc se calculer en ne considérant que des diagrammes du vide, calculés en présence du champ de fonds, $\widetilde{\Gamma}[0,\widetilde{\phi}] = \Gamma[\widetilde{\phi}]$.

En effet, si on fait le changement de variables $\phi \to \phi - \widetilde{\phi}$ dans (4.48), on trouve

$$\widetilde{Z}[J,\widetilde{\phi}] = \int \mathcal{D}\phi e^{\frac{i}{\hbar}(S[\phi]+J_A(\phi^A-\widetilde{\phi}^A))} = Z[J]e^{-\frac{i}{\hbar}J_A\widetilde{\phi}^A}.$$

Puisque $\widetilde{Z}[0,0] = Z[0]$, on trouve

$$\widetilde{W}[J,\widetilde{\phi}] = \frac{\hbar}{i}\ln\big(\frac{Z[J]}{\widetilde{Z}[0,0]}e^{-\frac{i}{\hbar}J_A\widetilde{\phi}^A}\big) = W[J] - J_A\widetilde{\phi}^A.$$

En appliquant $\dfrac{\delta^L}{\delta J_A}$ on trouve

$$\phi^A_{J,\widetilde{\phi}} = \phi^A_J - \widetilde{\phi}^A,$$

et puis, en combinant avec les 2 équations précédentes,

$$\widetilde{\Gamma}[\phi_{J,\widetilde{\phi}},\widetilde{\phi}] = \widetilde{W}[J,\widetilde{\phi}] - J_A\phi^A_{J,\widetilde{\phi}} = W[J] - J_A\phi^A_J = \Gamma[\phi_J] = \Gamma[\phi_{J,\widetilde{\phi}}+\widetilde{\phi}],$$

ce qui donne le résultat en substituant $J = J^{\phi,\widetilde{\phi}}$.

Remarque: On peut calculer $\widetilde{\Gamma}[0,\widetilde{\phi}] = \Gamma[\widetilde{\phi}]$ de 2 manières différentes.

(1) On peut traiter $\widetilde{\phi}$ de manière exacte, c-à-d en sommant sur tous les diagrammes 1PI du vide (sans points externes) en utilisant les règles de Feynman associées à $S[\phi+\widetilde{\phi}]$. Ceci implique par exemple d'utiliser le propagateur exact pour ϕ en présence de $\widetilde{\phi}$ et des vertex supplémentaires qui dépendent de $\widetilde{\phi}$. En général ce n'est que possible si $\widetilde{\phi}$ est suffisamment simple et on considérera plus loin le cas où $\widetilde{\phi}$ est constant.

(2) On peut traiter $\widetilde{\phi}$ de manière perturbative. Pour un ordre k donné en $\widetilde{\phi}$ on somme tous les diagrammes 1PI du vide pour ϕ qui contiennent k $\widetilde{\phi}$ en utilisant le propagateur habituel pour ϕ et en traitant tous les termes contenant des ϕ comme des nouveaux vertex externes.

energy interpretation of effective action; check with [29]

Chapter 5

Renormalization and asymptotic behavior

This chapter is based on [2], [25], [9], [10], [16].

5.1 Casimir effect

In addition to the computations below, additional considerations on the physical meaning of the renormalization procedure in the context of the Casimir effect can be found in chapter 15 of [30]. For the electromagnetic case, the presentation follows chapter 3.2.4 of [9], chapter 4.3.1 of [31], chapter 7 of [32] and section 6.3 of [33] (see also [34]).

Additional material on the standard Casimir effect can be found here, while more details on the scalar Casimir effect can be found here. The relation between low and high temperature expansion is discussed in [35].

5.1.1 Scalar Casimir effect

Casimir effect for scalar field in 1+1 dimension

Saturday, April 20, 2019 5:21 PM

Consider a massless scalar field in 1+1 dimensions.

$$S = \int dt \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \phi'^2 \right]$$

$$H = \int_{-\infty}^{+\infty} dx \left(\frac{1}{2} \pi^2 + \frac{1}{2} \phi'^2 \right)$$

We have quantized this field by going to momentum space

$$\phi(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega_k}} \left[a(k)\, e^{-i\omega_k t + ikx} + h.c. \right]$$

$$\pi(x,t) = -i \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}} \sqrt{\frac{\omega_k}{2}} \left[a(k)\, e^{-i\omega_k t + ikx} - h.c. \right] \qquad \omega(k) = \sqrt{k^2} = |k|$$

$$[\hat{a}(k), \hat{a}^+(k')] = \delta(k-k')$$

The Hamiltonian, using the symmetric ordering prescription, is

$$\hat{H} = \int_{-\infty}^{+\infty} dk\, \omega_k \left[\hat{a}^+(k)\hat{a}(k) + \frac{1}{2} \right]$$

The vacuum energy, also known as zero-point energy, is infinite,

$$E_{0M} = \langle 0|\hat{H}|0\rangle = \frac{1}{2}\int_{-\infty}^{+\infty} dk\,|k| = \int_{0}^{\infty} dk\,k = \infty$$

In order to get a better handle on this divergence, suppose we had quantized the field on an interval $(-\frac{L}{2},\frac{L}{2})$ of length L with periodic boundary conditions.

In this case, Fourier integrals are replaced by Fourier series

$$\phi(x,t) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2\omega_k L}}\left[e^{-i\omega_k t + kx}\,a(k) + h.c.\right]$$

$$k = \frac{2\pi}{L}n \qquad \omega_k = \frac{2\pi}{L}|n|$$

$$\pi(x,t) = -i\sum_{n\in\mathbb{Z}} \sqrt{\frac{\omega_k}{2L}}\left[e^{-i\omega_k t + kx}\,a(k) - h.c.\right]$$

$$H = \sum_{n\in\mathbb{Z}} \omega_k\left[\hat{a}^{\dagger}(k)\,\hat{a}(k) + \frac{1}{2}\right]$$

In the large interval limit, the sum goes to an integral, $\sum_{n\in\mathbb{Z}} \to \int_{-\infty}^{+\infty} dn = \frac{L}{2\pi}\int_{-\infty}^{+\infty} dk$

Hence, when interpreting the vacuum energy

[illegible]

on the infinite line as coming from the vacuum energy on a finite interval (with periodic boundary condition) in the limit where $L \to \infty$, the continuous result has to be multiplied by $\frac{L}{2\pi}$.

The vacuum energy is thus given by

$$\boxed{E_{0R} = \frac{L}{2\pi} \int_0^\infty dk\, k}$$

In case one imposes Dirichlet conditions on an interval of length L, we have instead

$$\phi(x,t) = \sum_{n>0} e^{-i\omega_k t} \phi_k \sin kx \quad , \quad k = \frac{\pi}{L} n$$

Defining oscillators $\qquad \omega_k = k$

$$a(k) = \sqrt{\frac{\omega_k L}{2}} \left(\phi(x) + i\, \pi(x) \right)$$

We now have

$$\hat{H} = \sum_{n>0} \omega(\vec{k}) \left(\hat{a}^{+}(\vec{k}) \hat{a}(\vec{k}) + \frac{1}{2} \right)$$

and the vacuum energy is

$E(L) = \langle 0 | \hat{H} | 0 \rangle = \frac{\pi}{L} \sum n$ (x) which again diverges

$E_D(L) = \langle 0|\hat{H}|0\rangle = \frac{\pi}{2L} \sum_{n>0} n \ (*)$ which again diverges.

Let us regularize both expressions by multiplying each term by $e^{-\delta k}$ and then consider $\delta \to 0$ at the end.

The integral becomes

$$E_{0R} = \frac{L}{2\pi} \int_0^\infty dk \underbrace{k e^{-\delta k}}_{\left(k\left(-\frac{1}{\delta}\right)e^{-\delta k}\right)' + \frac{1}{\delta} e^{-\delta k}}$$

$$= \frac{L}{2\pi\delta} \int_0^\infty dk \, e^{-\delta k} = \frac{L}{2\pi\delta} \left[-\frac{1}{\delta} e^{-\delta k} \right]_0^\infty = \boxed{\frac{L}{2\pi\delta^2}}$$

For the Dirichlet case, we get

$$E_{0D} = \sum_{n>0} \frac{1}{2} \frac{\pi}{L} n \, e^{-\frac{\pi}{L} n \delta}$$

$$\left[\quad \sum_{n\geq 0} e^{-nx} = \sum_{n\geq 0} \left(e^{-x}\right)^n = \frac{1}{1-e^{-x}} \right.$$

$$-\frac{d}{dx} \sum_{n\geq 0} e^{-nx} = \sum_{n\geq 0} n e^{-nx} = \frac{e^{-x}}{(1-e^{-x})^2}$$

$$= \frac{1}{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2} = \frac{1}{\left(2 \operatorname{sh} \frac{x}{2}\right)^2} \Big] \qquad x = \frac{\pi}{L}\delta$$

$$\underset{\sim}{E_{0\delta}} = \frac{\pi}{2L} \sum_{n>0} n\, e^{-\frac{\pi}{L} n\delta} = \boxed{\frac{\pi}{8L}\left(\mathrm{sh}\, \frac{\pi}{2L}\delta\right)^{-2}}$$

$$\mathrm{sh}\, x = x + \frac{x^3}{3!} + O(x^5)$$

$$E_{0\delta} = \frac{\pi}{8L}\left(\frac{1}{\frac{\pi}{2L}\delta + \frac{1}{3!}\left(\frac{\pi}{2L}\delta\right)^3 + O(\delta^5)}\right)^2$$

$$= \frac{\pi}{8L}\left(\frac{\pi}{2L}\delta\right)^{-2}\left(\frac{1}{1+\frac{1}{3!}\left(\frac{\pi}{2L}\delta\right)^2 + O(\delta^4)}\right)^2$$

$$= \frac{L}{2\pi}\delta^{-2}\left(1 - \frac{1}{3!}\left(\frac{\pi}{2L}\delta\right)^2 + O(\delta^4)\right)^2$$

$$= \boxed{\frac{L}{2\pi}\delta^{-2} - \frac{\pi}{24L} + O(\delta^2)}$$

It follows that $E_R = \lim_{\delta\to 0}\left(E_{0\delta} - E_{0\mathbb{R}}\right) = -\frac{\pi}{24L}$,

the difference between the vacuum energy of an interval of length L and the vacuum energy for the full line is finite.

This result can be obtained directly when using zeta function regularization for the divergent sum (*)

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \zeta(-1) = -\frac{1}{12}.$$

$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $\zeta(-1) = -\frac{1}{12}$.

The Casimir force is $F = -\frac{\partial E}{\partial L} = -\frac{\pi}{24 L^2}$.

When restoring factors of $\hbar$, we have $\omega \to \hbar\omega$. It follows that both the zero-point energy and the Casimir force is proportional to $\hbar$. (and also to c). It is thus a purely quantum mechanical effect.

This is a toy model for the Casimir force between two perfectly conducting parallel plates of area A where one finds instead an attractive force between the plates given by

$$\boxed{F = -\frac{\pi^2 A}{240 L^4}}$$

More [redacted] the inte[redacted]etation:

[redacted] off $e^{-\frac{\pi}{L} n \delta}$ [redacted] e^{-n/n_c} [redacted] $\frac{\pi}{L}\delta = \frac{1}{n_c} \Rightarrow$ [redacted] $\delta = \frac{L}{\pi n_c}$ [redacted] $\to \infty$

[redacted] n_c: critical [redacted] ode number [redacted] t which [redacted]

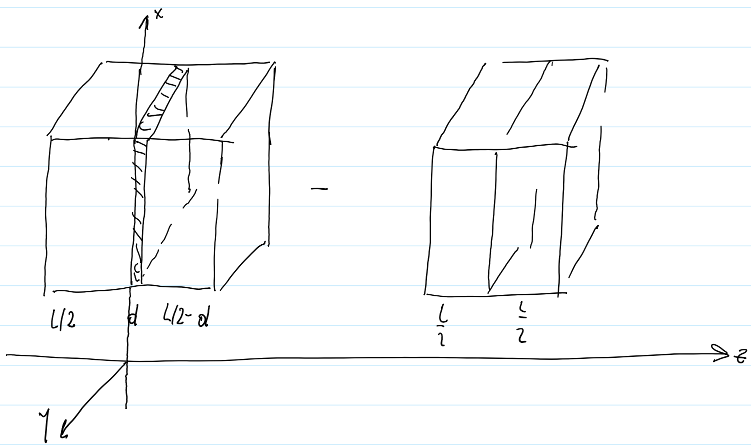
contributio[redacted] of mode [redacted] falls off [redacted] h a fac[redacted] $\frac{1}{2.7}$

5.1.2 Electromagnetic Casimir effect

Electromagnetic Casimir effect

Thursday, May 30, 2019 10:22 AM

For the electromagnetic effect, one considere the zero-point energy between 2 perfectly conducting metallic plates separated by a distance d. In order to compare and substract the result for empty Minkowski spacetime, the set-up is as follows:



The Casimir energy is then

$$E_c(d) = \lim_{L\to\infty}\left\{\left(\cancel{E(L/2)} + E(d) + E\left(\tfrac{L}{2}-d\right)\right) - \left(\cancel{E(L/2)} + E\left(\tfrac{L}{2}\right)\right)\right\}$$

$$= \lim_{L\to\infty}\left[E(d) + E\left(\tfrac{L}{2}-d\right) - E\left(\tfrac{L}{2}\right)\right]$$

• modes in cavity of volume Ad, $A=L^2$, $L\gg d$.

Perfectly conducting plates means that

$$\underbrace{\vec{n}\cdot\vec{B}}_{\vec{n}\cdot(\vec{\nabla}\times\vec{A})} = 0 = \vec{n}\times\underbrace{\vec{E}}_{-\vec{\pi}} \quad \text{on the plates} \Rightarrow \text{at } z=0 \ \& \ z=a$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}\pi^x\\\pi^y\\\pi^z\end{pmatrix} = \begin{pmatrix}-\pi^y\\\pi^x\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\left[\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}\times\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix}\right] = \partial_x A_y - \partial_y A_x$$

$V^a = (A^a, \pi^a)$, $a = 1,2$, $x^1 = x$, $x^2 = y$, $x^3 = z$

have to satisfy Dirichlet conditions at $z=0$, $z=d$

Suppose also periodic boundary conditions in x, y.

$$\Rightarrow V^a = \sum_{n_a \in \mathbb{Z}} \sum_{n>0} V^a_{k_a, k_3} \sin k_3 x^3 e^{i k_a x^a}$$

$$a = 1,2 \quad k_a = \frac{2\pi}{L} n_a, \quad k_3 = \frac{\pi}{d} n$$

If one wants $\vec{\nabla}\cdot\vec{\pi}$, $\vec{\nabla}\cdot\vec{A}$ to satisfy definite boundary conditions, one needs Neumann conditions on π^z, A^z, so that

$$V^3 = \sum_{n_a \in \mathbb{Z}} \left[V^3_{k_a, 0} + \sum_{n>0} V^3_{k_a, k_3} \cos k_3 x^3 \right] e^{i k_a x^a}$$

Gauge transformations are generated by $-\int d^3x \left[\vec{\nabla}\cdot\vec{\pi}\, \epsilon(\vec{x})\right]$.

Hence $\epsilon(\vec{x})$ also satisfies Dirichlet conditions.

Note that the constraint $\vec{\nabla}\cdot\vec{\pi}$ does not involve the $k_3 = 0$ mode.

$\delta_\epsilon A_i = \partial_i \epsilon \Rightarrow \delta_\epsilon A_3 = \partial_3 \epsilon$ does not affect the $k_3 = 0$ mode.

For $n>0$, there are only 2 physical polarizations while the $k_3 = 0$ zero-mode is physical. Since it only exists in the z-direction, there is only one mode of this type.

The frequencies are explicitly given by

$$\omega_{\vec{k}} = c \sqrt{\underbrace{k_a k^a}_{k_\perp^2} + (k_3)^2} = c \sqrt{\left(\frac{2\pi}{L} n_1\right)^2 + \left(\frac{2\pi}{L} n_2\right)^2 + \left(\frac{\pi}{d} n\right)^2}$$

For the zero-point energy between the plates,

$k_\perp$

For the zero-point energy between the plates,

$$\frac{1}{2}\hbar\sum_{\vec{k}}\omega_k = \frac{\hbar c}{2}\int L^2\frac{d^2k}{(2\pi)^2}\left[\sqrt{k_a k^a} + 2\sum_{n=1}^{\infty}\sqrt{k_a k^a + \frac{\pi^2 n^2}{d}}\right]$$

$$= \frac{\hbar c L^2}{2\pi}\int_0^\infty k_\perp dk_\perp\left[\frac{1}{2}k_\perp + \sum_{n=1}^{\infty}\sqrt{k_\perp^2 + \frac{\pi^2 n^2}{d^2}}\right] \quad (y)$$

The energy $E(\frac{L}{2}-d) - E(\frac{L}{2})$ is given by (2 polarizations)

$$\frac{\hbar c}{2}\int\frac{L^2 d^2 k}{(2\pi)^2}\left[\left(\frac{L}{2}-d\right)-\frac{L}{2}\right]\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}2\sqrt{k_\perp^2 + k_z^2} \qquad k_z = \frac{\pi}{d}n$$

$$= -d\frac{\hbar c}{2}\frac{L^2}{(2\pi)^2}\frac{(2\pi)}{2\pi}\int_0^\infty k_\perp dk_\perp \frac{\pi}{d}\int_0^\infty dn\, 2\cdot 2\sqrt{k_\perp^2 + \frac{\pi^2 n^2}{d^2}}$$

$$= -\frac{\hbar c L^2}{(2\pi)}\int_0^\infty k_\perp dk_\perp\int_0^\infty dn\sqrt{k_\perp^2 + \frac{\pi^2 n^2}{d^2}}$$

$$\frac{E_c(d)}{L^2} = \frac{\hbar c}{(2\pi)}\int_0^\infty k_\perp dk_\perp\left[\frac{1}{2}k_\perp + \sum_{n=1}^{\infty}\sqrt{k_\perp^2 + \frac{\pi^2 n^2}{d^2}} - \int_0^\infty dn\sqrt{k_\perp^2 + \frac{\pi^2 n^2}{d^2}}\right] \quad (x)$$

In order to get rid of ultraviolet (large k) divergences one introduces a cut-off function $f(k)$ with $f(k) = 1$ for $k \lesssim k_m$ and vanishing for $k \gg k_m$ (where k_m is of order the inverse of atomic size, where the perfect conductor approximation is no longer realistic).

Set in addition $u = \frac{d^2 k_\perp^2}{\pi^2}$ $\quad k_\perp = \frac{\pi}{d}\sqrt{u}$ $\quad dk_\perp = \frac{\pi}{d}\frac{1}{2\sqrt{u}}du$

$$\frac{E_c(d)}{L^2} = \underbrace{\frac{\hbar c}{2\,2\pi}\frac{\pi^3}{d^3}}\int_0^\infty du\left[\frac{\sqrt{u}}{2}f\left(\frac{\pi}{d}\sqrt{u}\right) + \sum_{n=1}^{\infty}\sqrt{u+n^2}\,f\left(\frac{\pi}{d}\sqrt{u+n^2}\right)\right.$$
$$\left. - \int_0^\infty dn\sqrt{u+n^2}\,f\left(\frac{\pi}{d}\sqrt{u+n^2}\right)\right]$$

$$= \frac{\hbar c\pi^2}{4d^3}\left[\frac{1}{2}F(0) + \sum_{n=1}^{\infty}F(n) - \int_0^\infty dn\, F(n)\right]$$

$$F(\ \ ) = \int^\infty \ldots$$

$$F(n) = \int_0^\infty du \sqrt{u+n^2}\, f\left(\frac{\pi}{d}\sqrt{u+n^2}\right) = \int_{n^2}^\infty du \sqrt{u}\, f\left(\frac{\pi}{d}\sqrt{u}\right)$$

Euler-Maclaurin formula: (Abramowitz & Stegun, 3.6.28)

$$\frac{1}{2}F(0) + \sum_{n=1}^\infty F(n) - \int_0^\infty dn\, F(n) = -\frac{1}{2!} B_2 F'(0) - \frac{1}{4!} B_4 F'''(0) + \dots$$

with Bernoulli numbers $\frac{y}{e^y - 1} = \sum_{\nu=0}^\infty B_\nu \frac{y^\nu}{\nu!}$

$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \dots$

$$F'(n) = -2n^2 f\left(\frac{\pi}{d}n\right), \quad F''(n) = -4n f\left(\frac{\pi}{d}n\right) - 2n^2 \frac{\pi}{d} f'\left(\frac{\pi}{d}n\right)$$

$$F'''(n) = -4 f\left(\frac{\pi}{d}n\right) - 8n\frac{\pi}{d} f'\left(\frac{\pi}{d}n\right) - 2n^2\left(\frac{\pi}{d}\right)^2 f''\left(\frac{\pi}{d}n\right)$$

Assumption: The cut-off function is 1 at the origin and all its derivatives vanish there.

$\Rightarrow$ only $F'''(0) = -4$ is non-vanishing.

$$\boxed{\frac{E_c(d)}{L^2} = \frac{\hbar c \pi^2}{4 d^3}\left(-\frac{1}{4!}\right)\left(-\frac{1}{30}\right)(-4) = -\frac{\pi^2}{720}\frac{\hbar c}{d^3}}$$

$$F_c(d) = -\frac{\partial E_c(d)}{\partial d} = -\frac{\pi^2}{240}\frac{\hbar c L^2}{d^4}$$

Tiny force which has been measured experimentally.

Alternative regularisation: start from (*)

$$\frac{E_c(d)}{L^2} = \frac{\hbar c}{(2\pi)} \int_0^\infty k_\perp dk_\perp \left[\frac{1}{2}k_\perp + \sum_{n=1}^\infty \sqrt{k_\perp^2 + \frac{\pi^2 n^2}{d^2}} - \int_0^\infty dn \sqrt{k_\perp^2 + \frac{\pi^2 n^2}{d^2}}\right]$$

and change variables $s^2 = k_\perp^2 + \frac{\pi^2 n^2}{d^2}$, $2s\,ds = 2k\,dk$

$$\frac{E_c(d)}{L^2} = \frac{\hbar c}{2\pi}\left\{ \sum_{n=0}' \int_{\frac{\pi n}{d}}^\infty ds\, s^2 - \int_0^\infty dn \int_{\frac{\pi n}{d}}^\infty ds\, s^2 \right\}$$

where a prime means that one has to include a factor 1/2 for the term at $n=0$.

The regularisation consists in replacing the integral over s by

$$\int_{\frac{\pi n}{d}}^{\infty} ds\, s^2 \longrightarrow \lim_{\alpha\to 0} \frac{\partial^2}{\partial\alpha^2} \int_{\frac{\pi n}{d}}^{\infty} ds\, e^{-\alpha s} = \lim_{\alpha\to 0} \frac{\partial^2}{\partial\alpha^2}\left(\frac{e^{-\alpha\frac{\pi n}{d}}}{\alpha}\right)$$

$$\Rightarrow \frac{E_c(d)}{L^2} = -\frac{\hbar c}{2\pi} \lim_{\alpha\to 0} \frac{\partial^2}{\partial\alpha^2} \frac{1}{\alpha}\left\{ \sum_{n=0}' e^{-\alpha\frac{\pi n}{d}} - \int_0^\infty dn\, e^{-\alpha\frac{\pi n}{d}} \right\}$$

$$= -\frac{\hbar c}{2\pi} \lim_{\alpha\to 0} \frac{\partial^2}{\partial\alpha^2} \frac{1}{\alpha} \Bigg[\underbrace{\frac{1}{1-e^{-\frac{\alpha\pi}{d}}} - \frac{1}{2} - \left(-\frac{d}{\pi\alpha}\right)\left[e^{-\frac{\alpha\pi n}{d}} \right]_{n=0}^{n=\infty}}_{\frac{e^{\frac{\alpha\pi}{2d}}}{e^{\frac{\alpha\pi}{2d}}}\frac{2-1+e^{-\frac{\alpha\pi}{d}}}{2\left(1-e^{-\frac{\alpha\pi}{d}}\right)} - \frac{d}{\pi\alpha}} \Bigg]$$

$$= -\frac{\hbar c}{2\pi} \lim_{\alpha\to 0} \frac{\partial^2}{\partial\alpha^2} \frac{1}{\alpha}\frac{1}{2}\left[\frac{\operatorname{ch}\frac{\alpha\pi}{2d}}{\operatorname{sh}\frac{\alpha\pi}{2d}} - \frac{2d}{\pi\alpha} \right]$$

$$\coth\frac{\alpha\pi}{2d} = \frac{1+\frac{1}{2!}\left(\frac{\alpha\pi}{2d}\right)^2+\frac{1}{4!}\left(\frac{\alpha\pi}{2d}\right)^4+O(\alpha^6)}{\frac{\alpha\pi}{2d}+\frac{1}{3!}\left(\frac{\alpha\pi}{2d}\right)^3+\frac{1}{5!}\left(\frac{\alpha\pi}{2d}\right)^5+O(\alpha^7)}$$

$$\operatorname{ch} x = 1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+O(x^6)$$
$$\operatorname{sh} x = x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+O(x^7)$$

$$= \frac{2d}{\pi\alpha}\left[\frac{1+\frac{1}{2}\left(\frac{\alpha\pi}{2d}\right)^2+\frac{1}{4!}\left(\frac{\alpha\pi}{2d}\right)^4+O(\alpha^6)}{1+\frac{1}{3!}\left(\frac{\alpha\pi}{2d}\right)^2+\frac{1}{5!}\left(\frac{\alpha\pi}{2d}\right)^4+O(\alpha^6)} \right]$$

$$= \frac{2d}{\pi\alpha}\left[1+\frac{1}{2}\left(\frac{\alpha\pi}{2d}\right)^2+\frac{1}{4!}\left(\frac{\alpha\pi}{2d}\right)^4-\frac{1}{3!}\left(\frac{\alpha\pi}{2d}\right)^2-\frac{1}{5!}\left(\frac{\alpha\pi}{2d}\right)^4-\frac{1}{2\,3!}\left(\frac{\alpha\pi}{2d}\right)^4+\frac{1}{3!3!}\left(\frac{\alpha\pi}{2d}\right)^4+O(\alpha^6) \right]$$

$$= \frac{2d}{\pi\alpha}\left[1+\frac{1}{3}\left(\frac{\alpha\pi}{2d}\right)^2+\frac{6\cdot5-6-6\cdot5\cdot2+5\cdot4}{6!}\left(\frac{\alpha\pi}{2d}\right)^4+O(\alpha^6) \right] \qquad \frac{-16}{6!} = -\frac{1}{3\cdot5\cdot3} = -\frac{1}{45}$$

$$\Rightarrow \frac{E_c(d)}{L^2} = -\frac{\hbar c}{4\pi} \lim_{\alpha\to 0} \frac{\partial^2}{\partial\alpha^2} \frac{1}{\alpha}\left[\frac{1}{3}\frac{\alpha\pi}{2d} - \frac{1}{45}\left(\frac{\alpha\pi}{2d}\right)^3 + O(\alpha^5) \right]$$

$$= -\frac{\hbar c}{4\pi}\frac{2\pi^3}{45\cdot 8 d^3} = -\frac{\hbar c}{720}\frac{\pi^2}{d^3}$$

Addendum: 1) polarization vectors, cf. Bordag et al., (3.74), (7.24)

Unit vectors $\perp$ to $\frac{\vec{k}}{k}$, $k = |\vec{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}$, $k_\perp = \sqrt{k_1^2 + k_2^2}$:

$$\vec{\epsilon}_k^{(1)} = \frac{1}{k_\perp}\begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix}, \quad \vec{\epsilon}_k^{(2)} = \frac{1}{k_\perp k}\begin{pmatrix} k_1 k_3 \\ k_2 k_3 \\ -k_\perp^2 \end{pmatrix} \qquad \| \vec{\epsilon}_k^{(2)} \| = \frac{1}{k_\perp^2 k^2} k_\perp^2 \underbrace{(k_3^2 + k_\perp^2)}_{k^2} = 1.$$

plane of incidence:

$(\vec{k}, \vec{n})$, $\vec{n}$: normal to boundary plane $(= \vec{1}_z)$

$$\vec{k} \times \vec{1}_n = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix}$$

$$\vec{\epsilon}_{\vec{k}}^{(1)} = \frac{\vec{k} \times \vec{1}_n}{\| \vec{k} \times \vec{1}_n \|} \quad \perp \text{ plane of incidence}$$

if $\vec{E} \parallel \epsilon_{\vec{k}}^{(1)}$: transverse electric mode

$\Rightarrow$ associated $\vec{B}$ is in plane of incidence ($\& \perp \vec{k}$)

$\vec{\epsilon}_k^{(2)}$: $\perp \vec{k}$ but in plane $(\vec{k}, \vec{n})$

$$\vec{k} \times (\vec{k} \times \vec{1}_n) = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \times \begin{pmatrix} +k_2 \\ -k_1 \\ 0 \end{pmatrix} = \begin{pmatrix} k_3 k_1 \\ k_3 k_2 \\ -k_1^2 - k_2^2 \end{pmatrix}$$

$$\vec{\epsilon}_{\vec{k}}^{(2)} = \frac{\vec{k} \times (\vec{k} \times \vec{1}_n)}{\| \vec{k} \times (\vec{k} \times \vec{1}_n) \|}$$

if $\vec{E} \parallel \vec{\epsilon}_k^{(2)}$: transverse magnetic mode orthonormal
$(\Rightarrow \vec{B} \parallel \epsilon_{\vec{k}}^{(1)})$

is $\vec{\epsilon}_{\vec{k}}^{(1)}, \epsilon_{\vec{k}}^{(2)}, \vec{1}_k$ orthonormal ?

$\vec{\epsilon}_{\vec{k}}^{(1)} \times \vec{\epsilon}_{\vec{k}}^{(2)} = \vec{1}_k$?

$$\frac{1}{k_\perp^2 k}\begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} k_1 k_3 \\ k_2 k_3 \\ -k_\perp^2 \end{pmatrix} = \frac{1}{k_\perp^2 k}\begin{pmatrix} +k_1 k_\perp^2 \\ k_2 k_\perp^2 \\ k_\perp^2 k_3 \end{pmatrix} = \frac{1}{k}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \quad \checkmark$$

[For a complete set of modes, see Bordag, (7.25) or
appendix of "Capacitor"]

2) Computation using zeta function regularisation, without subtraction of "empty space result" (Bordag, Chapter 7.2):

From (4): $\hbar = 1 = c$

$$\frac{E_0(d)}{L^2} = \frac{1}{2}\int_0^\infty \frac{k_\perp dk_\perp}{2\pi} a_\perp + \int_0^\infty \frac{k_\perp dk_\perp}{2\pi}\sum_{n=1}^\infty \sqrt{k_\perp^2 + \left(\frac{\pi}{d}n\right)^2}$$

first term: neglect because d independent & diverges.

2nd term: ζ-function regularisation

$$\frac{E_0^s(d)}{L^2} = \mu^{2s}\sum_{n=1}^\infty \int_0^\infty k_\perp \frac{dk_\perp}{2\pi}\left(k_\perp^2 + \left(\frac{\pi}{d}n\right)^2\right)^{(1-2s)/2}$$

$$\left(k_\perp = \frac{\pi n y}{d}\right) = \mu^{2s}\frac{1}{2\pi}\left(\frac{\pi}{d}\right)^{3-2s}\underbrace{\sum_{n=1}^\infty \frac{1}{n^{2s-3}}}_{\zeta_R(2s-3)}\int_0^\infty y\,dy\,(y^2+1)^{(1-2s)/2}$$

$\zeta_R(z)$: defined by the series for $Re(z) > 1$ ($Re(s) > 2$, we need $s=0$, $z=-3$)

analytic continuation gives a meromorphic function with a single pole at $z=1$ on real axis.

reflection relation: $\Gamma\left(\frac{z}{2}\right)\zeta_R(z) = \pi^{(2z-1)/2}\,\Gamma\left(\frac{1-z}{2}\right)\zeta_R(1-z)$

$$\Rightarrow \Gamma\left(-\tfrac{3}{2}\right)\zeta(-3) = \pi^{-7/2}\,\Gamma(1)\,\zeta_R(4)$$

$$\zeta_R(-3) = \pi^{-7/2}\frac{\pi^4}{90}\left(\Gamma\left(-\tfrac{3}{2}\right)\right)^{-1} \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$= \cancel{\pi^{-7/2}}\,\frac{3}{4}\,\cancel{\pi^{-1/2}}\,\cancel{\pi^4}\,\frac{1}{90} \qquad \Gamma\left(\tfrac{1}{2}\right) = -\tfrac{1}{2}\Gamma\left(-\tfrac{1}{2}\right) \Rightarrow \Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}$$

$$= \frac{1}{120} \qquad \Gamma\left(-\tfrac{1}{2}\right) = -\tfrac{3}{2}\Gamma\left(-\tfrac{3}{2}\right) \Rightarrow \Gamma\left(-\tfrac{3}{2}\right) = \tfrac{4}{3}\sqrt{\pi}$$

$$\int_0^\infty y\,dy\,(y^2+1)^{(1-2s)/2}\,, \qquad Re(s) > 3/2$$

$$= \cancel{\frac{1}{2}}\left[\frac{(y^2+1)^{(3-2s)/2}}{\frac{(3-2s)}{\cancel{2}}}\right]_0^\infty = -\frac{1}{3-2s}$$

$$\lim_{s\to 0} : \boxed{\frac{E_0(d)}{L^2} = \frac{1}{2\pi}\left(\frac{\pi}{d}\right)^3\left(-\frac{1}{3}\right)\frac{1}{120} = \boxed{-\frac{\pi^2}{720\,d^3}}}$$

5.1.3 Photons in a Casimir box revisited

5.1.3.1 Boundary conditions and mode decomposition

Consider coordinates x^i, $i = 1, 2, 3$ in Euclidean space. The electric field is $E^i = -\pi^i$, the magnetic field is $B^i = \epsilon^{ijk}\partial_j A_k$. The starting point is the first order action

$$S = \int dx^0 \Big[\int_V d^3x \; \partial_0 A_i \pi^i - H + \int_V d^3x \; A_0 \partial_i \pi^i \Big], \quad H = \frac{1}{2}\int_V d^3x (\pi^i \pi_i + B^i B_i). \tag{5.1}$$

Let $a = 1, 2$, $i = a, 3$, $V = L_1 L_2 d$ with L_a large, $k_3 = \frac{\pi n_3}{a}$, $k_a = \frac{2\pi n_a}{L_a}$ (with no summation over a). Perfectly conducting boundary conditions on parallel plates at $x^3 = 0$ and $x^3 = a$ require $B^3 = 0$ and $E^a = 0$ on the plates. Let

$$\psi^H_k = \sqrt{\frac{2}{V}} e^{ik_a x^a} \sin k_3 x^3, \; \psi^E_{k_a,0} = \frac{1}{\sqrt{V}} e^{ik_a x^a}, \; \psi^E_k = \sqrt{\frac{2}{V}} e^{ik_a x^a} \cos k_3 x^3, \tag{5.2}$$

and let us use $V^i(x)$ for either of the canonically conjugate variables $A^i(x)$ or $\pi^i(x)$. The boundary conditions are implemented through [1]

$$V^a(x) = i \sum_{n_a, n_3 > 0} V^a_k \psi^H_k, \quad V^3(x) = \sum_{n_a, n_3 \geqslant 0} V^3_k \psi^E_k, \tag{5.3}$$

where we take $V^a_{k_a,0} = 0$. Reality and parity conditions are

$$V^a_{k_a,k_3} = -V^{*a}_{-k_a,k_3}, \quad V^3_{k_a,k_3} = V^{*3}_{-k_a,k_3}, \quad V^a_{k_a,k_3} = -V^a_{k_a,-k_3}, \quad V^3_{k_a,k_3} = V^3_{k_a,-k_3}. \tag{5.4}$$

5.1.3.2 The particle

The mode at $n_i = 0$, $A_{3,0,0} = q, \pi^3_{0,0} = p$ is treated separately. It is not affected by the constraints nor by proper gauge transformations. Its Poisson brackets are canonical and its contribution to the Hamiltonian is that of a free particle of unit mass,

$$H_{n_i=0} = \frac{1}{2} p^2. \tag{5.5}$$

5.1.3.3 Polarization vectors

Take now $\alpha = 1, 2$, and $A = (\alpha, \|)$. In Euclidean momentum space k^i (minus the origin) let $k = \sqrt{k_i k^i}$ and consider an orthonormal frame $e_A{}^i(k)$ built out of two vectors normal to k^i and one vector parallel to k^i,

$$e_\|{}^i = \frac{k^i}{k}, \quad \epsilon^i{}_{jm} e_1{}^j e_2{}^m = e_\|{}^i, \quad e_A{}^i e^B{}_i = \delta^B_A, \quad e_A{}^i e^B{}_j = \delta^B_A, \tag{5.6}$$

so that the decomposition of a vector $v^i(k)$ in momentum space in this frame is $V^i(k) = V^A(k) e_A{}^i$ with inverse $V^A(k) = V^i e^A{}_i$.

The non-vanishing Poisson brackets for these modes are read off from the expansion of $\int_V d^3x \; \partial_0 A_i \pi^i$ and given by

$$\{A_{Ak}, \pi^{*B}_{k'}\} = \delta^B_A \prod_{i=1}^{3} \delta_{n_i, n'_i}. \tag{5.7}$$

[1] The factor i in front of the expansion of (A^a, π^a) is chosen for later convenience.

The contribution to the Hamiltonian from the $n_3 > 0$ modes is given by

$$H_{n_3 \neq 0} = \frac{1}{2} \sum_{n_a, n_3 > 0} (\pi_k^A \pi_{kA}^* + k^2 A_k^\alpha A_{k\alpha}^*). \tag{5.8}$$

Let $k_\perp = \sqrt{k_a k^a}$. The contribution from the $n_3 = 0$ modes is

$$H_{n_3=0} = \frac{1}{2} \sum_{n_a}{}' (\pi^3_{k_a,0} \pi^{*3}_{k_a,0} + k_\perp^2 A^3_{k_a,0} A^{*3}_{k_a,0}), \tag{5.9}$$

where the prime means that the mode with $n_i = 0$ is omitted. In order to implement the constraint $\partial_i \pi^i = 0$ and/or the Coulomb gauge condition $\partial^i A_i = 0$,

$$\partial_i V^i = -\sum_{n_a, n_3 > 0} k V_k^{\|} \psi_k^H = 0 \iff V_k^{\|} = 0. \tag{5.10}$$

Note that $V^a_{k_a,0} = 0$ implies $V^{\|}_{k_a,0} = 0$ and that, in order not to introduce spurious variables, one needs to expand A_0 as

$$A_0 = \sum_{n_a, n_3 > 0} A_{k,0} \psi_k^H. \tag{5.11}$$

By variations with respect to A_0, respectively $A_{k,0}$, one may then solve the constraints, or equivalently $\pi_k^{\|} = 0$ in the action, without the need to impose the Coulomb or any other gauge condition. As a consequence, in the first term in (5.8), one may limit the sum over A to one over α. The same is true for the kinetic term that gives rise to the canonical Poisson brackets. Because proper gauge transformations with gauge parameters satisfying Dirichlet conditions correspond to arbitrary shifts of $A_k^{\|}$, it follows that gauge invariant quantities reduced to the constraint surface do not depend on the variables $A_k^{\|}, \pi_k^{\|}$.

5.1.3.4 Adapted polarization vectors

Consider now the following choice of ${e_\alpha}^i$ (see e.g. [32]),

$${e_H}^i = \frac{1}{k_\perp} \begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix} \qquad {e_E}^i = \frac{1}{k_\perp k} \begin{pmatrix} k_1 k_3 \\ k_2 k_3 \\ -k_\perp^2 \end{pmatrix}, \tag{5.12}$$

so that the components V^H, V^E, $V^{\|}$ are given by

$$V^H_{k_a,k_3} = \frac{\epsilon^{ab} k_b}{k_\perp} V^a_{k_a,k_3}, \quad V^E_{k_a,k_3} = \frac{k_a k_3}{k_\perp k} V^a_{k_a,k_3} - \frac{k_\perp}{k} V^3_{k_a,k_3}, \quad V^{\|}_{k_a,k_3} = \frac{k_i}{k} V^i_{k_a,k_3}, \tag{5.13}$$

with inverse relations

$$V^a_{k_a,k_3} = \frac{\epsilon^{ab} k_b}{k_\perp} V^H_{k_a,k_3} + \frac{k^a k_3}{k_\perp k} V^E_{k_a,k_3} + k^a V^{\|}_{k_a,k_3}, \quad V^3_{k_a,k_3} = -\frac{k_\perp}{k} V^E_{k_a,k_3} + k_3 V^{\|}_{k_a,k_3}. \tag{5.14}$$

Note in particular that $V^3_{k_a,0} = -V^E_{k_a,0}$. Reality and parity conditions become

$$V^H_{k_a,k_3} = V^{*H}_{-k_a,k_3}, \quad V^E_{k_a,k_3} = V^{*E}_{-k_a,k_3}, \quad V^H_{k_a,-k_3} = -V^H_{k_a,k_3}, \quad V^E_{k_a,k_3} = V^E_{k_a,-k_3}. \tag{5.15}$$

Let $\lambda = (H, E)$. Oscillator variables are defined as

$$\begin{aligned} a^\lambda_{k_a,k_3} &= \sqrt{\frac{k}{2}} (A^\lambda_{k_a,k_3} + \frac{i}{k} \pi^\lambda_{k_a,k_3}), \\ A^\lambda_{k_a,k_3} &= \frac{1}{\sqrt{2k}} (a^\lambda_{k_a,k_3} + a^{*\lambda}_{-k_a,k_3}), \quad \pi^\lambda_{k_a,k_3} = -i \sqrt{\frac{k}{2}} (a^\lambda_{k_a,k_3} - a^{*\lambda}_{-k_a,k_3}), \end{aligned} \tag{5.16}$$

with the understanding that $a^H_{k_a,0} = 0$. Their non-vanishing Poisson brackets are

$$\{a^\lambda_k, a^{*\lambda'}_{k'}\} = -i\delta^{\lambda,\lambda'}\prod_{i=1}^{3}\delta_{n_i,n'_i}. \tag{5.17}$$

In terms of these oscillators, we have

$$\begin{aligned}
A^a &= i\sum_{n_a,n_3>0}\Big[\frac{\epsilon^{ab}k_b}{\sqrt{2k}k_\perp}(a^H_k\psi^H_k - \text{c.c.}) + \frac{k^a k_3}{\sqrt{2k}k_\perp k}(a^E_k\psi^H_k - \text{c.c.}) + k^a A^\parallel_k\psi^H_k\Big],\\
\pi^a &= \sum_{n_a,n_3>0}\Big[\frac{\sqrt{k}\epsilon^{ab}k_b}{\sqrt{2}k_\perp}(a^H_k\psi^H_k + \text{c.c.}) + \frac{k^a k_3}{\sqrt{2k}k_\perp}(a^E_k\psi^H_k + \text{c.c.}) + ik^a\pi^\parallel_k\psi^H_k\Big],\\
A^3 &= \sum_{n_a,n_3\geqslant 0}{}' \Big[-\frac{k_\perp}{\sqrt{2k}k}(a^E_k\psi^E_k + \text{c.c.}) + k_3 A^\parallel_k\psi^E_k\Big] + \frac{1}{\sqrt{V}}q,\\
\pi^3 &= \sum_{n_a,n_3\geqslant 0}{}' \Big[i\frac{k_\perp}{\sqrt{2k}}(a^E_k\psi^E_k - \text{c.c.}) + k_3\pi^\parallel_k\psi^E_k\Big] + \frac{1}{\sqrt{V}}p.
\end{aligned} \tag{5.18}$$

On the constraint surface, the full Hamiltonian is given by

$$H = \sum_{\lambda,n_a,n_3\geqslant 0}{}' \frac{1}{2}(\pi^\lambda_k\pi^{*\lambda}_k + k^2A^\lambda_kA^{*\lambda}_k) + \frac{1}{2}p^2 = \sum_{\lambda,n_a,n_3\geqslant 0}{}' \frac{k}{2}(a^\lambda_k a^{*\lambda}_k + a^{*\lambda}_k a^\lambda_k) + \frac{1}{2}p^2. \tag{5.19}$$

5.1.3.5 Bromwich-Borgnis fields

Consider now the real fields

$$\begin{aligned}
\phi^H &= \sum_{n_a,n_3>0}\frac{1}{\sqrt{2k}k_\perp}(a^H_k\psi^H_k + \text{c.c.}), \quad \pi^H = -i\sum_{n_a,n_3>0}\frac{\sqrt{k}}{\sqrt{2}k_\perp}(a^H_k\psi^H_k - \text{c.c.})\\
\phi^E &= -\sum_{n_a,n_3\geqslant 0}{}' \frac{1}{\sqrt{2k}kk_\perp}(a^E_k\psi^E_k + \text{c.c.}), \quad \pi^E = i\sum_{n_a,n_3\geqslant 0}{}' \frac{1}{\sqrt{2k}k_\perp}(a^E_k\psi^E_k - \text{c.c.})\\
\phi^G &= \sum_{n_a,n_3>0}\frac{1}{2}(A^\parallel_k\psi^H_k + \text{c.c.}), \quad \pi^G = \sum_{n_a,n_3>0}\frac{1}{2}(\pi^\parallel_k\psi^H_k + \text{c.c.}).
\end{aligned} \tag{5.20}$$

Let $\Lambda = (H, E, G)$ and φ^Λ stand for either ϕ^Λ or π^Λ. The fields φ^H, φ^G satisfy Dirichlet conditions while φ^E satisfies Neumann conditions. In these terms,

$$\begin{aligned}
A^a &= \epsilon^{ab}\partial_b\phi^H + \partial^a\partial_3\phi^E + \partial^a\phi^G, \quad A^3 = (-\triangle + \partial_3^2)\phi^E + \partial^3\phi^G + \frac{1}{\sqrt{V}}q,\\
\pi^a &= \epsilon^{ab}\partial_b\pi^H + \partial^a\partial_3\pi^E + \partial^a\pi^G, \quad \pi^3 = (-\triangle + \partial_3^2)\pi^E + \partial^3\pi^G + \frac{1}{\sqrt{V}}p,\\
B^a &= \epsilon^{ab}\partial_b(-\triangle)\phi^E + \partial^a\partial_3\phi^H, \quad B^3 = (-\triangle + \partial_3^2)\phi^H.
\end{aligned} \tag{5.21}$$

On the constraint surface where $\pi^G = 0$, one recovers the construction of [36, 37] (see also [38] section 32 and [21, 39, 33] for related more modern discussions).

5.1.3.6 Single scalar field formulation

In order to streamline the computation of the partition function and to discuss modular properties, it is useful to go one step further and introduce a formulation with a single scalar field on $z \in [-a, a]$ with periodic boundary conditions [40].

The variables defined by

$$\begin{aligned} &\phi_{k_a,k_3} = \frac{A^E_{k_a,k_3} - iA^H_{k_a,k_3}}{\sqrt{2}}, \quad \pi_{k_a,k_3} = \frac{\pi^E_{k_a,k_3} - i\pi^H_{k_a,k_3}}{\sqrt{2}}, \\ &\phi_{k_a,0} = A^E_{k_a,0}, \quad \pi_{k_a,0} = \pi^E_{k_a,0}, \end{aligned} \tag{5.22}$$

satisfy the reality conditions

$$\phi^*_{-k_a,0} = \phi_{k_a,0}, \quad \phi^*_{-k_a,-k_3} = \phi_{k_a,k_3}, \quad \pi^*_{-k_a,0} = \pi_{k_a,0}, \quad \pi^*_{-k_a,-k_3} = \pi_{k_a,k_3}. \tag{5.23}$$

The first expression for the Hamiltonian in (5.164) can then be written as

$$H = \sum_{n_i} \frac{1}{2}(\pi_k \pi^*_k + k^2 \phi_k \phi^*_k), \tag{5.24}$$

with the understanding that $\pi_{0,0} = p$, $\phi_{0,0} = q$ and the sum goes over all $n_i \in \mathbb{Z}$. In terms of appropriate oscillators, defined for $n_i \neq 0$,

$$a_{k_a,k_3} = \sqrt{\frac{k}{2}}(\phi_{k_a,k_3} + \frac{i}{k}\pi_{k_a,k_3}) = \begin{cases} \frac{a^E_{k_a,k_3} - ia^H_{k_a,k_3}}{\sqrt{2}}, & n_3 \neq 0, \\ a^E_{k_a,0}, \quad n_3 \neq 0 \end{cases}, \tag{5.25}$$

the Hamiltonian becomes

$$H = \sum_{n_i}{}' \frac{k}{2}(a^*_k a_k + a_k a^*_k) + \frac{1}{2}p^2. \tag{5.26}$$

The above are the mode decomposition, oscillators and Hamiltonian of a single real scalar field ϕ and its momentum π in a volume $V' = L_1 L_2 L_3$ where $L_3 = 2a$ with periodic boundary conditions in all directions and, in particular, with periodicity $2a$ in the x^3 direction.

$$\begin{aligned} \phi &= \frac{1}{\sqrt{V'}} \sum_{n_i} e^{ik_j x^j} \phi_{k_i} = \frac{1}{\sqrt{V'}}\Big(\sum_{n_i}{}' \frac{1}{\sqrt{2k}}[a_{k_i} e^{ik_j x^j} + a^*_{k_i} e^{-ik_j x^j}] + q\Big), \\ \pi &= \frac{1}{\sqrt{V'}} \sum_{n_i} e^{ik_j x^j} \pi_{k_i} = \frac{1}{\sqrt{V'}}\Big(-i\sum_{n_i}{}' \sqrt{\frac{k}{2}}[a_{k_i} e^{ik_j x^j} - a^*_{k_i} e^{-ik_j x^j}] + p\Big), \end{aligned} \tag{5.27}$$

for which the Hamiltonian is

$$H = \frac{1}{2}\int_{V'} d^3x \, (\pi^2 + \partial_i \phi \partial^i \phi). \tag{5.28}$$

5.1.4 Massless scalar field Casimir energy on $\mathbb{R}^{d-1} \times \mathbb{S}^1$

Suppose now that the spatial dimension x^d is small, while all other spatial dimensions x^I, for $I = 1, \dots, d-1$ are large, $L_I \gg L_d$. In other words, one considers a slab geometry with two infinite parallel hyperplanes separated by a distance L_d, or because of the periodic boundary conditions, a scalar field on the spatial manifold $\mathbb{R}^{d-1} \times \mathbb{S}^1_{L_d}$. Suppose also that one uses symmetric instead of normal ordering, so that

$$\widehat{H} = \frac{1}{2}\sum_{n_i \in \mathbb{Z}^d}{}' \omega_{k_i}(\widehat{a}^\dagger_{k_i}\widehat{a}_{k_i} + \widehat{a}_{k_i}\widehat{a}^\dagger_{k_i}) = \sum_{n_i \in \mathbb{Z}^d}{}' \omega_{k_i}(\widehat{a}^\dagger_{k_i}\widehat{a}_{k_i} + \frac{1}{2}). \tag{5.29}$$

The divergent zero-point energy is

$$E_0^d = \langle 0|\widehat{H}|0\rangle = \frac{1}{2}\sum_{n_i\in\mathbb{Z}^d}{}' \omega_{k_i}. \tag{5.30}$$

Transforming $d-1$ sums into integrals, and using ζ function regularization to give a meaning to this divergent quantity, we need to evaluate

$$E_0^d(s) = \frac{1}{2}\nu^{2s}\frac{V_{d-1}}{(2\pi)^{d-1}}\sum_{n_d\in\mathbb{Z}}\int d^{d-1}k\,[(\frac{2\pi n_d}{L_d})^2 + k_I k^I]^{-s+\frac{1}{2}}, \tag{5.31}$$

where $V_{d-1} = \prod_{I=1}^{d-1} L_I$. This is the same computation as in (2.268), up an overall factor of $(2\pi\nu^2)^s$ versus $\frac{1}{2}\nu^{2s}$ and the replacements $d \to d-1$, $s \to s-\frac{1}{2}$, $\beta = L_{d+1} \to L_d$. The result can thus be read off (2.273),

$$E_0^d(s) = \frac{1}{2}\nu^{2s}\frac{V_{d-1}}{L_d^{d-2s}}2^{2-2s}\pi^{\frac{d+1}{2}-2s}\frac{\Gamma(s-\frac{d}{2})\zeta(2s-d)}{\Gamma(s-\frac{1}{2})}, \tag{5.32}$$

which yields at $s=0$,

$$E_0^d(0) = -\xi(-d)\frac{V_{d-1}}{L_d^d}, \tag{5.33}$$

and thus, after using the reflection formula,

$$\boxed{E_0^d(0) = -\xi(d+1)\frac{V_{d-1}}{L_d^d}} = \begin{cases} -\frac{\pi}{6}\frac{1}{L_1} & \text{for } d=1 \\ -\frac{\zeta(3)}{2\pi}\frac{L_1}{L_2^2} & \text{for } d=2 \\ -\frac{\pi^2}{90}\frac{L_1 L_2}{L_3^3} & \text{for } d=3 \\ & \vdots \end{cases}. \tag{5.34}$$

Remarks:

(i) It is intriguing that the same dimensionless number $\xi(d+1)$ appears here in the Casimir energy on a circle and in the black body result in the large volume limit. This is related to a high/low temperature duality which will be explained in detail later.

(ii) On account of the previously discussed equivalence, the correct electromagnetic results with perfectly conducting plates at $x^3=0$ and $x^3=a$ are obtained through the replacement $L_3 = 2a$.

(iii) Note that, even with symmetric ordering,

$$P_i^0 = \langle 0|\widehat{P}_i|0\rangle = 0, \tag{5.35}$$

irrespective of the existence or not of a small spatial dimension, because either $\sum_{n_i\in\mathbb{Z}} k_i = 0$ or $\int \mathrm{d}k_i k_i = 0$.

(iv) The computation of the Casimir energy can also be done using a suitable integral representation. Starting from the representation

$$\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\, t^{s-1}e^{-t\lambda} = \frac{1}{\lambda^s}, \quad \mathfrak{Re}(s) > 0, \quad \lambda > 0, \tag{5.36}$$

at $s = -\frac{1}{2}$ (to be understood in the sense of analytic continuation), the zero point energy in (5.30) is written as

$$E_0^d = \frac{1}{2\Gamma(-\frac{1}{2})}\sum_{n_i\in\mathbb{Z}^d}{}'\int_0^\infty \mathrm{d}t\, t^{-\frac{3}{2}}e^{-t\omega_{k_i}^2} \tag{5.37}$$

After changing sums to integrals for the $d-1$ large dimensions and integrating over the angles, one gets

$$E_0^d = -\frac{1}{4\sqrt{\pi}}\frac{V_{d-1}}{(2\pi)^{d-1}}\frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\sum_{n_d\in\mathbb{Z}}\int_0^\infty \mathrm{d}t\, t^{-\frac{3}{2}}\int_0^\infty \mathrm{d}k k^{d-2} e^{-t\left[(\frac{2\pi}{L_d}n_d)^2+k^2\right]}. \tag{5.38}$$

The change of variables $k=\frac{2\pi}{L_d}\sqrt{y}$, $t\to t=(\frac{L_d}{2\pi})^2 t$ then leads to

$$E_0^d = -\frac{V_{d-1}}{L_d^d}\frac{\pi^{\frac{d}{2}}}{2\Gamma(\frac{d-1}{2})}\sum_{n_d\in\mathbb{Z}}\int_0^\infty \mathrm{d}t\, t^{-\frac{3}{2}}e^{-tn_d^2}\int_0^\infty \mathrm{d}y y^{\frac{d-1}{2}-1}e^{-ty}. \tag{5.39}$$

Performing the integral over y using (5.36) again gives

$$E_0^d = -\frac{V_{d-1}}{L_d^d}\frac{\pi^{\frac{d}{2}}}{2}\sum_{n_d\in\mathbb{Z}}\int_0^\infty \mathrm{d}t\, t^{-\frac{d}{2}-1}e^{-tn_d^2} \tag{5.40}$$

The integral term with $n_d=0$ is divergent, $\int_\epsilon^\infty \mathrm{d}t\, t^{-\frac{d}{2}-1}=\frac{2\epsilon^{-\frac{d}{2}}}{d}$ and neglected, while the terms with $n_d\neq 0$ are evaluated using (5.36) to yield

$$E_0^d = -\frac{V_{d-1}}{L_d^d}\pi^{\frac{d}{2}}\Gamma(-\frac{d}{2})\zeta(-d), \tag{5.41}$$

in agreement with (5.33).

5.1.5 Functional approach to massless scalar partition function on $\mathbb{R}^{d-1}\times\mathbb{T}^2$

We now consider the case where only the first $d-1$ spatial dimensions are large, while L_d is small, and where a chemical potential for linear momentum in the small dimension x^d is turned on. We start here with the functional approach.

In this case $I=1,\dots,d-1$, $V_{d-1}=\prod_{I=1}^{d-1}L_I$ and, when taking into account (2.264), we get instead of (2.268),

$$\zeta_{-\Delta_\mu}(s)=\frac{V_{d-1}}{(2\pi)^{d-1}}\int d^{d-1}k\sum_{(n_{d+1},n_d)\in\mathbb{Z}^2}{}^{'}[(\frac{2\pi n_{d+1}}{\beta}-\mu\frac{2\pi n_d}{L_d})^2+(\frac{2\pi n_d}{L_d})^2+k_Ik^I]^{-s}. \tag{5.42}$$

The computation then proceeds as in section 2.8.5, up to the replacements $d\to d-1$ and of the zeta function by a suitable Eisenstein series. More precisely, instead of (2.271), we now get

$$\zeta_{-\Delta_\mu}(s)=\frac{V_{d-1}}{(2\pi)^{d-1}}\frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}(\frac{2\pi}{\beta})^{d-1-2s}$$
$$\sum_{(n_{d+1},n_d)\in\mathbb{Z}^2}{}^{'}\int_0^\infty dy\, y^{d-2}[(n_{d+1}-n_d\frac{\beta\mu}{L_d})^2+n_d^2(\frac{\beta}{L_d})^2+y^2]^{-s}, \tag{5.43}$$

For later convenience, we also change parametrization and use $\alpha=\beta\mu$ so that we are now computing

$$\mathrm{Z}_{d,1}(\beta,\alpha)=\mathrm{Tr}\, e^{-\beta\widehat{H}+i\alpha\widehat{P}_d}. \tag{5.44}$$

Introducing the modular parameter

$$\tau=\frac{\alpha+i\beta}{L_d}, \tag{5.45}$$

the sum simplifies to

$$\sum_{(n_{d+1},n_d)\in\mathbb{Z}^2}{}' \int_0^\infty dy\, y^{d-2}[|n_{d+1}+n_d\tau|^2+y^2]^{-s}. \tag{5.46}$$

The integral is convergent for $\mathfrak{Re}(s) > \frac{d-1}{2}$ and can be performed through the change of variables $y = |n_{d+1}+n_d\tau|\sinh x$ which gives

$$\zeta_{-\Delta_\alpha}(s) = \frac{V_{d-1}}{(2\pi)^{d-1}} \frac{\pi^{\frac{d-1}{2}}\Gamma(s-\frac{d-1}{2})}{\Gamma(s)} (\frac{2\pi}{\beta})^{d-1-2s} \sum_{(n_{d+1},n_d)\in\mathbb{Z}^2}{}' |n_{d+1}+n_d\tau|^{d-1-2s}. \tag{5.47}$$

In terms of the $\mathrm{SL}(2,\mathbb{Z})$ real analytic Eisenstein series,

$$f_s(\tau,\bar\tau) = \sum_{(m,n)\in\mathbb{Z}^2/(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}, \quad \mathfrak{Re}(s) > 1, \tag{5.48}$$

with $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$, which are invariant under the modular transformations

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad a,b,c,d\in\mathbb{Z}, \quad ad-bc=1, \tag{5.49}$$

this can be written as

$$\zeta_{-\Delta_\alpha}(s) = \frac{V_{d-1}}{(2\pi)^{d-1}} \frac{\pi^{\frac{d-1}{2}}\Gamma(s-\frac{d-1}{2})}{\Gamma(s)\tau_2^{s-\frac{d-1}{2}}} (\frac{2\pi}{\beta})^{d-1-2s} f_{s-\frac{d-1}{2}}(\tau,\bar\tau), \tag{5.50}$$

When using the functional relation for the analytically continued Eisenstein series,

$$\pi^{-z}\Gamma(z)f_z(\tau,\bar\tau) = \pi^{z-1}\Gamma(1-z)f_{1-z}(\tau,\bar\tau). \tag{5.51}$$

for $z = s - \frac{d-1}{2}$, and also $\tau_2 = \frac{\beta}{L_d}$, we end up with

$$\zeta_{-\Delta_\alpha}(s) = \frac{\Gamma(\frac{d+1}{2}-s)}{2^{2s}\pi^{\frac{d+1}{2}}\Gamma(s)} \frac{V_{d-1}}{L_d^{d-1-2s}} \frac{f_{\frac{d+1}{2}-s}(\tau,\bar\tau)}{\tau_2^{\frac{d-1}{2}-s}}. \tag{5.52}$$

Again, since $\Gamma(s) \approx \frac{1}{s} + O(s^0)$, $\zeta_{-\Delta_\alpha}(0)$, if $\mathcal{Z}_{d,1}(\tau,\bar\tau) = \mathrm{Z}_{d,1}(\beta,\alpha)$, the partition function is

$$\ln \mathcal{Z}_{d,1}(\tau,\bar\tau) = \frac{\Gamma(\frac{d+1}{2})V_{d-1}}{2\pi^{\frac{d+1}{2}}L_d^{d-1}\tau_2^{\frac{d-1}{2}}} f_{\frac{d+1}{2}}(\tau,\bar\tau). \tag{5.53}$$

In terms of the completion of the real analytic Eisenstein series,

$$\xi(z;\tau,\bar\tau) = \frac{\Gamma(\frac{z}{2})f_{\frac{z}{2}}(\tau,\bar\tau)}{2\pi^{\frac{z}{2}}}, \tag{5.54}$$

which satisfies the reflection formula

$$\xi(z;\tau,\bar\tau) = \xi(1-z;\tau,\bar\tau), \tag{5.55}$$

this gives

$$\boxed{\ln \mathcal{Z}_{d,1}(\tau,\bar\tau) = \frac{V_{d-1}}{L_d^{d-1}\tau_2^{\frac{d-1}{2}}}\xi(d+1;\tau,\bar\tau)}. \tag{5.56}$$

Since $\xi(z;\tau,\bar{\tau})$ is invariant under modular transformations (5.49) while τ_2 transforms as

$$\tau_2 \to \frac{\tau_2}{|c\tau+d|^2}, \tag{5.57}$$

the partition function (5.54) transforms as

$$\boxed{\ln \mathcal{Z}_{d,1}(\tau',\bar{\tau}') = |c\tau+d|^{d-1} \ln \mathcal{Z}_{d,1}(\tau,\bar{\tau})}. \tag{5.58}$$

Remarks:

(i) The above result holds for $d > 1$ because if $d = 1$, i.e., for the free boson on $\mathbb{T}^2$, $\ln \mathcal{Z}_{1,1}(\tau,\bar{\tau}) = \frac{1}{2\pi} f_1(\tau,\bar{\tau})$, is not convergent and more care is needed to derive the correct result. This is reviewed below.

(ii) As discussed above, for $d = 3$, the result is directly relevant for the electromagnetic field with Casimir boundary conditions after the replacement $L_3 = 2a$.

(iii) If the chemical potential vanishes, $\alpha = 0$, $\tau = i\tau_2$, with $\tau_2 = \frac{\beta}{L_d}$ and $\mathcal{Z}_{d,1}(i\tau_2, -i\tau_2) = \mathcal{Z}_{d,1}(\tau_2) = Z_{d,1}(\beta)$, the result can be written in terms of an Epstein zeta function,

$$\zeta(s;1,\tau_2^2) = \sum_{(m^{d+1},m^d)\in\mathbb{Z}^2}{}' \frac{1}{[(m^d)^2 + (m^{d+1})^2\tau_2^2]^s}, \tag{5.59}$$

as

$$\ln \mathcal{Z}_{d,1}(\tau_2) = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d+1}{2}}} \frac{V_{d-1}\tau_2}{L_d^{d-1}} \zeta(\frac{d+1}{2};1,\tau_2^2). \tag{5.60}$$

If $a = 0 = d$, $b = 1 = -c$, the modular transformation reduces to temperature inversion $\tau_2 \to \frac{1}{\tau_2}$ with

$$\ln \mathcal{Z}_{d,1}(\frac{1}{\tau_2}) = \tau_2^{d-1} \ln \mathcal{Z}_{d,1}(\tau_2). \tag{5.61}$$

(iv) Setting $m^d = 0$ in (5.60) gives

$$\ln \mathcal{Z}^{\text{high}}_{d,1}(\tau_2) = \xi(d+1)\frac{V_{d-1}}{L_d^{d-1}\tau_2^d}, \tag{5.62}$$

which is the black body result (1.64). From the analysis of section 1.4.2, we know that this is the dominant contribution in the high temperature/large L_d limit $\tau_2 \ll 1$. This justifies a posteriori the use of a normal ordered Hamiltonian in section 1.4.2.

(v) From equation (5.61), it follows that in the low temperature/small L_d limit $\tau_2 \gg 1$, the dominant contribution is given by

$$\ln \mathcal{Z}^{\text{low}}_{d,1}(\tau_2) = \xi(d+1)\frac{V_{d-1}\tau_2}{L_d^{d-1}}. \tag{5.63}$$

The result for the Casimir energy with one small spatial dimension in (5.34), then follows by taking $-\frac{\partial}{\partial\beta}$ of this dominant contribution.

5.1.6 Canonical approach to massless scalar partition function on $\mathbb{R}^{d-1} \times \mathbb{T}^2$

In this section, we evaluate the partition function

$$\mathrm{Z}_{d,1}(\beta,\alpha) = \mathrm{Tr}\, e^{-\beta\widehat{H} + i\alpha\widehat{P}_d}, \tag{5.64}$$

directly in the operator formalism where $\widehat{H}$ and $\widehat{P}_d$ are given by

$$\widehat{H} = \sum_{n_i \in \mathbb{Z}^d} \omega_k \widehat{a}^\dagger_{k_i} \widehat{a}_{k_i} + E_0^d, \qquad \widehat{P}_d = \sum_{n_i \in \mathbb{Z}^d} k_d \widehat{a}^\dagger_{k_i} \widehat{a}_{k_i}, \tag{5.65}$$

and E_0^d is given in (5.34). It follows that

$$\mathrm{Z}_{d,1}(\beta,\alpha) = e^{-\beta E_0^d} \prod_{n_i \in \mathbb{Z}^d}{}' \sum_{N_{k_i} \in \mathbb{N}} e^{(-\beta\omega_k + i\alpha k_d) N_{k_i}} = e^{-\beta E_0^d} \prod_{n_i \in \mathbb{Z}^d}{}' \frac{1}{1 - e^{-\beta\omega_k + i\alpha k_d}}. \tag{5.66}$$

Turning again the sums into integrals in the large dimensions gives

$$\begin{aligned} \ln \mathrm{Z}_{d,1}(\beta,\alpha) &= -\beta E_0^d - \frac{V_{d-1}}{(2\pi)^{d-1}} \int_{-\infty}^{\infty} d^{d-1}k \sum_{n_d \in \mathbb{Z}} \ln \left[1 - e^{-\beta\omega_k + i\alpha k_d}\right] \\ &= -\beta E_0^d - \frac{V_{d-1}}{(2\pi)^{d-1}} \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \sum_{n_d \in \mathbb{Z}} \int_0^\infty dk\, k^{d-2} \ln[1 - e^{-\beta\sqrt{k^2+k_d^2} + i\alpha k_d}], \end{aligned} \tag{5.67}$$

where $k = k_I k^I$, the range of I is from 1 to $d-1$ and $V_{d-1} = \prod_{I=1}^{d-1} L_I$. For the sum over n_d, it is convenient to split $n_d = 0$ from the other terms. When using (1.60) for $d \to d-1$ for the former term, we get

$$\ln \mathrm{Z}_{d,1}(\beta,\alpha) = -\beta E_0^d + \frac{\Gamma(d-1)\zeta(d)}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})} \frac{V_{d-1}}{\beta^{d-1}} - \frac{V_{d-1}}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})} \sum_{n_d \in \mathbb{Z}}{}' I_d(\beta,\alpha;n_d), \tag{5.68}$$

where

$$I_d(\beta,\alpha;n_d) = \int_0^\infty dk\, k^{d-2} \ln[1 - e^{-\beta\sqrt{k^2+k_d^2} + i\alpha k_d}]. \tag{5.69}$$

Introducing the variable z as

$$z = \beta\sqrt{k^2 + k_d^2}, \quad k^{d-2}dk = \frac{1}{\beta^2\beta^{d-3}} (z^2 - \beta^2 k_d^2)^{\frac{d-3}{2}} z dz, \tag{5.70}$$

we get

$$I_d(\beta,\alpha;n_d) = -\frac{1}{\beta^{d-1}} \int_{\beta|k_d|}^\infty dz\, z (z^2 - \beta^2 k_d^2)^{\frac{d-3}{2}} \ln\left[1 - e^{-z + i\alpha k_d}\right]. \tag{5.71}$$

Expanding the logarithm as

$$\ln\left[1 - e^{-z+i\alpha k_d}\right] = -\sum_{l \in \mathbb{N}^*} \frac{e^{-lz + il\alpha k_d}}{l}, \tag{5.72}$$

we get, after the change of variables $z' = lz$,

$$I_d(\beta,\alpha;n_d) = -\frac{1}{\beta^{d-1}} \sum_{l \in \mathbb{N}^*} \frac{e^{il\alpha k_d}}{l^d} \int_{l\beta|k_d|}^\infty dz z (z^2 - l^2\beta^2 k_d^2)^{\frac{d-3}{2}} e^{-z}, \tag{5.73}$$

The integrals are given in terms of a modified Bessel function of the second kind as

$$\int_{l\beta|k_d|}^\infty dz z (z^2 - l^2\beta^2 k_d^2)^{\frac{d-3}{2}} e^{-z} = 2^{\frac{d-2}{2}} \frac{\Gamma(\frac{d-1}{2})}{\sqrt{\pi}} (l\beta|k_d|)^{\frac{d}{2}} K_{\frac{d}{2}}(l\beta|k_d|), \tag{5.74}$$

so that

$$\ln \mathrm{Z}_{d,1}(\beta,\alpha) = -\beta E_0^d + \frac{\Gamma(d-1)\zeta(d)}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})}\frac{V_{d-1}}{\beta^{d-1}} + 2\frac{V_{d-1}}{L_d^{\frac{d}{2}}\beta^{\frac{d-2}{2}}}\sum_{n_d\in\mathbb{Z}}{}' \sum_{l\in\mathbb{N}^*}(\frac{|n_d|}{l})^{\frac{d}{2}} K_{\frac{d}{2}}(2\pi l|n_d|\frac{\beta}{L_d})e^{2\pi i l n_d\frac{\alpha}{L_d}}. \quad (5.75)$$

Remarks:

(i) At low temperature/small distance $\frac{\beta}{L_d} \gg 1$, the leading term in the expansion of the partition function is directly related to the Casimir energy. The leading correction is the contribution of the modes with spatial frequencies $n_d = 0$. It coincides with the black body result (1.61) of a massless scalar field in $d-1$ spatial dimensions. On account of the equivalence of this expression with (1.64), it can also be written more compactly as $\xi(d)\frac{V_{d-1}}{\beta^{d-1}}$. It is independent of L_d and thus does not contribute to the Casimir pressure,

$$\mathrm{p}_{d,1}(\beta,\alpha) = \frac{1}{V_{d-1}}\frac{\partial(\beta^{-1}\ln \mathrm{Z}_{d,1}(\beta,\alpha))}{\partial L_d}. \quad (5.76)$$

The asymptotic expansion

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}e^{-x}(1+O(x^{-1})), \quad (5.77)$$

for large x, implies that all other terms are exponentially suppressed. It follows that low-temperature/small distance expansion of the Casimir pressure is

$$\boxed{\mathrm{p}^{\mathrm{low}}_{d,1}(\beta,\alpha) = -d\xi(d+1)\frac{1}{L_d^{d+1}} + \dots}, \quad (5.78)$$

where the dots denote exponentially suppressed terms. In the low-temperature/small distance expansion of the entropy,

$$\mathrm{S}_{d,1}(\beta,\alpha) = (1-\beta\partial_\beta)\ln \mathrm{Z}_{d,1}(\beta,\alpha), \quad (5.79)$$

on the other hand, the first term in (5.75) proportional to the Casimir energy in d spatial dimensions drops out since it is linear in β and the leading term now comes from the lower dimensional scalar field, i.e., the modes with $n_d = 0$,

$$\boxed{\mathrm{S}^{\mathrm{low}}_{d,1}(\beta,\alpha) = d\xi(d)\frac{V_{d-1}}{\beta^{d-1}} + \dots}. \quad (5.80)$$

Since the two leading terms in the low temperature/small intervall expansion of (5.75) do not depend on α they are the same when starting from $\ln Z_{d,1}(\beta,\mu)$. It follows that the leading terms in the expansion of

$$p_{d,1}(\beta,\mu) = \frac{1}{V_{d-1}}\frac{\partial(\beta^{-1}\ln Z_{d,1}(\beta,\mu))}{\partial L_d}, \quad S_{d,1}(\beta,\mu) = (1-\beta\partial_\beta)\ln Z_{d,1}(\beta,\mu), \quad (5.81)$$

are still given by the right hand sides of (5.78) and of (5.80).

(ii) When changing the double sum in the last term to a sum over $m = ln_d \in \mathbb{Z}^*$ and introducing the divisor sum

$$\sigma_s(m) = \sum_{n|m} n^s, \quad (5.82)$$

the exponentially suppressed terms, i.e., the last line of (2.78), may be written as

$$2\frac{V_{d-1}}{L_d^{\frac{d}{2}}\beta^{\frac{d-2}{2}}}\sum_{m\in\mathbb{Z}}{}' \sigma_{-d}(m)|m|^{\frac{d}{2}} K_{\frac{d}{2}}(2\pi m\frac{\beta}{L_d})e^{2\pi i m\frac{\alpha}{L_d}}. \quad (5.83)$$

When taking into account the explicit expression for the vacuum energy E_0^d in (5.34), the partition function may be re-written as

$$\boxed{\ln \mathrm{Z}_{d,1}(\beta,\alpha) = \frac{V_{d-1}}{L_d^{d-1}}\Big[\xi(d+1)\frac{\beta}{L_d} + \xi(d)(\frac{L_d}{\beta})^{d-1}}$$
$$\boxed{+2(\frac{L_d}{\beta})^{\frac{d-2}{2}}\sum_{m\in\mathbb{Z}}{}' \sigma_{-d}(m)|m|^{\frac{d}{2}}K_{\frac{d}{2}}(2\pi m\frac{\beta}{L_d})e^{2\pi i m\frac{\alpha}{L_d}}\Big]}. \quad (5.84)$$

The equivalence of the functional Lagrangian and the canonical approaches to computing the partition function then implies that this last result is the same than (5.53). For $s = \frac{d+1}{2}$, this shows

$$f_s(\tau,\bar\tau) = 2\zeta(2s)\Big[\tau_2^s + \frac{\xi(2s-1)}{\xi(2s)}\tau_2^{1-s}$$
$$+\frac{2}{\xi(2s)}\tau_2^{\frac{1}{2}}\sum_m{}' \sigma_{1-2s}(m)|m|^{\frac{2s-1}{2}}K_{\frac{2s-1}{2}}(2\pi m\tau_2)e^{2\pi i m\tau_1}\Big], \quad (5.85)$$

which is the Fourier expansion of $f_s(\tau,\bar\tau)$, traditionally derived using Poisson resummation (see e.g. [41], Appendix A).

(iii) When expressed in terms of inverse temperature and chemical potential, the generating set of transformations of the modular group become

$$\tau' = \tau + 1 \iff \begin{cases} (\frac{\beta}{L_d})' = \frac{\beta}{L_d} \\ (\frac{\alpha}{L_d})' = \frac{\alpha}{L_d} + 1 \end{cases} \qquad \tau' = -\frac{1}{\tau} \iff \begin{cases} (\frac{\beta}{L_d})' = \frac{L_d\beta}{\alpha^2+\beta^2} \\ (\frac{\alpha}{L_d})' = -\frac{L_d\alpha}{\alpha^2+\beta^2} \end{cases}. \quad (5.86)$$

Under the first of these transformations, $\ln \mathrm{Z}_{d,1}(\beta,\alpha)$ in (5.84) is manifestly invariant, as required by (2.14) for $c = 0, a = b = d = 1$. For the second of these transformations, we get

$$\ln \mathrm{Z}_{d,1}(\beta',\alpha') = \frac{(\alpha^2+\beta^2)^{\frac{d-1}{2}}}{L_d^{d-1}} \ln \mathrm{Z}_{d,1}(\beta,\alpha). \quad (5.87)$$

When transposing and using the explicit expression in (5.84) for the LHS, this gives

$$\ln \mathrm{Z}_{d,1}(\beta,\alpha) = \frac{V_{d-1}}{(\alpha^2+\beta^2)^{\frac{d-1}{2}}}\Big[\xi(d+1)\frac{L_d\beta}{\alpha^2+\beta^2} + \xi(d)(\frac{\alpha^2+\beta^2}{L_d\beta})^{d-1}$$
$$+2(\frac{\alpha^2+\beta^2}{L_d\beta})^{\frac{d-2}{2}}\sum_{m\in\mathbb{Z}}{}' \sigma_{-d}(m)|m|^{\frac{d}{2}}K_{\frac{d}{2}}(2\pi m\frac{L_d\beta}{\alpha^2+\beta^2})e^{-2\pi i m\frac{L_d\alpha}{\alpha^2+\beta^2}}\Big]. \quad (5.88)$$

For a high temperature/large interval expansion, it is more convenient to use μ rather than α, in terms of which the previous expression becomes

$$\ln Z_{d,1}(\beta,\mu) = \frac{V_{d-1}}{L_d^{d-1}(1+\mu^2)^{\frac{d-1}{2}}}\Big[\frac{\xi(d+1)}{1+\mu^2}(\frac{\beta}{L_d})^{-d} + \xi(d)(1+\mu^2)^{d-1} +$$
$$+2(\frac{\beta}{L_d})^{-\frac{d}{2}}(1+\mu^2)^{\frac{d-2}{2}}\sum_{m\in\mathbb{Z}}{}' \sigma_{-d}(m)|m|^{\frac{d}{2}}K_{\frac{d}{2}}(2\pi m\frac{L_d}{\beta}\frac{1}{1+\mu^2})e^{-2\pi i m\frac{L_d}{\beta}\frac{1}{1+\mu^2}}\Big]. \quad (5.89)$$

The high temperature/large interval expansion is then defined by $\frac{\beta}{L_d} \ll 1$ at fixed μ. The leading contributions are given by the first two terms, while the others are exponentially suppressed,

$$\ln Z_{d,1}^{\mathrm{high}}(\beta,\mu) = \frac{V_{d-1}L_d}{\beta^d}\xi(d+1)(1+\mu^2)^{-\frac{d+1}{2}} + \frac{V_{d-1}}{L_d^{d-1}}\xi(d)(1+\mu^2)^{\frac{d-1}{2}} + \ldots. \quad (5.90)$$

The associated expansion of the Casimir pressure is

$$p_{d,1}^{\text{high}}(\beta,\mu)=\frac{\xi(d+1)}{\beta^{d+1}}(1+\mu^2)^{-\frac{d+1}{2}}-\frac{(d-1)\xi(d)}{\beta L_d^d}(1+\mu^2)^{\frac{d-1}{2}}+\dots, \tag{5.91}$$

while for the entropy, one finds

$$S_{d,1}^{\text{high}}(\beta,\mu)=\frac{V_{d-1}L_d}{\beta^d}(d+1)\xi(d+1)(1+\mu^2)^{-\frac{d+1}{2}}+\frac{V_{d-1}}{L_d^{d-1}}\xi(d)(1+\mu^2)^{\frac{d-1}{2}}+\dots. \tag{5.92}$$

This can be written as

$$S_{d,1}^{\text{high}}(\beta,\mu)=(d+1)\varepsilon_{\text{vac}}^d\frac{V_d}{\beta^d}(1+\mu^2)^{-\frac{d+1}{2}}+\varepsilon_{\text{vac}}^{d-1}\frac{V_{d-1}}{L_d^{d-1}}(1+\mu^2)^{\frac{d-1}{2}}+\dots. \tag{5.93}$$

5.1.7 Massless scalar partition function on $\mathbb{T}^2$

In this section, we derive the well-known result [42, 43] (see e.g. [44, 12, 45] for reviews) for a torus in Euclidean spacetime, that is to say we derive the partition function of a massless scalar field in a one spatial dimension, $d=1$, $L_1=L$, with periodic boundary conditions. As we have seen in section 5.1.5, a naive application of the functional approach leads one to the divergent expression

$$\ln\mathcal{Z}_1(\tau,\bar\tau)=\frac{1}{2\pi}f_1(\tau,\bar\tau). \tag{5.94}$$

5.1.7.1 Canonical approach

No such divergences occur in the canonical approach. In the analysis of section 1.4.1, the full Hamiltonian, including the $n=0$ mode, which is a free particle, is now given by

$$H=\frac{1}{L}\frac{p^2}{2}+\frac{1}{2}\sum_{n\in\mathbb{Z}}{}'\,\omega_n(a_n^*a_n+a_na_n^*),\qquad \omega_n=\frac{2\pi|n|}{L}. \tag{5.95}$$

After the conventional redefinition $p=\sqrt{4\pi}a_0$, the quantum Hamiltonian including the zero point energy E_0^1 in (5.34) can be written as

$$\widehat{H}=-\frac{\pi}{6L}+\frac{2\pi}{L}\big(\widehat{a}_0^2+\sum_{n\in\mathbb{Z}}{}'\,|n|\widehat{a}_n^\dagger\widehat{a}_n\big). \tag{5.96}$$

Using (1.56), we get

$$\mathrm{Z}_1(\beta,\alpha)=\sqrt{\frac{L}{2\pi\beta}}e^{\frac{\pi\beta}{6L}}\prod_{n\in\mathbb{Z}}{}'\sum_{N_n\in\mathbb{N}}e^{\frac{2\pi}{L}(-\beta|n|+i\alpha n)N_n}=\sqrt{\frac{L}{2\pi\beta}}e^{\frac{\pi\beta}{6L}}\prod_{n\in\mathbb{Z}}{}'\frac{1}{1-e^{\frac{2\pi}{L}(-\beta|n|+i\alpha n)}}. \tag{5.97}$$

Defining

$$q=e^{2\pi i\tau},\quad \bar q=e^{-2\pi i\bar\tau}, \tag{5.98}$$

where τ and $\bar\tau$ are given in (5.45), we have

$$\ln\mathcal{Z}_1(\tau,\bar\tau)=-\frac{1}{2}\ln(2\pi)+\frac{\pi\tau_2}{6}-\frac{1}{2}\ln(\tau_2)-\sum_{n\in\mathbb{N}^*}\ln\left[(1-q^n)(1-\bar q^n)\right]. \tag{5.99}$$

In terms of Dedekind's eta function,

$$\eta(q) = q^{\frac{1}{24}} \prod_{n\in\mathbb{N}^*} (1 - q^n), \tag{5.100}$$

and up to irrelevant numerical factors, the modular invariant partition function can then be written compactly as

$$\boxed{\mathcal{Z}_1(\tau,\bar{\tau}) = \frac{1}{\sqrt{\tau_2}|\eta(q(\tau))|^2}}. \tag{5.101}$$

5.1.7.2 Dimensional continuation

A different way to give a meaning the non-convergent expression of the partition function (5.94) obtained in the functional or heat kernel approach is to study the limit $s \to 1$ of the real analytic Eisenstein series $f_s(\tau,\bar{\tau})$ by starting from its Fourier expansion [41] given in equation (5.85),

$$\begin{aligned} f_s(\tau,\bar{\tau}) = 2\zeta(2s)\tau_2^s + 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)}\tau_2^{1-s} \\ + 2\frac{\pi^s}{\Gamma(s)}\tau_2^{\frac{1}{2}}\sum_{n\in\mathbb{Z}}{}' \sum_{m\in\mathbb{Z}}{}' |\frac{n}{m}|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|nm|\tau_2)e^{2\pi i n|m|\tau_1}. \end{aligned} \tag{5.102}$$

Only the second term diverges in the limit $s \to 1$, while the first term on the right hand side becomes

$$\frac{\pi^2}{3}\tau_2 = -\frac{\pi}{12}\ln|q|^2. \tag{5.103}$$

Using $K_{\frac{1}{2}}(z) = e^{-z}\sqrt{\frac{\pi}{2z}}$, the term on the last line becomes

$$\begin{aligned} \pi\sum_{n\in\mathbb{Z}}{}' \sum_{m\in\mathbb{Z}}{}' \frac{1}{|m|}e^{-2\pi|nm|\tau_2+2\pi i n|m|\tau_1} &= 2\pi \sum_{m\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*}\frac{1}{m}(e^{2\pi i n m\tau} + e^{-2\pi i n m\bar{\tau}}) \\ &= -2\pi\sum_{n\in\mathbb{N}^*}\left[\ln(1-e^{2\pi i n\tau}) + \ln(1-e^{-2\pi i n\bar{\tau}})\right] \\ &= -2\pi\ln\prod_{n\in\mathbb{N}^*}(1-q^n)\prod_{n\in\mathbb{N}^*}(1-\bar{q}^n). \end{aligned} \tag{5.104}$$

These terms combine into

$$-2\pi\ln\prod_{n\in\mathbb{N}^*}q^{\frac{1}{24}}(1-q^n)\prod_{n\in\mathbb{N}^*}\bar{q}^{\frac{1}{24}}(1-\bar{q}^n) = -2\pi\ln|\eta(q)|^2. \tag{5.105}$$

For the divergent term, we consider the expansion around $s = 1$ using

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1), \tag{5.106}$$

so that

$$\begin{aligned} 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)}\tau_2^{1-s} &= \frac{\pi}{s-1} + \pi\left(3\gamma + \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - \ln\tau_2\right) + O(s-1) \\ &= \frac{\pi}{s-1} + \pi\left(2\gamma - 2\ln 2 - \ln\tau_2\right) + O(s-1). \end{aligned} \tag{5.107}$$

We thus get

$$\lim_{s\to 1}\left(f_s(\tau,\bar\tau)-\frac{\pi}{s-1}\right)=2\pi\left(\gamma-\ln 2-\ln\sqrt{\tau_2}|\eta(q)|^2\right). \tag{5.108}$$

Therefore, once we remove the divergence, the real analytic Eisenstein series for $s=1$ contains the result for the partition function of the $d=1$ case. Note that the divergence in the second term of the expansion in (5.102), comes from the $n_1=0$ term in the mode expansion of the field (at fixed time), as shown in the canonical approach in the previous section. In $d>1$ this mode is a scalar field in $d-1$ spatial dimensions. For $d=1$, it is a free particle, which gives the factor $\sqrt{\tau_2}$ needed for modular invariance. Associated with the divergence, the finite, τ_2 independent term, is undetermined and needs to be fixed by a normalization condition.

How this computation appears from the viewpoint of the integral representation connected to the world-line approach is discussed in [43].

5.1.8 Casimir effect at finite temperature. Alternative derivation

Casimir effect at finite temperature

Saturday, June 1, 2019 10:43 AM

$$F_C = -\frac{\partial}{\partial d}\mathcal{F} \qquad \mathcal{F}: \text{Helmholtz free energy.}$$

Justification: $W[0] = -E_0 T$

Peskin: 11.43.

$\langle\Omega|e^{-iHT}|\Omega\rangle^{J} = Z[J]$

$\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}(S+J\phi)} = e^{\frac{i}{\hbar}W[J]}$

$J=0$ $\quad W[0] = -E_0 T, \quad E_0 = \langle\Omega|H|\Omega\rangle$

$$\boxed{E_0 = \frac{i}{T}\ln Z[0]}$$

$$\mathcal{F} = -\frac{1}{\beta}\ln Z(\beta)$$

$$= -\frac{1}{\beta}\ln\Big(\prod_{\vec{k}}\sum_{n=0}^{\infty} e^{-\beta\hbar\omega_{\vec{k}}(n+\frac{1}{2})}\Big)$$

$$= -\frac{1}{\beta}\ln\prod_{\vec{k}} e^{-\beta\hbar\omega_{\vec{k}}\frac{1}{2}}\sum_{n=0}^{\infty} e^{-\beta\hbar\omega_{\vec{k}} n} = -\frac{1}{\beta}\ln\prod_{\vec{k}} e^{-\beta\hbar\omega_{\vec{k}}\frac{1}{2}}\frac{1}{1-e^{-\beta\hbar\omega_{\vec{k}}}}$$

$$= \sum_{\vec{k}}\Big[\underbrace{\frac{1}{2}\hbar\omega_{\vec{k}}}_{\text{zero temperature result, just computed}} + \frac{1}{\beta}\ln(1-e^{-\beta\hbar\omega_{\vec{k}}})\Big] = -\frac{L^2\hbar c\pi^2}{720 d^3} + \mathcal{F}_{c,2}(d,T) \qquad (B)$$

put a $\frac{1}{2}$ for $n=0$

$$\mathcal{F}_{c,2}(d,T) = 2L^2\frac{1}{\beta}\Big\{\int\frac{d^2k}{(2\pi)^2}\sum_{n=0}^{\infty}{}^{(1)}\ln\Big(1-\exp\Big(-\hbar\beta c\sqrt{k^2+\big(\tfrac{n\pi}{d}\big)^2}\Big)\Big)$$

$$+\Big[\big(\tfrac{L}{2}-d\big)-\tfrac{L}{2}\Big]\int\frac{dk_z}{2\pi}\ln\Big(1-\exp\Big(-\hbar\beta c\sqrt{k^2+k_z^2}\Big)\Big)\Big\}$$

(X)

$$\text{Defining } b(d,T,n) = \frac{1}{2}\frac{1}{\beta}\int_{n^2}^{\infty} ds\,\ln\Big[1-\exp\Big(-\frac{\pi\hbar}{d}\beta c\sqrt{s}\Big)\Big]$$

$$\Big(s = \frac{d^2}{\pi^2}k^2 + n^2 \qquad ds = 2\frac{d^2}{\pi^2}k\,dk\Big)$$

$k_z = n\frac{\pi}{d}$

$dk_z = dn\,\frac{\pi}{d}$

$$\mathcal{F}_{c,2}(d,T) = \cancel{2}L^2\frac{1}{\cancel{2\pi}}\frac{\pi^{\cancel{2}}}{d^2}\Big[\sum_{n=0}^{\infty}{}' b(d,T,n) - \int_0^{\infty} dn\, b(d,T,n)\Big]$$

$$= \frac{L^2\pi}{d^2}\Big[\frac{1}{2}b(d,T,0) + \sum_{n=1}^{\infty}(d,T,n) - \int_0^{\infty} dn\, b(d,T,n)\Big] \qquad (X)$$

$$b(d,T,0) = \frac{1}{2\beta}\int_0^{\infty} ds\,\ln\Big(1-\exp\Big(-\frac{\pi\hbar}{d}\beta c\sqrt{s}\Big)\Big) \qquad x = \frac{\pi\hbar}{d}\beta c\sqrt{s}$$

$$dx = \frac{\pi\hbar}{d}\beta c\frac{1}{2\sqrt{s}}\,ds$$

$$= \frac{d^2}{\pi^2\hbar^2c^2\beta^3}\int_0^{\infty} dx\; x\ln(1-e^{-x})$$

$$= \frac{d^2}{\pi^2\hbar^2c^2\beta^3}\underbrace{\int_0^\infty dx\, x \ln(1-e^{-x})}_{\left[\frac{x^2}{2}\ln(1-e^{-x})\right]_0^\infty - \int_0^\infty dx \frac{x^2}{2}\frac{e^{-x}}{1-e^{-x}}}$$

$$dx = \frac{\pi\hbar}{d}\beta c \frac{1}{2\sqrt{s}}\, ds$$

$$x\,dx = \left(\frac{\pi\hbar}{d}\beta c\right)^2 \frac{1}{2}\, ds$$

$$= -\frac{d^2}{2\pi^2\hbar^2c^2\beta^3}\int_0^\infty dx \frac{x^2}{e^{x}-1} \qquad \zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt \frac{t^{s-1}}{e^t-1}$$

$$= -\frac{d^2}{\cancel{2}\pi^2\hbar^2c^2\beta^3}\zeta(3)\underbrace{\cancel{\Gamma(3)}}_{=2} = -\frac{d^2}{\pi^2\hbar^2c^2\beta^3}\zeta(3)$$

| to be multiplied by $\frac{L^2\pi}{d^2}\frac{1}{2}$

For the first term, we note that it is given by the contribution of a massless scalar in 2d to the Helmholtz free energy.

$$\left(-\frac{1}{\beta}\right)\left(\frac{L^2}{(2\pi)^2}\right)\int d^2k \ln(1-e^{-\beta\omega_{\vec{k}}}) = \underbrace{\left(-\frac{1}{\beta}\right)\frac{L^2}{2\pi}\zeta(3)\beta^{-2}}_{(y)}$$

when $\hbar = 1 = c$

For the last term, we note from (x) that it is given by (-2) the contribution of a massless scalar in 3d to the Helmholtz free energy:

[Bordag et al, (5.36, 37, 39, 40, 41) leading contribution also agrees with (y); $\delta(d,T,\mu)$, $\mu>0$ is exponentially surpressed]

$$(x) = -\frac{2}{\beta}\underbrace{\frac{(V=L^2d)}{(2\pi)^3}\int d^3k \ln(1-e^{-\beta\omega_{\vec{k}}})}$$

$$= (-)\frac{2}{\beta}(-)\ln Z(\beta, \text{massless scalar}, d=3)$$

$$= \frac{2}{\beta}\frac{L^2d}{(2\pi)^3}\frac{2\pi^{3/2}}{\frac{1}{2}\pi^{1/2}}\frac{1}{3}\beta^{-3}\Gamma(4)\zeta(4)$$

$$= \frac{2}{\beta}\left[\frac{L^2d\,\pi^2}{90\beta^3}\right]$$

see section 1.4. $\frac{\pi^4}{90}$

$$\left(\cancel{\Gamma(4)}\zeta(4) = \cancel{3!}\frac{\pi^4}{90}\right) = \frac{L^2\pi}{d^2}\left[\frac{2}{\beta}\left(\frac{d}{\pi\hbar\beta c}\right)^3\zeta(4)\right]$$

$$= \frac{L^2\pi}{d^2}\left[\frac{2}{\beta}\left(\frac{d}{\pi\hbar\beta c}\right)^3 \zeta(4)\right]$$

when keeping factors of $\hbar$ and c.

$$\Rightarrow \boxed{\mathcal{F}_{c,2}(d,T) = \frac{L^2\pi}{d^2}\left[-\frac{1}{2\beta}\left(\frac{d}{\pi\hbar\beta c}\right)^2\zeta(3) + \sum_{n=1}^{\infty} b(d,T,n) + \frac{2}{\beta}\left(\frac{d}{\pi\hbar\beta c}\right)^3 \zeta(4)\right]} \quad (\chi)$$

Using (β) (χ), (x), we can write

$\hbar = 1 = c$

$$\frac{\mathcal{F}}{L^2} = -\frac{\pi^2}{720 d^3} + \frac{\pi}{d^2}\left[\sum_{n=0}^{\prime} b(d,T,n)\right] + \frac{\pi^2\beta^{-4} d}{45}$$

$\mathcal{C}$ - black body result

$$\shortparallel$$

$$\frac{\pi^2}{720 d^3 t^4}$$

$$b(d,T,n) = \frac{1}{2}\frac{1}{\beta}\int_{n^2}^{\infty} ds \ \ln\left(1 - e^{-\frac{\pi\beta}{d}\sqrt{s}}\right)$$

$$z = \frac{\pi\beta}{d}\sqrt{s} \qquad dz = \frac{\pi\beta}{2d\sqrt{s}}\,ds = \frac{1}{2}\frac{\pi^2\beta^2}{d^2 z}\,ds$$

$$ds = 2\frac{d^2}{\pi^2\beta^2} z \ dz$$

$$\boxed{k_B T_{eff} = \frac{1}{2d}}$$

$$\boxed{t = \frac{T_{eff}}{T} = \frac{k_B T_{eff}}{k_B T} = \frac{\beta}{2d}}$$

$$\Rightarrow \frac{\pi^2}{720 d^3 t^4} = \frac{\pi^2\beta^{-4}\, 16\, d^4}{720 d^3} = \frac{\pi^2\beta^{-4} d\, \cancel{4}}{\cancel{180}\ 45}$$

$$\frac{\mathcal{F}}{L^2} = -\frac{\pi^2}{720 d^3} + \frac{\beta^{-3}}{\pi}\left[\sum_{n=0}^{\prime}\int_{2\pi n t}^{\infty} dz \ z \ \ln\left(1-e^{-z}\right)\right] + \frac{\pi^2}{720 d^3 t^4}$$

(= 7.86 of Bordag)

$$\int_{2\pi n t}^{\infty} dz \ z \ \ln\left(1-e^{-z}\right) = -\sum_{l=1}^{\infty}\frac{1}{l}\int_{2\pi n t}^{\infty} dz \ z \ e^{-lz}$$

$f \qquad g'$

$$\frac{d}{dz}\left[z \frac{e^{-lz}}{-l}\right] + \frac{e^{-lz}}{l}$$

$$= -\sum_{l=1}^{\infty} \frac{1}{l^2} 2\pi n t\, e^{-l 2\pi n t} + \sum_{l=1}^{\infty} \frac{1}{l^3} \left[e^{-lz} \right]_{2\pi n t}^{\infty}$$

$$- \sum_{l=1}^{\infty} \frac{1}{l^3} e^{-2\pi l n t}$$

$$= -\sum_{l=1}^{\infty} \left(\frac{2\pi n t}{l^2} + \frac{1}{l^3} \right) e^{-2\pi l n t}$$

$$= -t^3 \sum_{l=1}^{\infty} \frac{1}{l^3 t^3} (1 + 2\pi n l t)\, e^{-2\pi l n t}$$

$$t^3 = \frac{\beta^3}{(2d)^3}$$

$$\sum_{n=0}^{\prime} x^n = \frac{1}{1-x} - \frac{1}{2}$$

$$\sum_{n=0}^{\prime} -t^3 \sum_{l=1}^{\infty} \frac{1}{l^3 t^3} (1 + 2\pi n l t)\, e^{-2\pi l n t}$$

$$\sum_{n=0}^{\prime} n x^n = \sum_{n=0} n x^n$$

$$= -t^3 \sum_{l=1}^{\infty} \frac{1}{l^3 t^3} \left(\frac{1}{1-e^{-2\pi l t}} - \frac{1}{2} + \underbrace{2\pi l t}\, e^{-2\pi l t} \frac{1}{(1-e^{-2\pi l t})^2} \right)$$

$$= x \frac{d}{dx} \sum_{n=0} x^n$$

$$= -\frac{t^3}{2} \sum_{l=1}^{\infty} \frac{\coth \pi l t}{t^3 l^3} + \frac{\pi}{t^2 l^2} \frac{1}{\mathrm{sh}^2(\pi t l)} \quad \checkmark$$

$$= x \frac{d}{dx} \frac{1}{1-x}$$

$$x \frac{1}{(1-x)^2}$$

$$x = e^{-2\pi l t}$$

Indeed: $$\frac{1}{t^3 l^3} \frac{e^{+\pi t l} + e^{-\pi t l}}{e^{+\pi t l} - e^{-\pi t l}} + \frac{\pi}{t^2 l^2} \frac{4}{(e^{\pi t l} - e^{-\pi t l})^2}$$

$$\frac{1}{1-e^{-2\pi l t}} - \frac{1}{2} = \frac{e^{\pi l t}}{e^{\pi l t} - e^{-\pi l t}} - \frac{1}{2} = \frac{e^{\pi l t} - \frac{1}{2}(e^{\pi l t} - e^{-\pi l t})}{e^{\pi l t} - e^{-\pi l t}}$$

$$= \frac{1}{2} \frac{e^{\pi l t} + e^{-\pi l t}}{e^{\pi l} - e^{-\pi l t}}$$

$$-\frac{t^3}{2} \frac{\beta^{-3}}{\pi} = -\frac{1}{2}\left(\frac{\beta}{2d}\right)^3 \frac{\beta^{-3}}{\pi} = -\frac{1}{16}\, d^3 \frac{1}{\pi}$$

$$\boxed{\frac{F}{L^2} = -\frac{\pi^2}{720 d^3}\left[1 + \frac{45}{\pi^3}\sum_{\ell=1}^{\infty}\left(\frac{\coth \pi \ell t}{t^3 \ell^3} + \frac{\pi}{t^2 \ell^2 \sinh^2 \pi \ell t}\right) - \frac{1}{t^4}\right]}$$

Bordag (7.81)

NB: In this computation, when one does first the sum over n and then the sum over ℓ, one looses the control on the contribution of the lower dimensional massless scalar to the partition function.

5.1.9 Low temperature limit. Alternative

Low temperature limit

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$$e^{-\frac{\pi}{\alpha}\sqrt{s}} = \frac{1}{e^{\frac{\pi}{\alpha}\sqrt{s}}} \qquad \alpha \to 0 \qquad e^{-\infty}\ \text{small}$$

low temperature limit: $\underbrace{\frac{d}{\hbar\beta c}}_{\alpha} \ll 1$

$$\sum_{n=1}^{\infty} b(d,T,n) = \frac{1}{2\beta}\sum_{n=1}^{\infty}\int_{n^2}^{\infty} ds\ \ln\left(1-\exp\left(-\frac{\pi\hbar\beta c}{d}\sqrt{s}\right)\right)$$

$$\ln(1+x) = \sum_{m=1} (-)^{m-1}\frac{x^m}{m}$$

$$\ln(1-x) = -\sum_{m=1}\frac{x^m}{m}$$

$$= -\frac{1}{2\beta}\sum_{n=1}^{\infty}\int_{n^2}^{\infty} ds \sum_{m=1}^{\infty}\frac{1}{m}\exp\left(-m\frac{\pi\hbar\beta c}{d}\sqrt{s}\right)$$

$$s' = \sqrt{s} \qquad s'ds' = \tfrac{1}{2}ds$$

$$= -\sum_{m=1}^{\infty}\frac{1}{m\beta}\sum_{n=1}^{\infty}\int_{n}^{\infty} ds'\ s'\exp\left(-m\frac{\pi\hbar\beta c}{d}s'\right)$$

$$\left[-\frac{d}{m\pi\hbar\beta c}s'\exp\left(-m\frac{\pi\hbar\beta c}{d}s'\right)\right]_n^{\infty}$$

$$+\frac{d}{m\pi\hbar\beta c}\int_n^{\infty} ds'\ e^{-m\frac{\pi\hbar\beta c}{d}s'}$$

$$= -\sum_{m=1}^{\infty}\frac{1}{m\beta}\cdot\sum_{n=1}^{\infty}\left[\frac{d}{m\pi\hbar\beta c}\,n\exp\left(-m\frac{\pi\hbar\beta c}{d}n\right)+\left(\frac{d}{m\pi\hbar\beta c}\right)^2\exp\left(-m\frac{\pi\hbar\beta c}{d}n\right)\right]$$

$$= -\sum_{m=1}^{\infty}\frac{1}{m\beta}\left(\frac{d}{m\pi\hbar\beta c}\right)^2\sum_{n=1}^{\infty}\left(n\frac{m\pi\hbar\beta c}{d}+1\right)\exp\left(-nm\frac{\pi\hbar\beta c}{d}\right)$$

$$\approx -\frac{1}{\beta}\left(\frac{d}{\pi\hbar\beta c}\right)^2\left(\frac{\pi\hbar\beta c}{d}+1\right)e^{-\frac{\pi\hbar\beta c}{d}}.$$

Total result at low temperature $\quad \frac{c^2\pi}{d^2}\ \underbrace{-\frac{1}{2\beta}}\left(\frac{d}{\pi\hbar\beta c}\right)^2\underbrace{\zeta(3)}$

$$\boxed{F_c(d,T) = -\frac{L^2\pi^2\hbar c}{720\, d^3}\left[1 + 360\,\zeta(3)\left(\frac{d}{\pi\hbar\beta c}\right)^3 - \left(\frac{2d}{\hbar\beta c}\right)^4 + 720\left(\frac{d}{\pi\hbar\beta c}\right)^3\left(1+\frac{\pi\hbar\beta c}{d}\right)e^{-\frac{\pi\hbar\beta c}{d}}\right]} \quad (R)$$

$$\frac{L^2\pi}{d^2}\ \frac{\pi\hbar\beta c}{d}\cdot\frac{1}{2\beta}\left(\frac{d}{\pi\hbar\beta c}\right)^3\zeta(3)$$

$$\frac{\pi\hbar\beta c}{d} \to \gg 1$$

NB: It follows that the leading correction to the zero temperature result for the free energy comes from the massless scalar.

$$\frac{L^2\pi}{d^2}\ \frac{2}{\beta}\left(\frac{d}{\pi\hbar\beta c}\right)^3\underset{\frac{\pi^4}{90}}{\underset{\shortparallel}{\zeta(4)}} = -\frac{L^2\pi^2\hbar c}{720\,d^3}\left[-\left(\frac{2d}{\hbar\beta c}\right)^4\right]$$

$$= \frac{L^2 d\,\pi^2}{45\ \hbar^3c^3\beta^4} \qquad \frac{L^2\pi^2 d}{\hbar^3c^3\beta^4}\ \frac{\cancel{16}\ \cancel{4}}{\cancel{720}\ \cancel{180}\ 45} \checkmark$$

$$+\cancel{L^2}\cancel{\pi^2}\cancel{\hbar}\cancel{c}\ \frac{1}{(\cancel{\pi\hbar}/\beta\cancel{d})^3}\left(1+\cancel{\frac{\pi\hbar\beta c}{d}}\right)\cancel{e^{-\frac{\pi\hbar\beta c}{d}}} \leftrightarrow \frac{\cancel{L^2}\cancel{\pi}}{\cancel{d^2}}\left[+\frac{1}{\cancel{\beta}}\left(\frac{\cancel{d}}{\cancel{\pi\hbar\beta c}}\right)^2\left(\cancel{\pi\hbar\beta c/d+1}\right)\cancel{\exp\left(-\frac{\pi\hbar\beta c}{d}\right)}\right]$$

$$\Rightarrow F_c(d,T) = -\frac{L^2\pi^2\hbar c}{240\,d^4}\left[1 + 360\,\cancel{\zeta(3)\left(\frac{d}{\pi\hbar\beta c}\right)^3} - \left(\frac{2d}{\hbar\beta c}\right)^4 + 720\left(\frac{d}{\pi\hbar\beta c}\right)^3\cancel{\left(1+\frac{\pi\hbar\beta c}{d}\right)}e^{-\frac{\pi\hbar\beta c}{d}}\right.$$

$\left[-\frac{1}{3}d\frac{\partial}{\partial d}\text{ of argument}\right]$

$$-\frac{1}{3}\,360\,\cancel{\zeta(3)\left(\frac{d}{\pi\hbar\beta c}\right)^3}\cdot 3 + \frac{4}{3}\left(\frac{2d}{\hbar\beta c}\right)^4$$

$$-\frac{3}{3}\,720\,\cancel{\left(\frac{d}{\pi\hbar\beta c}\right)^3\left(1+\frac{\pi\hbar\beta c}{d}\right)}e^{-\frac{\pi\hbar\beta c}{d}}$$

$\frac{4}{3}-1=\frac{1}{3}$

$$+\frac{1}{3}\,720\left(\frac{d}{\pi\hbar\beta c}\right)^3\cancel{\frac{\pi\hbar\beta c}{d}}\,e^{-\frac{\pi\hbar\beta c}{d}}$$

$$\left.-\frac{1}{3}\,720\left(\frac{d}{\pi\hbar\beta c}\right)^3\left(\cancel{1}+\frac{\pi\hbar\beta c}{d}\right)e^{-\frac{\pi\hbar\beta c}{d}}\left(\frac{\pi\hbar\beta c}{d}\right)\right]$$

$$= -\frac{L^2\pi^2\hbar c}{240 d^4}\left[1+\frac{1}{3}\left(\frac{2d}{\pi\beta c}\right)^4 - 240\,\frac{d}{\pi\hbar\beta c}\,e^{-\frac{\pi\hbar\beta}{d}}\right]$$

$$\frac{16}{3}\left(\frac{d}{\pi\beta c}\right)^4$$

$\frac{16}{3} = \frac{48}{9}$ agrees with Sernelius

NB: The scalar does not contribute to the Casimir force!

Low temperature limit from Bordag (7.81):

$$t = \frac{T_{eff}}{T} \gg 1$$

small parameter: $\frac{1}{t} = x$ $\qquad \frac{1}{t^2 \sinh^2 \alpha l t} \to 0$

$$\frac{e^{\pi l t} + e^{-\pi l t}}{e^{\pi l t} - e^{\pi l t}} = \frac{\cancel{e^{\pi l t}}(1+e^{-2\alpha l t})}{\cancel{e^{\pi l t}}(1-e^{-2\alpha l t})} \to 1$$

$$\frac{F}{L^2} = -\frac{\pi^2}{720 d^3}\left[1+\frac{45}{\pi^3}\zeta(3)\left(\frac{T}{T_{eff}}\right)^3 - \left(\frac{T}{T_{eff}}\right)^4\right] \qquad \frac{T}{T_{eff}} = \frac{2d}{\beta}$$

leading exponentially surpressed correction

$$+\frac{180}{\pi^2}\left(\frac{T}{T_{eff}}\right)^2 e^{-2\pi T_{eff}/T} = \frac{720}{\pi^2}\frac{d^2}{\beta^2}e^{-\pi\beta/d}$$

agrees with previous result (R)

Casimir pressure:

$$\frac{F}{L^2} = -\frac{\pi^2}{240 d^4}\left[1 + \frac{1}{3}\left(\frac{T}{T_{eff}}\right)^4\right]$$

$T_{eff} = \frac{2d}{\beta}$

$\Rightarrow 16/3!$ as in Servelius

exp correction:
$$-\frac{120}{\pi}\frac{T}{T_{eff}}\, e^{-2\pi T_{eff}/T}$$
$$= -240\frac{d}{\pi\beta}\, e^{-\pi\beta/d}$$ also agrees!

redo computation $\longrightarrow$ more complicated see above

$$-\frac{\partial}{\partial d}\quad -\frac{\pi^2}{720 d^3}\left[1 - \left(\frac{2d}{\beta}\right)^4 + 720\frac{d^2}{\cancel{\pi\beta^2}}\cancel{e}^{-\pi\beta/d}\right]$$

$$= -\frac{\pi^2}{240 d^4}\left[1 - \left(\frac{2d}{\beta}\right)^4 + 720\frac{d^2}{\cancel{\pi\beta^2}}\cancel{e}^{-\pi\beta/d}\right]$$

$$+\frac{\pi^2}{720 d^3}\left[-4\left(\frac{2d}{\beta}\right)^3\cdot\frac{2}{\beta} + \cancel{720\cdot 2\frac{d}{\pi^2\beta^2}e^{-\pi\beta/d}} + \cancel{720\frac{d^2}{\pi^2\beta^2}\frac{\pi\beta}{d^2}}e^{-\pi\beta/d}\right]$$

$$= -\frac{\pi^2}{240 d^4}\left[1 - \left(\frac{2d}{\beta}\right)^4 + \frac{4}{3}\cdot\frac{2}{\beta}\cdot d\left(\frac{2d}{\beta}\right)^3 + e^{-\pi\beta/d}\left(\cancel{720\frac{d^2}{\pi^2\beta^2}} - \frac{720\cdot 2}{3}\frac{d^2}{\pi^2\beta^2} \cancel{- \frac{720}{3}\frac{d}{\pi\beta}}\right)\right.$$

$$\left[1 + \frac{1}{3}\left(\frac{2d}{\beta}\right)^4 + 240\frac{d^2}{\cancel{\pi^2\beta^2}}\cancel{e}^{-\pi\beta/d}\right] \quad \checkmark$$

Casimir entropy: $S(a,T) = -\frac{\partial F(a,T)}{\partial T}$

$$S(a,T) = \frac{3\zeta_R(3)}{8\pi a^2}\left(\frac{T}{T_{eff}}\right)^2\left[1 - \frac{4\pi^3}{135\zeta_R(3)}\frac{T}{T_{eff}} + \frac{8\pi^2}{3\zeta_R(3)}\left(\frac{T_{eff}}{T}\right)^2 e^{-2\pi\frac{T_{eff}}{T}}\right]$$

$S(a,Y) > 0 \quad \forall\, a, Y; \quad S(a,Y) \to 0$ when $Y \to 0$

5.1.10 Poisson summation formula

Poisson summation formula

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Standard formulas for Fourier transform:

$$c(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\, b(x)\, e^{-ikx}, \qquad b(x) = \int_{-\infty}^{+\infty} dk\, c(k)\, e^{ikx}$$

$$\Rightarrow \boxed{\sum_{m=-\infty}^{+\infty} b(m) = 2\pi \sum_{n=-\infty}^{+\infty} c(2\pi n)}$$

Proof:

Let $x \in [-\frac{L}{2}, \frac{L}{2}]$, extend $\phi(x)$ to a periodic function of period L

$$\phi(x) = \sum_{n=-\infty}^{+\infty} \phi_k\, e^{ikx}, \quad k = \frac{2\pi}{L} n \qquad \phi_{k'} = \frac{1}{L}\int_{-L/2}^{+L/2} dx\, \phi(x)\, e^{-ik'x}$$

Take now for $\phi(x)$ the periodic delta function for which all coefficients are $\frac{1}{L}$:

$$\sum_{m=-\infty}^{+\infty} \delta(x - Lm) = \sum_{n=-\infty}^{+\infty} \frac{1}{L} e^{ikx} \quad \Big| \int_{-\infty}^{+\infty} dx\, b(x) \qquad \delta\left(\frac{x}{L} - m\right)$$

$$\Rightarrow \sum_{m=-\infty}^{+\infty} b(Lm) = \sum_{n=-\infty}^{+\infty} \frac{1}{L} \int_{-\infty}^{+\infty} dx\, b(x)\, e^{ikx} = \sum_{n=-\infty}^{+\infty} \frac{1}{L} \int_{-\infty}^{+\infty} dx\, b(x)\, e^{-ikx} = \frac{2\pi}{L} \sum_{n=-\infty}^{+\infty} c(k)$$

The formula follows by setting $L = 1$.

Application:

For $b(x) = e^{-\frac{1}{2}\alpha x^2} \Rightarrow c(k) = \sqrt{\frac{1}{2\pi\alpha}}\, e^{-\frac{1}{2}\frac{1}{\alpha}k^2}$

$$\sum_{m=-\infty}^{+\infty} e^{-\frac{1}{2}\alpha m^2} = \sqrt{\frac{2\pi}{\alpha}} \sum_{n=-\infty}^{+\infty} e^{-\frac{1}{2}\frac{1}{\alpha}(2\pi)^2 n^2}$$

If $\alpha = 2\beta$:

$$\boxed{\sum_{m=-\infty}^{+\infty} e^{-\beta m^2} = \sqrt{\frac{\pi}{\beta}} \sum_{n=-\infty}^{+\infty} e^{-\frac{1}{\beta}\pi^2 n^2}}$$

For an even function $b(-x) = b(x)$, $\left[c(k) = \frac{1}{2\pi}\left[\int_0^{\infty} dx\, b(x) e^{-ikx} + \int_{-\infty}^{0} dx\, b(x) e^{-ikx}\right] = \frac{1}{\pi}\int_0^{\infty} dx\, b(x) \cos kx\right]$

Furthermore, $c(k) = c(-k)$ is also even and

$$\boxed{\frac{1}{2} b(0) + \sum_{m=1}^{\infty} b(m) = \pi\, c(0) + 2\pi \sum_{n=1}^{\infty} c(2\pi n)}$$

5.1.11 High temperature limit. Alternative

High temperature limit

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In the high temperature limit, $\frac{d}{\hbar\beta c} \gg 1$, one has to perform the sum over n in (x)

$$\mathcal{F}_{c,2}(d,T) = \frac{L^2\pi}{d^2}\left[\frac{1}{2}\, b(d,T,0) + \sum_{n=1}^{\infty} b(d,T,n) - \int_0^\infty dn\, b(d,T,n)\right] \quad (x)$$

$$\left(= \frac{L^2\pi}{d^2}\left[-\frac{1}{2\beta}\left(\frac{d}{\pi\hbar\beta c}\right)^2 \zeta(3) + \sum_{n=1}^{\infty} b(d,T,n) + \frac{2}{\beta}\left(\frac{d}{\pi\hbar\beta c}\right)^3 \zeta(4)\right]\right)$$

$$b(d,T,n) = \frac{1}{2\beta}\int_{n^2}^{\infty} ds\, \ln\left(1 - e^{-\frac{\pi\hbar\beta c}{d}\sqrt{s}}\right)$$

According to the Poisson summation formula for even functions,

if $c(d,T,k) = \frac{1}{\pi}\int_0^\infty dx\, \cos kx\, b(d,T,x)$

it follows that

$$\frac{1}{2}\, b(d,T,0) + \sum_{n=1}^{\infty} b(d,T,n) = \pi\, c(d,T,0) + \sum_{m=1}^{\infty} 2\pi\, c(d,T,2\pi m)$$

Since $\pi\, c(d,T,0) = \int_0^\infty dx\, b(d,T,x)$, the first term cancels precisely the empty space result that one has subtracted from the sum in (x), so that one remains with

$$\mathcal{F}_{c,2}(d,T) = \left(\frac{L\pi}{d}\right)^2 2 \sum_{m=1}^{\infty} c(d,T,2\pi m)$$

$$c(d,T,k) = \frac{1}{\pi\beta}\int_0^\infty dx\, \cos kx\, \frac{1}{2}\int_{x^2}^{\infty} ds\, \ln\left(1 - e^{-\frac{\pi\hbar\beta c}{d}\sqrt{s}}\right)$$

$$= \frac{1}{2\pi\beta}\left[\frac{\sin kx}{k}\int_{x^2}^{\infty} ds\, \ln\left(1 - e^{-\frac{\pi\hbar\beta c}{d}\sqrt{s}}\right)\right]_0^\infty \to 0$$

$$+ \frac{1}{\cancel{2}\pi k\beta}\int_0^\infty dx\, \sin kx\, \cancel{2}x \ln\left(1 - e^{-\frac{\pi\hbar\beta c}{d}x}\right)$$

Gradstein, Ryzhik (1983) 4.383 №2 page 584

$$\int_0^\infty \ln(1-e^{-\beta x})\cos bx\,dx = \frac{\beta}{2b^2} - \frac{\pi}{2b}\operatorname{cth}\left(\frac{\pi b}{\beta}\right) \qquad \operatorname{Re}\beta>0,\ b>0$$

$$\frac{d}{db}: \quad \int_0^\infty \ln(1-e^{-\beta x})(-x)\sin bx\,dx = \frac{d}{db}\left[\frac{\beta}{2b^2} - \frac{\pi}{2b}\operatorname{cth}\left(\frac{\pi b}{\beta}\right)\right]$$

we need: $\beta \to \frac{\pi\hbar\beta c}{d}$ $\quad b \to k$

$$\Rightarrow c(d,T,k) = -\frac{1}{\pi k\beta}\frac{d}{dk}\left[\frac{\frac{\pi\hbar\beta c}{d}}{2k^2} - \frac{\pi}{2k}\operatorname{cth}\left(\frac{\pi k d}{\pi\hbar\beta c}\right)\right]$$

$$= -\frac{1}{2\pi k\beta}\frac{d}{dk}\left[\frac{\pi\hbar\beta c}{d}\frac{1}{k^2} - \frac{\pi}{k}\operatorname{cth}\left(\frac{kd}{\hbar\beta c}\right)\right]$$

$$= \frac{1}{\pi k\beta}\frac{\pi\hbar\beta c}{d}\frac{1}{k^4} - \frac{\pi}{k^3 2\pi\beta}\operatorname{cth}\left(\frac{kd}{\hbar\beta c}\right) + \frac{\pi}{k\,2\pi k\beta}\frac{d}{\hbar\beta c}\operatorname{sh}^{-2}\frac{kd}{\hbar\beta c}$$

$$\operatorname{cth}'(x) = \left(\frac{\operatorname{ch}(x)}{\operatorname{sh}(x)}\right)' = \frac{\operatorname{sh}^2x-\operatorname{ch}^2x}{\operatorname{sh}^2x} = -\frac{1}{\operatorname{sh}^2x} \qquad \operatorname{cth}x = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{1+e^{-2x}}{1-e^{-2x}} = 1+2e^{-2x}+O(e^{-4x})$$

$$= \frac{\hbar c}{dk^4} - \frac{1}{2\beta k^3}\left(1+2e^{-2\frac{kd}{\hbar\beta c}}+O\left(e^{-\frac{4kd}{\hbar\beta c}}\right)\right) - \frac{1}{2k^2}\frac{1}{\beta}\frac{d}{\hbar\beta c}\frac{4}{e^{\frac{2kd}{\hbar\beta c}}+e^{-\frac{2kd}{\hbar\beta c}}-2}$$

$$= \frac{\hbar c}{dk^4} - \frac{1}{2\beta k^3}\left(1+2e^{-\frac{2kd}{\hbar\beta c}}+O\left(e^{-\frac{4kd}{\hbar\beta c}}\right)\right) - \frac{2}{k^2}\frac{1}{\beta}\frac{d}{\hbar\beta c}e^{-2\frac{kd}{\hbar\beta c}}\left(\frac{1}{1-2e^{-\frac{2kd}{\hbar\beta c}}+e^{-\frac{4kd}{\hbar\beta c}}}\right)$$

$$= \frac{\hbar c}{dk^4} - \frac{1}{2\beta k^3} - e^{-\frac{2kd}{\hbar\beta c}}\left[\frac{1}{\beta k^3} + \frac{2d}{\beta^2k^2\hbar c}\right] + O\left(e^{-4\frac{kd}{\hbar\beta c}}\right)$$

Now one has to do the sum over n:

$$F_{c,2}(d,T) = \left(\frac{L\pi}{d}\right)^2 2\sum_{n=1}^\infty \left[\frac{\hbar c}{d(2\pi n)^4} - \frac{1}{2\beta(2\pi n)^3}\right.$$
$$\left. - e^{-\frac{4\pi n d}{\hbar\beta c}}\left[\frac{1}{\beta(2\pi n)^3}+\frac{2d}{(2\pi n)^2\beta^2\hbar c}\right] + O\left(e^{-4\frac{2\pi n d}{\hbar\beta c}}\right)\right]$$

$$= \left(\frac{L\pi}{d}\right)^2 2\frac{\hbar c}{d(2\pi)^4}\zeta(4) - \left(\frac{L\pi}{d}\right)^2 2\frac{1}{2\beta(2\pi)^3}\zeta(3) - \left(\frac{L\pi}{d}\right)^2 2e^{-4\pi\frac{d}{\hbar\beta c}}\left[\frac{1}{\beta(2\pi)^3}+\frac{2d}{(2\pi)^2\beta^2\hbar c}\right] + O\left(e^{-\frac{8\pi d}{\hbar\beta c}}\right)$$

$$= \frac{L^2\pi^2}{d^2}2\frac{\hbar c}{d\,16\,\pi^4}\frac{\pi^4}{90} - \frac{L^2\pi^2}{d^2}\frac{1}{\beta\,8\pi^3}\zeta(3) - e^{-4\pi\frac{d}{\hbar\beta c}}\left[\frac{L^2\pi^2}{d^2}2\frac{1}{\beta\,8\pi^3} + \frac{L^2\pi^2\cancel{2}\cdot\cancel{2}d}{d^2\,4\pi^2\beta^2\hbar c}\right] + O\left(e^{-\frac{8\pi d}{\hbar\beta c}}\right)$$

$$= L^2\left[\pi^2\frac{\hbar c}{720d^3} - \zeta(3)\frac{1}{8\pi d^2\beta} - e^{-4\pi\frac{d}{\hbar\beta c}}\left[\frac{1}{4\pi\beta d^2} + \frac{1}{d\beta^2\hbar c}\right] + O\left(e^{-\frac{8\pi d}{\hbar\beta c}}\right)\right]$$

- agrees with Greiner (6.40) up to a mistake of π in in his $\zeta(3)$ term

- agrees with Bordag (7.95), (7.97) up to a mistake of π in the last term

(in Bordag : $\frac{T}{T_{eff}} = \frac{2\pi d}{(\hbar c)\beta}$

$$- \frac{1}{2d^2\beta} \frac{T}{T_{eff}} e^{-2\pi T/T_{eff}}$$

$$= - e^{-4\pi d/\hbar\beta c} \frac{1}{2d^2\beta} \frac{4\pi d}{\hbar\beta c}$$ factor π ?)

In the full free energy, the zero temperature contribution precisely cancels with the first term so that one remains with

$$\boxed{\mathcal{F}_c(d,T) = L^2\left[-\zeta(3)\frac{1}{8\pi d^2\beta} - e^{-4\pi\frac{d}{\hbar\beta c}}\left(\frac{1}{4\pi\beta d^2} + \frac{1}{d\beta^2\hbar c}\right) + \mathcal{O}\left(e^{-8\pi\frac{d}{\hbar\beta c}}\right)\right]}$$

NB: When using the Poisson formula, one looses the information on the contribution of the massless scalar.

5.1.12 Inversion symmetry

Inversion symmetry

Sunday, February 9, 2020 10:47 PM

Low and high temperature expansions are connected by inversion symmetry (cf. Brown & Macly 1969). More precisely, start again from the exact result for the total free energy

$$\mathcal{F}_c(d,T) = \frac{c^2\pi}{d^2}\left[-\frac{\pi}{720 d} + \sum_{n=0}^{\infty}{}' \, b(d,T,n) - \int_0^\infty dn\, b(d,T,n)\right]$$

where the first term is the zero temperature result and the last term the subtraction of the empty space black body result, and

$$b(d,T,n) = \frac{1}{2\beta}\int_{n^2}^{\infty} ds\, \ln\left(1 - e^{-\frac{\pi\hbar\beta c}{d}\sqrt{s}}\right)$$

$$-2\pi\left(\frac{d}{\pi\beta\hbar c}\right)^3\zeta(4) = -2\left(\frac{d}{\beta\hbar c}\right)^3\pi^{-2}\frac{\pi^4}{90}$$

For $\ln Z(\beta) = -\beta F(\beta)$ this gives, when using (x) of 5.1.3,

$$\ln Z(\beta) = \frac{c^2}{d^2}\left[\frac{\pi^2}{720}\frac{\beta\hbar c}{d} \underbrace{-\pi\beta\sum_{n=0}^{\infty}{}' \, b(d,T,n)}_{} - \left(\frac{d}{\beta\hbar c}\right)^3\frac{\pi^2}{45}\right]$$

Let us concentrate on this middle piece and redo the computation of section 5.1.3 keeping $\hbar$, c:

$$\frac{\pi}{d^2}\left[\sum_{n=0}^{\infty}{}' \, b(d,T,n)\right] = \frac{\pi}{2\beta d^2}\sum_{n=0}^{\infty}{}' \int_{n^2}^{\infty} ds\, \ln\left(1 - e^{-\frac{\pi\hbar\beta c\sqrt{s}}{d}}\right)$$

change of variable $z = \frac{\pi\beta\hbar c}{d}\sqrt{s}$, $t = \frac{\beta\hbar c}{2d}$

$$dz = \frac{\pi\beta\hbar c}{2d}\frac{1}{\sqrt{s}}\,ds \Rightarrow z\,dz = \frac{(\pi\beta\hbar c)^2}{2d^2}\,ds$$

$$\frac{\pi}{d^2}\Big[\sum_{n=0}^{\prime} b(d,T,n)\Big] = \frac{\cancel{\pi}}{\cancel{2}\beta \cancel{d^2}}\frac{\cancel{2}\cancel{d^2}}{\pi(\beta\hbar c)^2}\sum_{n=0}^{\prime}\int_{2\pi n t}^{\infty} dz\, z\, \ln(1-e^{-z})$$

series expansion

$$\ln(1-x) = -\sum_{l=1}^{\infty}\frac{x^l}{l}$$

$$= -\frac{1}{\pi\beta^3(\hbar c)^2}\sum_{n=0}^{\prime}\sum_{l=1}^{\infty}\frac{1}{l}\int_{2\pi n t}^{\infty} dz\, \underset{f}{z}\, \underset{g'}{e^{-lz}}$$

$$\frac{d}{dz}\Big[\underset{f}{z}\,\underset{g}{\frac{e^{-lz}}{-l}}\Big] + \underset{-f'g}{\frac{e^{-lz}}{l}} = \frac{d}{dz}\Big(-\frac{1}{l^2 l}e^{-lz}\Big)$$

$$= -\frac{1}{\pi\beta^3(\hbar c)^2}\sum_{n=0}^{\prime}\sum_{l=1}^{\infty}\Big(2\pi n t\, e^{-2\pi n t l}\frac{1}{l^2} + \frac{1}{l^3}e^{-2\pi n t l}\Big)$$

$$= -\frac{t^3}{\pi\beta^3(\hbar c)^2}\sum_{n=0}^{\prime}\sum_{l=1}^{\infty}\frac{1}{l^3 t^3}(1+2\pi n l t)\, e^{-2\pi n t l}$$

As in section 5.1.3

$$\sum_{n=0}^{\prime} -t^3\sum_{l=1}^{\infty}\frac{1}{l^3t^3}(1+2\pi n l t)\, e^{-2\pi l n t}$$

$$= -\frac{t^3}{2}\sum_{l=1}^{\infty}\Big(\frac{\coth \pi l t}{t^3 l^3} + \frac{\pi}{t^2l^2}\frac{1}{\sinh^2(\pi t l)}\Big)$$

$$\Rightarrow \frac{\pi}{d^2}\sum_{n'=0} b(d,T,n) = -\frac{1}{\pi\beta^3(\hbar c)^2}\frac{t^3}{2}\Big(\sum_{l=1}^{\infty}\frac{\coth \pi l t}{t^3 l^3} + \frac{\pi}{t^2l^2}\frac{1}{\sinh^2(\pi t l)}\Big)$$

$$\rightarrow -\pi\beta \sum_{\mu'} b(d,T,\mu) = \frac{d^2}{\pi\beta^2(\hbar c)^2}\,\frac{t^3}{2}\,\Big(\qquad \Big)$$

$$\Rightarrow \boxed{\ln Z/|\beta| = \frac{L^2}{d^2}\left\{\frac{\pi^2}{360}t + \frac{t}{8\pi}\sum_{l=1}^{\infty}\left[\frac{\coth \pi l t}{t^3 l^3} + \frac{\pi}{t^2 l^2}\,\frac{1}{\sinh^2(\pi t l)}\right] - \frac{\pi^2}{360 t^3}\right\}}$$

In the discussion by Brown & Maclay on the Casimir effect based on an image method, they work with the variable $\xi = \frac{d}{\beta\hbar c}$. Since $t = \frac{\beta\hbar c}{2d}$

$$t = \frac{1}{2\xi} \Leftrightarrow \xi = \frac{1}{2t}$$

The important function is

$$f(\xi) = -\frac{1}{4\pi^2}\sum_{l,m=1}^{\infty}\frac{(2\xi)^4}{[l^2+(2\xi)^2m^2]^2}$$

which exhibits the inversion property

$$\boxed{f\left(\frac{1}{4\xi}\right)} = -\frac{1}{4\pi^2}\sum_{l,m=1}^{\infty}\frac{\left(\frac{1}{2\xi}\right)^4}{\left[l^2+\left(\frac{1}{2\xi}\right)^2m^2\right]^2}$$

$$= -\frac{1}{4\pi^2}(2\xi)^{-4}\sum_{l,m=1}^{\infty}\frac{1}{\left(\frac{1}{2\xi}\right)^4\left[\left(\frac{1}{2\xi}\right)^2 l^2+m^2\right]^2}$$

$$= (2\xi)^{-4}\left\{-\frac{1}{4\pi^2}\sum_{l,m=1}^{\infty}\frac{(2\xi)^4}{[m^2+\left(\frac{1}{2\xi}\right)^2 l^2]}\right\}$$

$$= \boxed{(2\xi)^{-4} f(\xi)} \iff \boxed{f\left(\frac{t}{2}\right) = t^4 f\left(\frac{1}{2t}\right)}$$

$$f\left(\frac{1}{2t}\right) = t^{-4}\left\{-\frac{1}{4\pi^2}\sum_{l,m=1}^{\infty}\frac{1}{[l^2+t^{-2}m^2]^2}\right\}$$

$$= -\frac{1}{4\pi^2}\underbrace{\sum_{l,m=1}^{\infty}\frac{1}{[t^2l^2+m^2]^2}}$$

The sum $\sum_{m=1}^{\infty}\frac{1}{[t^2l^2+m^2]^2}$ can be done by contour integral: (Maclay-Brown (29)).

$$\sum_{m=1}^{\infty}\frac{1}{[t^2l^2+m^2]^2} = \frac{\pi}{4t^3l^3}\coth(\pi l t) + \frac{\pi^2}{4t^2l^2}\,\mathrm{ch}^{-2}(\pi l t) - \frac{1}{2t^4l^4}$$

$$\sum_{l=1}^{\infty} l^{-4} = \zeta(4) = \frac{\pi^4}{90}, \quad \frac{1}{8\pi^2 t^4}\frac{\pi^4}{90} = \frac{\pi^2}{t^4 720}$$

$$\boxed{f\left(\frac{1}{2t}\right) = -\sum_{l=1}^{\infty}\left[\frac{1}{16\pi t^3 l^3}\coth(\pi l t) + \frac{1}{16t^2l^2}\,\mathrm{sh}^{-2}(\pi l t)\right] + \frac{\pi^2}{t^4 720}}$$

$$\frac{t}{8\pi}\sum_{l=1}^{\infty}\left(\frac{\coth \pi l t}{t^3 l^3} + \frac{\pi}{t^2l^2}\,\mathrm{sh}^{-2}\pi l t\right) - \frac{\pi^2}{360t^3} = -2t\, f\left(\frac{1}{2t}\right) = -\frac{2}{t^3} f\left(\frac{t}{2}\right)$$

$$\boxed{\ln Z = \frac{L^2}{d^2}\left[\frac{\pi^2 t}{360} - 2t\, f\left(\frac{1}{2t}\right)\right] = \frac{L^2}{d^2}\left[\frac{\pi^2 t}{360} - \frac{2}{t^3} f\left(\frac{t}{2}\right)\right]}$$

Low temperature limit: $t \gg 1$

$$\coth \pi l t = \frac{e^{\pi l t} + e^{-\pi l t}}{e^{\pi l t} - e^{-\pi l t}} = \frac{1 + e^{-2\pi l t}}{1 - e^{-2\pi l t}} \to 1$$

$$\frac{1}{sh^2 \pi \ell t} = \frac{4}{(e^{\pi \ell t} - e^{-\pi \ell t})^2} = \frac{4}{e^{2\pi \ell t}(1 - e^{-2\pi \ell t})} \to 0$$

$$f\left(\frac{1}{2t}\right) \simeq -\frac{1}{16\pi t^3}\zeta(3) + \frac{\pi^2}{t^4 720}$$

$$\ln Z \simeq \frac{L^2}{a^2}\left[\frac{\pi^2 t}{360} + \frac{2t}{16\pi t^3}\sum_{\ell=1}^{\infty}\frac{1}{\ell^3} - \frac{\pi^2}{t^3 360}\right]$$

$$\boxed{\ln Z = \frac{L^2}{a^2}\left[\frac{\pi^2 t}{360} + \frac{1}{8\pi t^2}\zeta(3) - \frac{\pi^2}{t^3 360}\right]}$$

High temperature limit $t \ll 1$

$$f\left(\frac{1}{2t}\right) = t^{-4} f\left(\frac{t}{2}\right)$$

$$f\left(\frac{t}{2}\right) = t^4 f\left(\frac{1}{2t}\right)$$

$t = \frac{1}{x}$ $\quad t \ll 1 \Rightarrow x \gg 1$

$$f\left(\frac{1}{2x}\right) \simeq -\frac{1}{16\pi x^3}\zeta(3) + \frac{\pi^2}{x^4 720}$$

$$f\left(\frac{t}{2}\right) \simeq -\frac{1}{16\pi} t^3 \zeta(3) + t^4 \frac{\pi^2}{720} \qquad t \ll 1$$

$$\ln Z = \frac{L^2}{a^2}\left[\frac{\pi^2 t}{360} - 2/t^3 f\left(\frac{t}{2}\right)\right]$$

$$\ln Z \simeq \frac{L^2}{a^2}\left[\cancel{\frac{\pi^2 t}{360}} + \underbrace{\frac{1}{8\pi}\zeta(3)} - \cancel{\frac{t\pi^2}{360}}\right]$$

$$\boxed{\ln Z \simeq \frac{L^2}{a^2}\frac{1}{8\pi}\zeta(3)}$$

5.2 1-loop effective action for scalar field

5.2.1 Effective potential

We start from the action

$$S[\phi] = -\int d^4x[\nu + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4], \tag{5.109}$$

where the constant ν, which does not change the dynamics, is introduced to clarify the discussion on renormalization that follows. As we have seen above, the associated effective action is

$$\Gamma[\phi] = S[\phi] - \frac{\hbar}{2i}\mathrm{Tr}\ln\left(\delta^4(x,y) + \mathcal{D}^{-1}(x,y)\frac{g}{2}\phi^2(y)\right) + O(\hbar^2). \tag{5.110}$$

Using the Taylor expansion $\ln(1+x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}(-)^{n-1}\frac{x^n}{n}$, we thus have

$$\begin{aligned}\ln\left(\delta^4(x,y) + K(x,y)\right) &= K(x,y) - \frac{1}{2}\int d^4z_1 K(x,z_1)K(z_1,y) + \dots \\ &= \sum_{n=1}\frac{(-)^{n-1}}{n}\int dz_1\dots dz_{n-1}K(x,z_1)\dots K(z_{n-1},y).\end{aligned}$$

In the particular case where $K(x,y) = \frac{1}{(2\pi)^4}\int d^4p e^{ip(x-y)}K(p)$, the convolution product becomes the ordinary product of Fourier transforms,

$$\begin{aligned}\ln\left(\delta^4(x,y) + K(x,y)\right) &= \sum_{n=1}\frac{(-)^{n-1}}{n}\frac{1}{(2\pi)^4}\int d^4p e^{ip(x-y)}K(p)^n \\ &= \frac{1}{(2\pi)^4}\int d^4p e^{ip(x-y)}\ln\left(1+K(p)\right).\end{aligned}$$

For a constant classical field, this holds because $K(x,y) = \mathcal{D}^{-1}(x,y)\frac{g}{2}\bar{\phi}^2$ and thus $K(p) = \frac{\frac{g}{2}\bar{\phi}^2}{p^2+m^2-i\epsilon}$. We thus find

$$\Gamma[\bar{\phi}] = S[\bar{\phi}] - \frac{\hbar}{2i}\int d^4x\int\frac{d^4p}{(2\pi)^4}\ln\left(1+K(p)\right) + O(\hbar^2). \tag{5.111}$$

The effective potential is defined as minus the effective action evaluated for a constant classical field, i.e., a classical field that does not depend on x^μ, $\phi(x) = \bar{\phi}$, up to a volume factor,

$$\Gamma[\bar{\phi}] = -\int d^4x V_{\text{eff}}[\bar{\phi}] = -V_{\text{eff}}[\bar{\phi}](2\pi)^4\delta^4(0). \tag{5.112}$$

Taking the trace consists in putting $x=y$ and integrating over x. We then find

$$\Gamma[\bar{\phi}] = -\int d^4x\Big[\nu + \frac{1}{2}m^2\bar{\phi}^2 + \frac{g}{4!}\bar{\phi}^4 + \frac{\hbar}{2i}\int\frac{d^4p}{(2\pi)^4}[\ln\left(1+\frac{\frac{g}{2}\bar{\phi}^2}{p^2+m^2-i\epsilon}\right)] + O(\hbar^2), \tag{5.113}$$

and thus

$$\Gamma[\bar{\phi}] = -\int d^4x\Big[\nu + \frac{1}{2}m^2\bar{\phi}^2 + \frac{g}{4!}\bar{\phi}^4 + \frac{\hbar}{2i}\int\frac{d^4p}{(2\pi)^4}[\ln\frac{p^2+(m^2+\frac{g}{2}\bar{\phi}^2)-i\epsilon}{p^2+m^2-i\epsilon}] + O(\hbar^2), \tag{5.114}$$

or, when defining $J(\sigma^2) = \int\frac{d^4p}{(2\pi)^4}\ln\left(p^2+\sigma^2-i\epsilon\right)$, this gives

$$V_{\text{eff}}[\bar{\phi}] = \nu + \frac{1}{2}m^2\bar{\phi}^2 + \frac{g}{4!}\bar{\phi}^4 + \frac{\hbar}{2i}[J(\mu^2) - J(m^2)] + O(\hbar^2), \tag{5.115}$$

with $\mu^2 = m^2 + \frac{g}{2}\bar{\phi}^2$.

5.2.2 Computing the divergent integral

To compute $J(\sigma^2)$, one replaces p^0 by ip^4. One thus rotates the integration contour by $\pi/2$ in the complex p^0 plane. One thus goes from the space of p^μ's with a Lorentz metric to the space of p^A's with $A = 1, \dots 4$ and an Euclidean metric. This gives

$$J(\sigma^2) = \frac{i}{(2\pi)^4}\int dp^A \ln(p^B p_B + \sigma^2) = \frac{i}{(2\pi)^4}\int_0^\infty dk \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^\pi d\chi \left|\frac{\partial p^A}{\partial(k,\phi,\theta,\chi)}\right| \ln(k^2+\sigma^2). \tag{5.116}$$

The Jacobian to go to spherical coordinates in 4 dimensions is given by $|\frac{\partial p^A}{\partial(k,\phi,\theta,\chi)}| = k^3 \sin^2\theta \sin\chi$ so that the integral is

$$J(\sigma^2) = \frac{i}{(2\pi)^4} 2\pi^2 \int_0^\infty dk k^3 \ln(k^2+\sigma^2). \tag{5.117}$$

This integral diverges, but it can be made convergent by differentiating sufficiently many times with respect to σ^2:

$$J'(\sigma^2) = \frac{i}{8\pi^2}\int_0^\infty dk \frac{k^3}{k^2+\sigma^2}, \tag{5.118}$$

diverges quadratically since the integrand is proportional to k,

$$J''(\sigma^2) = -\frac{i}{8\pi^2}\int_0^\infty dk \frac{k^3}{(k^2+\sigma^2)^2}, \tag{5.119}$$

diverges logarithmically , and finally

$$\begin{aligned} J'''(\sigma^2) &= \frac{i}{4\pi^2}\int_0^\infty dk \frac{k}{(k^2+\sigma^2)^3} k^2 \\ &= \frac{i}{4\pi^2}\Big([-\frac{1}{4(k^2+\sigma^2)^2} k^2]_0^\infty + \int_0^\infty dk \frac{2k}{4(k^2+\sigma^2)^2} \Big) \\ &= \frac{i}{4\pi^2} [\frac{-1}{4(k^2+\sigma^2)}]_0^\infty \\ &= \frac{i}{16\pi^2\sigma^2}. \end{aligned} \tag{5.120}$$

When using $(x \ln x - x)' = \ln x$ and $(\frac{x^2}{2}\ln x - \frac{x^2}{4})' = x\ln x$, one finds

$$\begin{aligned} J''(\sigma^2) &= \frac{i}{16\pi^2}\ln\sigma^2 + \bar{C}, \\ J'(\sigma^2) &= \frac{i}{16\pi^2}(\sigma^2\ln\sigma^2 - \sigma^2) + \bar{C}\sigma^2 + 2iB, \\ J(\sigma^2) &= \frac{i}{32\pi^2}\sigma^4\ln\sigma^2 + 2iC\sigma^4 + 2iB\sigma^2 + 2i\bar{A}, \end{aligned} \tag{5.121}$$

for constants (independent of $\bar{\phi}$) $\bar{A}, B, C, \bar{C}$ that diverge. One thus finds

$$V_{\text{eff}}[\bar{\phi}] = \nu + \frac{1}{2}m^2\bar{\phi}^2 + \frac{g}{4!}\bar{\phi}^4 + \frac{\hbar}{64\pi^2}\mu^4\ln\mu^2 + \hbar C\mu^4 + \hbar B\mu^2 + \hbar A + O(\hbar^2), \tag{5.122}$$

with $A = \bar{A} - \frac{\hbar}{2i}J(m^2)$.

Defining renormalized coupling constants through

$$\begin{cases} \nu_R = \nu + \hbar A + m^2\hbar B + m^4\hbar C, \\ m_R^2 = m^2 + g\hbar B + 2m^2 g\hbar C, \\ g_R = g + 6g^2\hbar C, \end{cases} \tag{5.123}$$

one finds

$$V_{\text{eff}}[\bar{\phi}] = \nu_R + \frac{1}{2}m_R^2\bar{\phi}^2 + \frac{g_R}{4!}\bar{\phi}^4 + \frac{\hbar}{64\pi^2}\mu_R^4 \ln \mu_R^2 + O(\hbar^2). \tag{5.124}$$

We thus see that $V_{\text{eff}}[\bar{\phi}]$ can be made finite to order $\hbar$ if the renormalized coupling constants are assumed to be finite, which in turn means that the bare, starting point, coupling constants contain a divergent part,

$$\begin{cases} \nu = \nu_R - \hbar A - m_R^2\hbar B - m_R^4\hbar C + O(\hbar^2), \\ m^2 = m_R^2 - g_R\hbar B - 2m_R^2 g_R\hbar C + O(\hbar^2), \\ g = g_R - 6g_R^2\hbar C + O(\hbar^2). \end{cases} \tag{5.125}$$

An equivalent point of view is to say that one obtains a finite result to order $\hbar$ for $V_{\text{eff}}[\bar{\phi}]$ by adding to the Lagrangian infinite "counterterms" of order $\hbar$,

$$\begin{cases} \nu \to \nu_R - \hbar A - m_R^2\hbar B - m_R^4\hbar C, \\ \frac{1}{2}m^2\bar{\phi}^2 \to \frac{1}{2}(m_R^2 - g_R\hbar B - 2m_R^2 g_R\hbar C)\bar{\phi}^2, \\ \frac{g}{4!}\bar{\phi}^4 \to \frac{g_R - 6g_R^2\hbar C}{4!}\bar{\phi}^4. \end{cases} \tag{5.126}$$

These counterterms cancel the divergences in $V_{\text{eff}}[\bar{\phi}]$ produced by one-loop diagrams.

5.2.3 Renormalized coupling constant at 1-loop

In order to derive an explicit expression for g_R and thus for C, one introduces an ultraviolet cut-off Λ, that is to say an upper limit on the norm of the momentum space vector,

$$J_\Lambda(\sigma^2) = \frac{i}{8\pi^2}\int_0^\Lambda dk k^3 \ln(k^2 + \sigma^2). \tag{5.127}$$

Changing variables $k = \sqrt{x}$, $dk = \frac{dx}{2\sqrt{x}}$, one finds

$$\begin{aligned} J_\Lambda(\sigma^2) &= \frac{i}{16\pi^2}\int_0^{\Lambda^2} dx x \ln(x + \sigma^2) \\ &= \frac{i}{16\pi^2}\Big(\int_0^{\Lambda^2} dx (x + \sigma^2)\ln(x + \sigma^2) - \sigma^2 \int_0^{\Lambda^2} dx \ln(x + \sigma^2)\Big) \\ &= \frac{i}{16\pi^2}\Big([\frac{y^2}{2}\ln y - \frac{y^2}{4}]_{\sigma^2}^{\Lambda^2+\sigma^2} - \sigma^2[y\ln y - y]_{\sigma^2}^{\Lambda^2+\sigma^2}\Big) \\ &= \frac{i}{16\pi^2}\Big(\frac{(\Lambda^2+\sigma^2)^2}{2}\ln(\Lambda^2+\sigma^2) - \frac{(\Lambda^2+\sigma^2)^2}{4} - \frac{\sigma^4}{2}\ln(\sigma^2) + \frac{\sigma^4}{4} \\ &\qquad - \sigma^2[(\Lambda^2+\sigma^2)\ln(\Lambda^2+\sigma^2) - (\Lambda^2+\sigma^2) - \sigma^2\ln\sigma^2 + \sigma^2]\Big). \end{aligned} \tag{5.128}$$

On account of (5.121), $2iC$ can be found from the divergent term multiplying σ^4 in this expression. It is given by $-\frac{i}{32\pi^2}\ln\left(\Lambda^2 + \sigma^2\right)$, which implies

$$C = -\frac{1}{64\pi^2}\ln\frac{\Lambda^2 + m^2}{m^2} \tag{5.129}$$

and, when using (5.125),

$$\boxed{g = g_R + \hbar g_R^2\frac{3}{32\pi^2}\ln\frac{\Lambda^2 + m_R^2}{m_R^2}}. \tag{5.130}$$

Note that the finite part contained in C is ambiguous. It is chosen here in such a way as to have a dimensionless quantity in the logarithm.

5.2.4 Structure of 1-loop divergences of effective action

We have shown that

$$\Gamma[\phi] = S[\phi] + \hbar\Gamma^{(1)}[\phi] + O(\hbar^2), \tag{5.131}$$

$$\Gamma^{(1)}[\phi] = -\frac{1}{2i}\mathrm{Tr}\ln\{\delta^d(x,y) + \mathcal{D}^{-1}(x,y)V''[\phi(y)]\}. \tag{5.132}$$

More explicity, in d space-time dimensions,

$$\begin{aligned}\boxed{\Gamma^{(1)}[\phi] =} &-\frac{1}{2i}\int d^dx \sum_{n=1}\frac{(-)^{n-1}}{n}\int d^dz_1\dots d^dz_{n-1}\mathcal{D}^{-1}(x,z_1)V''[\phi(z_1)]\mathcal{D}^{-1}(z_1,z_2)V''[\phi(z_2)]\dots\\ &\dots\mathcal{D}^{-1}(z_{n-1},x)V''[\phi(x)]\\ &= \boxed{-\frac{1}{2i}\sum_{n=1}\frac{(-)^{n-1}}{n}\int d^dz_1\dots d^dz_n V''[\phi(z_1)]\mathcal{D}^{-1}(z_1,z_2)V''[\phi(z_2)]\dots}\\ &\boxed{\dots\mathcal{D}^{-1}(z_{n-1},z_n)V''[\phi(z_n)]\mathcal{D}^{-1}(z_n,z_1)},\end{aligned} \tag{5.133}$$

when setting $z_n = x$. In terms of diagrams, with $V''[\phi(x)] = \frac{g}{2}\phi^2(x)$,we get

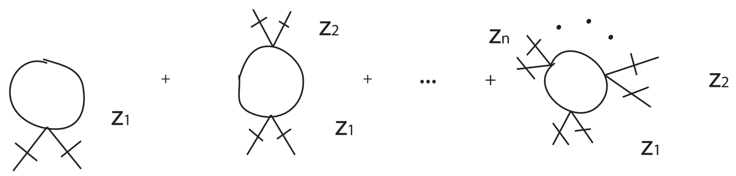


Figure 5.1: 1-loop contribution to effective action for $\frac{g}{4!}\phi^4$

Introducing the Fourier transform of the propagators,

$$\begin{aligned}\Gamma^{(1)}[\phi] = -\frac{1}{2i}\sum_{n=1}\frac{(-)^{n-1}}{n}\int d^dz_1\dots d^dz_n\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_n}{(2\pi)^d}e^{ip_1(z_1-z_2)}\dots e^{ip_n(z_n-z_1)}\\ \frac{1}{p_1^2+m^2-i\varepsilon}\cdots\frac{1}{p_n^2+m^2-i\varepsilon}V''[\phi(z_1)]\dots V''[\phi(z_n)].\end{aligned} \tag{5.134}$$

and using

$$e^{ip_1(z_1-z_2)}\dots e^{ip_n(z_n-z_1)} = e^{iz_1(p_1-p_n)}e^{iz_2(p_2-p_1)}\dots e^{iz_n(p_n-p_{n-1})},$$

one can do the triangular change of integration variables with unit Jacobian $p_1 = q$, $p_2 = q + q_2$, …, $p_n = q + q_2 \cdots + q_n$, which yields

$$\begin{aligned}\Gamma^{(1)}[\phi] = -\frac{1}{2i}\sum_{n=1}\frac{(-)^{n-1}}{n}\int d^dz_1\dots d^dz_n V''[\phi(z_1)]\dots V''[\phi(z_n)]\int\frac{d^dq_2}{(2\pi)^d}\cdots\frac{d^dq_n}{(2\pi)^d}\\ e^{-iz_1(q_2+\dots+q_n)}e^{iz_2q_2}\dots e^{iz_nq_n}\ \gamma^{(n)}(q_2,\dots,q_n),\\ \gamma^{(n)}(q_2,\dots,q_n) = \int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+m^2-i\varepsilon}\frac{1}{(q+q_2)^2+m^2-i\varepsilon}\cdots\frac{1}{(q+q_2+\dots q_n)^2+m^2-i\varepsilon}.\end{aligned} \tag{5.135}$$

[One may also provide a more symmetrical form:

$$\Gamma^{(1)}[\phi] = -\frac{1}{2i}\sum_{n=1}\frac{(-)^{n-1}}{n}\int d^d z_1 \dots d^d z_n V''[\phi(z_1)]\dots V''[\phi(z_n)]\int \frac{d^d q_1}{(2\pi)^d}\frac{d^d q_2}{(2\pi)^d}\dots\frac{d^d q_n}{(2\pi)^d}$$
$$e^{iz_1q_1}e^{iz_2q_2}\dots e^{iz_nq_n}\ \widetilde{\gamma}^{(n)}(q_1,\dots,q_n), \quad (5.136)$$

with

$$\widetilde{\gamma}^{(n)}(q_1,\dots,q_n) = (2\pi)^d\delta^d(q_1+q_2+\dots+q_n)\gamma^{(n)}(q_2,\dots,q_n).] \quad (5.137)$$

After going Euclidean, $q^0 = iq^d$, with "spherical" coordinates of radius κ, the integrand of $\gamma^{(n)}(q_2,\dots,q_n)$ behaves as $\frac{\kappa^{d-1}}{\kappa^{2n}}$ for large κ. The integrals are thus convergent for $d-1-2n < -1$ or $n > \frac{d}{2}$.

In 4 spacetime dimensions, only $\gamma^{(1)}$ and $\gamma^{(2)}$ are divergent. Furthermore, when doing a Taylor expansion in terms of the external momenta $\gamma^{(2)}(q_2) = \gamma^{(2)}(0) + q_2^A\dfrac{\partial\gamma^{(2)}}{\partial q_2^A} + \dots$. But

$$\frac{\partial\gamma^{(2)}}{\partial q_2^A} = \int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2-i\varepsilon}\frac{(-2)(q^A+q_2^A)}{((q+q_2)^2+m^2-i\varepsilon)^2}\sim\frac{\kappa^4}{\kappa^6}$$

converges.

We thus have $\gamma^{(1)} = A + \gamma^{(1)}_{fin}$ and $\gamma^{(2)}(q_2) = B + \gamma^{(2)}_{fin}(q_2)$ with A, B divergent constants. This gives

$$\boxed{\Gamma^{(1)}_{div}[\phi]} = -\frac{1}{2i}\int d^4z_1 V''[\phi(z_1)]A + \frac{1}{4i}\int\frac{d^4q_2}{(2\pi)^4}d^4z_1 d^4z_2\ e^{-iz_1q_2}e^{iz_2q_2}\ BV''[\phi(z_1)]V''[\phi(z_2)]$$
$$\boxed{= \int d^4z_1\left\{-\frac{A}{2i}V''[\phi(z_1)] + \frac{B}{4i}(V''[\phi(z_1)])^2\right\}}.$$

To first order in $\hbar$ and for $d=4$:

- the divergences are given by an integral of polynomials in the fields (and their derivatives in case $V[\phi]$ contains derivatives of fields),
- the effective action can be made finite by adding infinite counterterms of order $\hbar$ to the starting point Lagrangian,
- if $V[\phi]$ is at most quartic, the canonical dimension of the divergences is less or equal to the canonical dimension of the terms in the starting point Lagrangian.

If $V[\phi] = \frac{g}{4!}\phi^4$, $V''[\phi] = \frac{g}{2}\phi^2$ and the 1-loop divergences can be absorbed by a redefinition of the mass and the coupling constant.

5.2.5 Regularization and renormalization through ultraviolet cut-off

Cut-off regularization and renormalization

Thursday, April 30, 2015 10:30 AM

$$\gamma^{(1)} = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2+m^2-i\epsilon} = A + \gamma^{(1)}_{fin}$$

$$\gamma^{(2)}(q^2) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2+m^2-i\epsilon} \frac{1}{(q+q_2)^2+m^2-i\epsilon} = B + \gamma^{(2)}_{fin}(q_2)$$

$\Rightarrow$ A = divergent part of $J'_\Lambda(m^2)$

B = $-$ divergent part of $J''_\Lambda(m^2)$

$$J'_\Lambda(\sigma^2) = \frac{i}{16\pi^2}\Big[(\Lambda^2+\sigma^2)\cancel{\ln(\Lambda^2+\sigma^2)} + \frac{\Lambda^2+\cancel{\sigma^2}}{\cancel{2}} - \frac{\Lambda^2+\cancel{\sigma^2}}{\cancel{2}} - \sigma^2\cancel{\ln}\,\sigma^2$$
$$- \cancel{\frac{\sigma^2}{2}} + \cancel{\frac{\sigma^2}{2}} - \Big[(\Lambda^2+\sigma^2)\cancel{\ln(\Lambda^2+\sigma^2)} - (\Lambda^2+\cancel{\sigma^2}) - \sigma^2\cancel{\ln}\,\sigma^2 + \cancel{\sigma^2}\Big]$$
$$- \sigma^2\Big[\ln(\Lambda^2+\sigma^2) + \cancel{1} - \cancel{1} - \ln\sigma^2 - \cancel{1} + \cancel{1}\Big]\Big]$$

$$J'_\Lambda(\sigma^2) = \frac{i}{16\pi^2}\Big[\Lambda^2 - \sigma^2 \ln\frac{\Lambda^2+\sigma^2}{\sigma^2}\Big]$$

$$J''_\Lambda(\sigma^2) = \frac{i}{16\pi^2}\Big[-\ln\frac{\Lambda^2+\sigma^2}{\sigma^2} + \cancel{\sigma^2}\frac{\cancel{\sigma^2}}{\Lambda^2+\sigma^2}\,\frac{\cancel{\sigma^2}+\Lambda^2-\cancel{\sigma^2}}{\cancel{\sigma^4}}\Big]$$

$$= \frac{i}{16\pi^2}\Big[-\ln\frac{\Lambda^2+\sigma^2}{\sigma^2} + \frac{\Lambda^2}{\Lambda^2+\sigma^2}\Big]$$

$$\Rightarrow \boxed{B = +\frac{i}{16\pi^2}\ln\frac{\Lambda^2+m^2}{m^2}}, \quad \boxed{A = \frac{i}{16\pi^2}\Big[\Lambda^2 - m^2\ln\frac{\Lambda^2+m^2}{m^2}\Big]}$$

(Direct computation: $J'_\Lambda(\sigma^2) = \dfrac{i}{8\pi^2}\displaystyle\int_0^\Lambda dk\, k^3 \frac{1}{k^2+\sigma^2}$

(Direct computation : $J^1_\Lambda(\sigma^2) = \frac{i}{8\pi^2}\int_0^\Lambda dk\, k^3 \frac{1}{k^2+\sigma^2}$

$$J^4_\Lambda(\sigma^2) = -\frac{i}{8\pi^2}\int_0^\Lambda dk\, k^3 \frac{1}{(k^2+\sigma^2)^2}$$

$k=\sqrt{x}$, $dk = \frac{1}{2\sqrt{x}}$

$$J^1_\Lambda(\sigma^2) = \frac{i}{16\pi^2}\int_0^{\Lambda^2} dx\, \frac{x}{x+\sigma^2},$$

$$= \frac{i}{16\pi^2}\left[x \ln(x+\sigma^2)\right]_0^{\Lambda^2} - \int_0^{\Lambda^2} dx\, \ln(x+\sigma^2)$$

$$= \frac{i}{16\pi^2}\left[\Lambda^2 \ln(\Lambda^2+\sigma^2) - \left[y\ln y - y\right]_{\sigma^2}^{\Lambda^2+\sigma^2}\right]$$

$$= \frac{i}{16\pi^2}\left[\cancel{\Lambda^2}\ln(\cancel{\Lambda^2}+\sigma^2) - (\cancel{\Lambda^2}+\sigma^2)\ln(\Lambda^2+\sigma^2) + \Lambda^2+\cancel{\sigma^2} + \sigma^2\ln\sigma^2 - \cancel{\sigma^2}\right]$$

$$= +\frac{i}{16\pi^2}\left[\Lambda^2 - \sigma^2 \ln\frac{\Lambda^2+\sigma^2}{\sigma^2}\right]$$

$$J^4_\Lambda(\sigma^2) = -\frac{i}{16\pi^2}\int_0^{\Lambda^2} dx\, \frac{x}{(x+\sigma^2)^2} = -\frac{i}{16\pi^2}\left(\left[-\frac{x}{(x+\sigma^2)}\right]_0^{\Lambda^2} + \int_0^{\Lambda^2} dx\, \frac{1}{x+\sigma^2}\right)$$

$$= -\frac{i}{16\pi^2}\left[-\frac{\Lambda^2}{\Lambda^2+\sigma^2} + \ln\frac{\Lambda^2+\sigma^2}{\sigma^2}\right]$$

)

$$\Gamma^{(1)}_{div} = \int d^4x \left(-\frac{A}{2i} V''[\phi(x)] + \frac{B}{4i}\left(V''[\phi(x)]\right)^2\right)$$

$$= \int d^4x \left(-\frac{A}{2i}\frac{g}{2}\phi^2(x) + \frac{B}{4i}\frac{g^2}{4}\phi^4(x)\right)$$

$$S + \hbar \Gamma^{(1)}_{div} = -\int d^4x \; \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m_R^2\phi^2 + \frac{g_R}{4!}\phi^4$$

$$m_R^2 = m^2 + \frac{\hbar}{2i} A g \,, \qquad g_R = g - \frac{\hbar}{4i} B 6 g^2$$

$$\boxed{\begin{aligned} m_R^2 &= m^2 + \hbar \frac{g}{32\pi^2}\left[\Lambda^2 - m^2 \ln \frac{\Lambda^2 + m^2}{m^2}\right] \\ g_R &= g - \hbar \frac{3g^2}{32\pi^2} \ln \frac{\Lambda^2 + m^2}{m^2} \end{aligned}}$$

The result for the coupling constant agrees with (5.22) computed using the effectif potential.

5.2.6 Dimensional regularization and renormalization

It follows from definition (2.98) that the divergent integrals are

$$B = \gamma^{(2)}(0) = i\Phi(m, 4-2\epsilon, 2), \quad A = \gamma^{(1)} = i\Phi(m, 4-2\epsilon, 1). \tag{5.138}$$

In d dimensions, if $c = 1$, from $x^0 = ct$ one deduces that the dimension of time is the dimension of length, while if $\hbar = 1$, the action is dimensionless, but $[S] = [E][T]$, so the dimension of energy is the dimension of inverse length, but $E = \sqrt{p^2 + m^2}$ so the dimension of energy is the dimension of mass. When everything is converted to the dimension of mass $[m] = 1$, x^μ has dimensions of inverse mass, $[x^\mu] = -1$, $[p_\mu] = 1$, $[\partial_\mu] = 1$. In order for the action $S = -\frac{1}{2}\int d^dx[\partial_\mu\phi\partial^\mu\phi + m^2\phi^2 + \frac{g}{4!}\phi^4]$ to be dimensionless, one needs $[\phi] = \frac{d-2}{2}$, $[g] = -d + 2(d-2) = d - 4$.

Let us now suppose now that $d = 4$, where g is dimensionless. In dimensional regularization, one works instead in $d = 4 - 2\epsilon$ dimensions. It follows that $[g] = -2\epsilon$. One introduces a parameter μ with the dimension of mass, $[\mu] = 1$, and replaces g by the dimensionless coupling $g\mu^{2\epsilon}$. In other words, the dimensionally regularized theory is defined by the action

$$S = -\frac{1}{2}\int d^{4-2\epsilon}x[\partial_\mu\phi\partial^\mu\phi + m^2\phi^2 + \frac{g\mu^{2\epsilon}}{4!}\phi^4], \tag{5.139}$$

which reduces to the standard theory when $\epsilon \to 0$. It then follows from (5.138) that

$$-\frac{g_R}{4!} = -\frac{g\mu^{2\epsilon}}{4!} + \hbar\frac{B}{4i}\frac{(g\mu^{2\epsilon})^2}{4}. \tag{5.140}$$

When using (2.103), this gives

$$g_R = (g\mu^{2\epsilon})(1 - \hbar\frac{3}{32\pi^2}(g\mu^{2\epsilon})\mu^{-2\epsilon}\frac{1}{\epsilon}). \tag{5.141}$$

In the same way, it now follows from (5.138) that $-\frac{1}{2}m_R^2 = -\frac{1}{2}m^2 - \frac{A}{2i}\frac{g\mu^{2\epsilon}}{2}$. When using (2.104), this gives

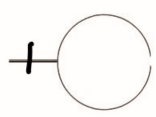
$$m_R^2 = m^2(1 - \hbar\frac{(g\mu^{2\epsilon})\mu^{-2\epsilon}}{32\pi^2}\frac{1}{\epsilon}). \tag{5.142}$$

Both of these results are in agreement with the previous computations in terms of an ultraviolet cut-off.

5.2.7 Exercises

5.2.7.1 Field renormalization and tadpoles

Study the 1-loop divergences of $V = \frac{g}{3!}\phi^3$ in $d = 6$. Show that there is a divergence that is linear in ϕ. This divergence is called a "tadpole" because it is associated to the diagram



Show that the other divergences can be absorbed by a renormalization of the mass, the coupling constant and the field of the theory.

Answer:

$$\Gamma^{(1)}_\infty[\phi] = \int d^6x\,\left[gA\phi - \frac{g^2}{4}(B\phi^2 + C\partial_\mu\phi\partial^\mu\phi) + \frac{g}{3!}D\phi^3\right], \tag{5.143}$$

with A, B, C, D divergent constants.

5.2.7.2 No tadpoles in 4 dimensions for ϕ^4

Why is it not necessary to ask for the absence of tadpoles in $d = 4$ when $V[\phi] = \frac{g}{4!}\phi^4$? Hint: How does the action behave under $\phi \to -\phi$?

5.3 Renormalizable theories

On dit qu'une théorie est renormalisable si elle (ce qui veut dire l'action effective) peut être rendue finie en rajoutant un nombre fini de contretermes différents au Lagrangien de départ. L'effet de ces contretermes revient à redéfinir la normalisation des champs, et un nombre fini de masses et de constantes de couplages, pour autant que toutes les constantes de couplage nécessaires aient été incorporées dans le Lagrangien de départ, voir par exemple l'ajout de la constante ν pour le calcul du potentiel effectif.

Par exemple, pour le champ scalaire, ceci veut dire que si on calcule l'action effective en utilisant un cut-off ultraviolet Λ pour couper l'intégration sur les hautes énergies, et on remplace le champ ϕ, la masse m et la constante de couplage $g(= \lambda)$ par des quantités renormalisées, on obtient un résultat fini $\Gamma_R[\phi_R, m_R, g_R]$ à la limite où le cut-off ultraviolet est enlevé:

$$\boxed{\Gamma_R[\phi_R, m_R, g_R] = \lim_{\Lambda\to\infty} \Gamma_\Lambda[Z^{1/2}(\Lambda, m_R, g_R)\phi_R, m(\Lambda, m_R, g_R), g(\Lambda, m_R, g_R)]}. \tag{5.144}$$

On a vu que, à une boucle, ceci est possible en $d = 4$ pour $V[\phi] = \frac{g}{4!}\phi^4$ par une redéfinition de la masse et et la constante de couplage uniquement. Donc, à cet ordre, $\phi = \phi_R$, $Z = 1$.

C'est aussi possible à une boucle en $d = 6$ pour $V[\phi] = \frac{g}{3!}\phi^3$, si on rajoute un terme $k\phi$ au Lagrangien de départ. Explicitement, par exemple $Z = 1 - \hbar\frac{g_R^2 C}{2}$. En effet, les termes comportant des dérivées du champ dans $S[\phi] + \hbar\Gamma^{(1)}_\infty[\phi]$ sont

$$\int d^6x \big[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \hbar\frac{g^2 C}{4}\partial_\mu\phi\partial^\mu\phi \big].$$

En remplaçant ϕ par $\phi = Z^{1/2}\phi_R$ dans $\Gamma[\phi]$ ou $S[\phi]$ (ce qui revient au même à cet ordre), on absorbe donc cette partie divergente. De même, $k = k_R - \hbar g_R A$ pour absorber la partie divergente linéaire en ϕ.

5.4 Normalization conditions

Demander que Γ_R soit finie pour ϕ_R, m_R, g_R fini ne fixe que la partie infinie de ϕ, m, g et termes de $\phi_R, m_R, g_R, \Lambda$. Par exemple, dans $g = g_R + \hbar\frac{3g_R^2}{32\pi^2}\ln\frac{\Lambda^2+m^2}{m^2}$, on peut modifier le terme en $\hbar$ par une contribution finie tout en gardant Γ_R fini au premier ordre. Pour fixer cette ambiguité, on impose des conditions de normalisations.

Pour $d = 4$, $V[\phi] = \frac{g}{4!}\phi^4$, on peut choisir par exemple que Γ_R est normalisée en termes des quantités renormalisées de la même manière que $S[\phi]$ est normalisée en termes des quantités non-renormalisées de départ:

$$\Gamma_R(x, y) = (\Box - m_R^2)\delta(x, y), \tag{5.145}$$

$$\Gamma_R(x_1, \dots, x_4) = -g_R\delta(x_1, x_2)\delta(x_1, x_3)\delta(x_1, x_4). \tag{5.146}$$

Notons que la première équation impose 2 conditions de normalisations, une sur la masse et une sur la fonction d'onde.

Pour $d = 6$, $V[\phi] = \frac{g}{3!}\phi^3$, on impose des conditions analogues avec comme condition supplémentaire l'absence de tadpoles:

$$\Gamma_R(x) = 0. \tag{5.147}$$

On demande donc que qu'il ne reste dans Γ_R aucun terme linéaire en ϕ et que Γ_R, comme l'action de départ commence par des termes quadratiques. En particulier, ceci impose $k_R = 0$ à l'ordre 0 en $\hbar$ et $A = \gamma^{(1)}$ à l'ordre 1 en $\hbar$.

5.5 Asymptotic behavior

5.5.1 Dilatation invariance

Si on regarde la théorie à plus petite échelle,

$$x' = \frac{1}{\lambda}x, \tag{5.148}$$

$$\phi^{A\prime}(x') = \lambda^{\Delta^A}\phi^A(x), \tag{5.149}$$

on a vu que l'action

$$S = \int d^4x\,\mathcal{L},\ \mathcal{L} = -[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4], \tag{5.150}$$

reste invariante si $\Delta_\phi = 1$ et si $m = 0$. La transformation infinitésimale associée est donnée par $\delta_{\delta\lambda}\phi = \delta\lambda[\phi + x\cdot\frac{\partial}{\partial x}\phi]$, et donc, si on regarde la théorie à plus petite échelle ($\delta\lambda = 1$)

$$\delta\phi = (1 + x\cdot\frac{\partial}{\partial x})\phi. \tag{5.151}$$

En présence du terme de masse, on trouve

$$\begin{aligned}\delta S &= -\int d^4x[\partial^\mu\phi\partial_\mu(\phi + x\cdot\frac{\partial}{\partial x}\phi) + \frac{g}{3!}\phi^3(\phi + x\cdot\frac{\partial}{\partial x}\phi) + m^2\phi(\phi + x\cdot\frac{\partial}{\partial x}\phi)]\\ &= -\int d^4x[2\partial^\mu\phi\partial_\mu\phi + m^2\phi^2 + \frac{g}{3!}\phi^4 + x\cdot\frac{\partial}{\partial x}(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4)\\ &= \int d^4x\,(4 + x\cdot\frac{\partial}{\partial x})\mathcal{L} + \int d^4x\,m^2\phi^2,\end{aligned} \tag{5.152}$$

ce qui donne

$$\delta S + m\frac{\partial}{\partial m}S = 0. \tag{5.153}$$

Comme la transformation est linéaire, on s'attend donc naïvement à trouver pour l'action effective, $\delta\Gamma = 0$ si $m = 0$ et

$$\delta\Gamma + m\frac{\partial}{\partial m}\Gamma = 0, \tag{5.154}$$

en répétant le raisonnement qui conduit aux identités de Ward. (Notons aussi que $\frac{\partial}{\partial m}\Gamma = \frac{\partial}{\partial m}W[J^\phi]$.) On va montrer maintenant que la renormalisation, i.e., l'absorption des divergences, implique des corrections quantiques.

5.5.2 Callan-Symanzik equation

Pour respecter l'analyse dimensionnelle habituelle on peut s'arranger, en ajustant la partie finie que l'on soustrait avec la partie divergente, pour avoir

$$Z(\Lambda, m_R, g_R) = Z(\frac{\Lambda}{m_R}, g_R),\ g(\Lambda, m_R, g_R) = g(\frac{\Lambda}{m_R}, g_R),\ \frac{m}{m_R}(\Lambda, m_R, g_R) = \frac{m}{m_R}(\frac{\Lambda}{m_R}, g_R), \tag{5.155}$$

et de même pour les quantités renormalisées en termes des quantités nues. Définissons $\Gamma_{\Delta,\Lambda}[\phi, m, g] = \frac{1}{2}m\frac{\partial\Gamma_\Lambda}{\partial m}[\phi, m, g]$ et varions m à g, ϕ, Λ fixé dans la relation (5.144):

$$\lim_{\Lambda\to\infty} 2\frac{dm}{m}\Gamma_{\Delta,\Lambda}[\phi, m, g] = \Big[dm_R\frac{\partial}{\partial m_R} + dg_R\frac{\partial}{\partial g_R} + \int d^4x\, \delta\phi_R(x)\frac{\delta}{\delta\phi_R(x)}\Big]\Gamma_R[\phi_R, m_R, g_R]. \tag{5.156}$$

Comme $\Gamma_R[\phi_R, m_R, g_R]$ est de dimension totale nulle (il n'y a plus de Λ qui peut invalider l'analyse dimensionnelle), on a aussi,

$$\Big[m_R\frac{\partial}{\partial m_R} + \int d^4x\,(1 + x\cdot\frac{\partial}{\partial x})\phi_R(x)\frac{\delta}{\delta\phi_R(x)}\Big]\Gamma_R[\phi_R, m_R, g_R] = 0. \tag{5.157}$$

En utilisant $\delta\phi(x) = 0 = \frac{1}{2}Z^{-1/2}dZ\phi_R(x) + Z^{1/2}\delta\phi_R(x)$ ce qui donne $\delta\phi_R(x) = -\frac{1}{2}d(\ln Z)\phi_R(x)$, en multipliant la relation (5.156) par $\frac{m_R}{dm_R}$ et en substituant (5.157), on trouve

$$\lim_{\Lambda\to\infty} 2\frac{\partial m}{\partial m_R}\frac{m_R}{m}\Gamma_{\Delta,\Lambda}[\phi, m, g] =$$
$$= \Big[m_R\frac{\partial g_R}{\partial m_R}\frac{\partial}{\partial g_R} - \int d^4x\,(1 + \frac{1}{2}m_R\frac{\partial\ln Z}{\partial m_R} + x\cdot\frac{\partial}{\partial x})\phi_R(x)\frac{\delta}{\delta\phi_R(x)}\Big]\Gamma_R[\phi_R, m_R, g_R]. \tag{5.158}$$

Pour simplifier le membre de droite, on définit alors

$$\beta(g_R) = m_R\frac{\partial g_R}{\partial m_R} = -\Lambda\frac{\partial g_R}{\partial\Lambda},\ \gamma(g_R) = \frac{1}{2}m_R\frac{\partial\ln Z}{\partial m_R} = -\frac{1}{2}\Lambda\frac{\partial\ln Z}{\partial\Lambda} \tag{5.159}$$

où la limite $\Lambda \to \infty$ est entendue.

Pour comprendre le membre de gauche, on couple à l'action de départ $\phi^2(x)$ avec une source externe $K(x)$,

$$S_K = -\int d^4x[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{4!}\phi^4 + \frac{1}{2}m^2(1 + K(x))\phi^2(x)] \tag{5.160}$$

ce qui donne $\Gamma_{\Delta,\Lambda} = \frac{1}{2}m\frac{\partial\Gamma_\Lambda}{\partial m} = \int d^4x\frac{\delta\Gamma_{\Lambda,K}}{\delta K(x)}\big|_{K=0}$. Dans le calcul de $\Gamma^{(1)}[\phi]$, il y a alors un nouveau terme car $V_K''[\phi(x)] = \frac{g}{2}\phi^2(x) + m^2K(x)$. On a donc,

$$\Gamma^{(1)}_{\infty,K} = \int d^4x\big(-V_K''[\phi(x)]\frac{A}{2i} + (V_K''[\phi(x)])^2\frac{B}{4i}\big) \tag{5.161}$$

avec A, B des constantes divergeantes, ce qui donne

$$\frac{\delta\Gamma^{(1)}_{\infty,K}}{\delta K(x)}\big|_{K=0} = -m^2\frac{A}{2i} + gm^2\phi^2(x)\frac{B}{4i}. \tag{5.162}$$

Le premier terme est absorbé par un terme du type $\int d^4x\nu K(x)$ dans le Lagrangien. Ce terme découple et ne nous intéresse pas, cf. "constante cosmologique" dans le calcul du potentiel effectif ou discussion des tadpoles]. Le deuxième terme peut être absorbée dans une redéfinition de $K(x)$, $K(x) = Z_K K_R(x)$ avec $Z_K = (1-\hbar g\frac{B}{4i})$. On a donc aussi que la partie linéaire en $K(x)$ de $\Gamma_{\Lambda,K}[Z_K K_R(x), Z^{1/2}\phi_R(x), m(), g()] = \Gamma_{R,K}[K_R, \phi_R, m_R, g_R]$ est finie à une boucle pour $\Lambda \to \infty$. On définit alors

$$\begin{aligned}\Gamma_{R,\Delta}[\phi_R, m_R, g_R] &= \int d^4x \frac{\delta\Gamma_{R,K}}{\delta K_R(x)}|_{K_R=0} = \\ &= \int d^4x Z_K \frac{\delta\Gamma_{\Lambda,K}}{\delta K(x)}|_{K=0} = Z_K \Gamma_{\Lambda,\Delta}[Z^{1/2}\phi_R, m(), g()]\,, \quad (5.163)\end{aligned}$$

Le membre de gauche de (5.158) devient alors $2\frac{\partial m}{\partial m_R}\frac{m_R}{m}\frac{1}{Z_K}\Gamma_{R,\Delta}[\phi_R, m_R, g_R]$. Si on définit

$$1 + \delta(g_R) = \frac{1}{Z_K}\frac{m_R}{m}\frac{\partial m}{\partial m_R} = -\frac{1}{Z_K}\frac{\Lambda}{m}\frac{\partial m}{\partial \Lambda} \quad (5.164)$$

l'équation (5.158) devient

$$\begin{aligned}&\boxed{2(1+\delta(g_R))\Gamma_{R,\Delta}[\phi_R, m_R, g_R] =} \\ &= \boxed{\Big[\beta(g_R)\frac{\partial}{\partial g_R} - \int d^4x\,(1+\gamma(g_R) + x\cdot\frac{\partial}{\partial x})\phi_R(x)\frac{\delta}{\delta\phi_R(x)}\Big]\Gamma_R[\phi_R, m_R, g_R]}\,. \quad (5.165)\end{aligned}$$

C'est l'équation de Callan-Symanzik. On voit que la relation naïve est violée par les termes $\delta(g_R)$, $\gamma(g_R)$, $\beta(g_R)$ qui sont d'ordres $\hbar$ et dus au fait que la relation entre quantités renormalisées et nues, nécessaire pour absorber les divergences, fait intervenir Λ.

5.5.3 High energy behavior and massless theory

En termes de transformées de Fourier,

$$\frac{\delta\Gamma}{\delta\phi(x_1)\ldots\delta\phi(x_n)} = \Gamma(x_1\ldots x_n) = \int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_n}{(2\pi)^4}\exp ip_1x_1\ldots\exp ip_nx_n\widetilde{\Gamma}(p_1,\ldots,p_n)\,. \quad (5.166)$$

Regardons la transformée de Fourier de

$$\begin{aligned}\Gamma_\Delta(x_1,\ldots,x_n) &= -\frac{1}{2}m^2\int d^4x\langle 0|T\widehat{\phi}^2(x)\widehat{\phi}(x_1)\ldots\widehat{\phi}(x_n)|0\rangle_R^{1PI} = \\ &= \int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_n}{(2\pi)^4}\exp ip_1x_1\ldots\exp ip_nx_n\widetilde{\Gamma}_\Delta(0;p_1,\ldots,p_n)\,. \quad (5.167)\end{aligned}$$

En effet, dans l'action, on a le terme

$$\int d^4x\; -\frac{1}{2}m^2(1+K(x))\phi^2(x) \quad (5.168)$$

et

$$\Gamma_{\Lambda,\Delta} = \frac{1}{2}m\frac{\partial\Gamma_\Lambda}{\partial m} = \int d^4x\frac{\delta\Gamma_\Lambda}{\delta K(x)}|_{K=0}\,, \quad (5.169)$$

ce qui donne, en transformée de Fourier,

$$\widetilde{\Gamma}_{\Lambda,\Delta}(p) = \int d^4x \exp -ipx\frac{\delta\Gamma_\Lambda}{\delta K(x)}\,, \quad (5.170)$$

et donc $\Gamma_{\Lambda,\Delta} = \widetilde{\Gamma}_{\Lambda,\Delta}(0)$; l'insertion se fait à moment zéro. On utilise $\int d^4x\, \phi(x)\Gamma(x) = \int \frac{d^4p}{(2\pi)^4}\widetilde{\phi}(-p)\widetilde{\Gamma}(p)$ et aussi

$$\begin{aligned}
&\int d^4x(1+\gamma(g_R)+x\cdot\frac{\partial}{\partial x})\phi(x)\frac{\delta}{\delta\phi(x)} \\
=&\int d^4x(1+\gamma(g_R)+x\cdot\frac{\partial}{\partial x})\int\frac{d^4p}{(2\pi)^4}\exp ipx\,\widetilde{\phi}(p)\int d^4p'\frac{\delta\widetilde{\phi}(p')}{\delta\phi(x)}\frac{\delta}{\delta\widetilde{\phi}(p')} \\
=&\int d^4x\int\frac{d^4p}{(2\pi)^4}(1+\gamma(g_R)+p\cdot\frac{\partial\exp ipx}{\partial p})\widetilde{\phi}(p)\int d^4p'\exp -ip'x\frac{\delta}{\delta\widetilde{\phi}(p')} \\
=&\int d^4x\int\frac{d^4p}{(2\pi)^4}(1+\gamma(g_R)-4-p\cdot\frac{\partial}{\partial p})\widetilde{\phi}(p)\exp ipx\int d^4p'\exp -ip'x\frac{\delta}{\delta\widetilde{\phi}(p')} \\
=&\int d^4p(1+\gamma(g_R)-4-p\cdot\frac{\partial}{\partial p})\widetilde{\phi}(p)\frac{\delta}{\delta\widetilde{\phi}(p)}\,.
\end{aligned}$$

L'équation de Callan-Symanzik devient alors, en appliquant $\frac{\delta^n}{\delta\widetilde{\phi}(-p_1)\ldots\delta\widetilde{\phi}(-p_n)}$ et en annulant $\widetilde{\phi}(p)$,

$$2(1+\delta(g_R))\widetilde{\Gamma}_{R,\Delta}(0;p_1,\ldots,p_n) = [\beta(g_R)\frac{\partial}{\partial g_R} - n(1+\gamma(g_R)) - \sum_{k=1}^{n} p_k\cdot\frac{\partial}{\partial p_k}]\widetilde{\Gamma}_R(p_1,\ldots,p_n)\,. \tag{5.171}$$

En utilisant $V''_K[\phi(z_1)] = -\frac{1}{2}g\phi^2(z_1) - m^2K(z_1)$, $\phi^2(z_1) = \int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\widetilde{\phi}(-p_1)\widetilde{\phi}(-p_2)\exp -i(p_1+p_2)z_1$, $\int d^4z_1 \exp i(q_1-p_1-p_2)z_1 = (2\pi)^4\delta^4(q_1-(p_1+p_2))$, on trouve que

$$\widetilde{\Gamma}^{(1)}(p_1,\ldots,p_{2n}) \sim \widetilde{\gamma}^{(n)}(p_1+p_2,\ldots,p_{2n-1}+p_{2n})\,, \tag{5.172}$$

$$\widetilde{\Gamma}^{(1)}_{\Delta}(0;p_1,\ldots,p_{2n}) \sim \widetilde{\gamma}^{(n+1)}(0;p_1+p_2,\ldots,p_{2n-1}+p_{2n})\,, \tag{5.173}$$

Si tous les moments externes deviennent grands ensembles $p\to\lambda p$, en utilisant $\delta(\lambda p) = \frac{1}{\lambda}\delta(p)$ et le comportement de $\widetilde{\gamma}^{(n)}$, on trouve

$$\widetilde{\Gamma}^{(1)}(\lambda p_1,\ldots,\lambda p_{2n}) = O(\frac{1}{\lambda^{2(n-1)+4}}), \tag{5.174}$$

$$\widetilde{\Gamma}^{(1)}_{\Delta}(0;\lambda p_1,\ldots,\lambda p_{2n}) = O(\frac{1}{\lambda^{2n+4}})\,. \tag{5.175}$$

La renormalisation n'affecte que $\widetilde{\gamma}^{(1)}(p_1)_{div} = (2\pi)^4\delta^4(p_1)\gamma^{(1)}_{div} = O(\lambda^{-4})$ et $\widetilde{\gamma}^{(2)}(p_1,p_2)_{div} = (2\pi)^4\delta^4(p_1+p_2)\gamma^{(2)}_{div}(0) = O(\lambda^{-4})$ et ne change donc pas le fait que le membre de gauche de l'équation Callan-Symanzik décroît plus vite et est donc négligeable à haute énergie. On dénotant par $\widetilde{\Gamma}^{as}_R(p_1,\ldots,p_n)$ le comportement dominant à haute énergie de $\widetilde{\Gamma}_R(p_1,\ldots,p_n)$, on trouve donc

$$\boxed{[\beta(g_R)\frac{\partial}{\partial g_R} - n(1+\gamma(g_R)) - \sum_{k=1}^{n} p_k\cdot\frac{\partial}{\partial p_k}]\widetilde{\Gamma}^{as}_R(p_1,\ldots,p_n) = 0}\,. \tag{5.176}$$

Comme l'analyse dimensionnelle usuelle s'applique pour la théorie renormalisée, faire $p\to\lambda p$ dans $\Gamma^{as}_R(p_i;g_R,m_R)$ est équivalent, à un facteur global près, à faire $m_R\to\lambda^{-1}m_R$. On peut alors considérer les fonctions de Green $\widetilde{\Gamma}^{as}_R(p_i;g_R)$ solutions de (5.176) comme la définition des fonctions de Green de la théorie de masse nulle.

5.5.4 Running coupling constant

En oubliant l'indice R par la suite et en introduisant

$$\lambda\frac{d}{d\lambda}g(\lambda) = \beta(g(\lambda)),\ g(1) = g \Longleftrightarrow \lambda = \exp\int_g^{g(\lambda)}\frac{dg'}{\beta(g')}\,, \tag{5.177}$$

$$\lambda\frac{d}{d\lambda}z(\lambda) = \gamma(g(\lambda))z(\lambda),\ z(1) = 1 \Longleftrightarrow z(\lambda) = \exp\int_1^{\lambda}\frac{d\lambda'\gamma(g(\lambda'))}{\lambda'} = \exp\int_g^{g(\lambda)}\frac{dg'\gamma(g')}{\beta(g')}, \tag{5.178}$$

l'équation (5.176), qui est valable pour toutes les valeurs des moments externes et de g, se réécrit

$$\lambda\frac{d}{d\lambda}[\lambda^{-n}(z(\lambda))^{-n}\widetilde{\Gamma}^{as}(\lambda^{-1}p_i; g(\lambda))] = 0, \tag{5.179}$$

ce qui est équivalent à

$$\widetilde{\Gamma}^{as}(\lambda p_i; g) = \lambda^{-n}(z(\lambda))^{-n}\widetilde{\Gamma}^{as}(p_i; g(\lambda)) \tag{5.180}$$

À partir de $x' = \lambda^{-1}x, \phi'(x') = \lambda\phi(x)$, on trouve qu'en transformée de Fourier, $p' = \lambda p, \widetilde{\phi}'(p') = \lambda^{-3}\widetilde{\phi}(p)$. Au niveau quantique, on définit $\widetilde{\phi}'(p') = \lambda^{-3}z(\lambda)\widetilde{\phi}(p)$, ce qui justifie la terminologie "dimension anomale" pour $z(\lambda)$. En termes de la fonctionnelle

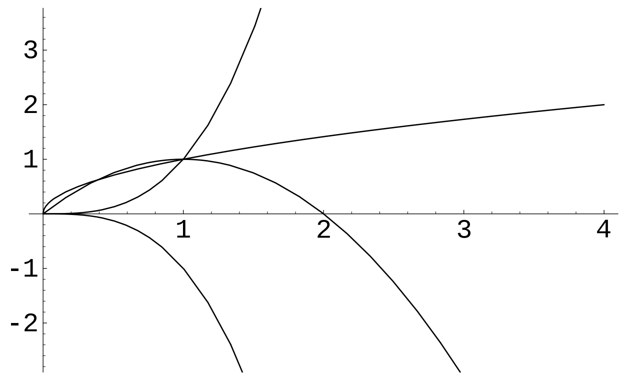
$$\widetilde{\Gamma}^{as}[\widetilde{\phi}; g] = \sum_{n=2}\frac{1}{n!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_n}{(2\pi)^4}\widetilde{\phi}(-p_1)\ldots\widetilde{\phi}(-p_n)\widetilde{\Gamma}^{as}(p_1,\ldots,p_n; g)\,, \tag{5.181}$$

l'équation (5.180) s'exprime par

$$\boxed{\widetilde{\Gamma}^{as}[\widetilde{\phi}'; g] = \widetilde{\Gamma}^{as}[\widetilde{\phi}; g(\lambda)]}. \tag{5.182}$$

On a donc montré que l'invariance par dilatation de la théorie de masse nulle peut être rétablie au niveau quantique en introduisant une dimension anomale pour les champs, sans importance pour les éléments de matrice S, et une "constante de couplage" qui varie avec l'énergie.

Le comportement de la constante de couplage en fonction de l'énergie est déterminé par la fonction $\beta(g)$:



- **Singularité à énergie finie:** $\beta(g) > 0$ et croît suffisamment rapidement pour que $\int^{\infty}\frac{dg'}{\beta(g')} < \infty$, par exemple $\beta(g) = g^3$. Dans ce cas, $g(\lambda)$ croît de 0 à ∞ et atteint $g = \infty$ pour une valeur finie de l'énergie, $\lambda_\infty = \exp\int_g^\infty\frac{dg'}{\beta(g')}$.

- **Singularité à énergie infinie:** $\beta(g) > 0$ et $\int^\infty \frac{dg'}{\beta(g')} \to \infty$, par exemple, $\beta(g) = \sqrt{g}$. Dans ce cas, $g(\lambda)$ croît de 0 à ∞ et atteint $g = \infty$ pour une valeur infinie de l'énergie, $\lambda_\infty = \infty$.
- **Point fixe ultraviolet:** $\beta(g) > 0$ pour $0 < g < g^*$, $\beta(g^*) = 0$, $\beta(g) < 0$ pour $g^* < g$, par exemple, $\beta(g) = -g(g-2)$. Dans ce cas, $g(\lambda)$ croît pour $0 < g < g^*$ et $g(\lambda)$ décroît pour $g^* < g$ et $g(\lambda) \to g^*$, peu importe de quel côté de g^* on commence. Si le zéro de $\beta(g)$ est simple, alors $\beta(g) \to -a(g-g^*)$ pour $g \to g^*$ avec $a > 0$. La solution de $\lambda\frac{d}{d\lambda}g = -a(g-g^*)$ est $g(\lambda) - g^* = (g(1) - g^*)\lambda^{-a}$. En supposant que $\gamma(g) \to \gamma(g^*)$, on trouve comme dimension anomale à haute énergie $z(\lambda) = \lambda^{\gamma(g^*)}$.
- **Liberté asymptotique:** $\beta(g) = -bg^n$, $b > 0$, $n > 2$ pour g petit. En intégrant l'équation différentielle, on trouve $g(\lambda) = g[1 + g^{n-1}(n-1)b\ln\lambda]^{-\frac{1}{n-1}}$. On a donc $g(\lambda) \to 0$ pour $\lambda \to \infty$. Dans ce cas, l'hypothèse sur laquelle se base le calcul perturbatif, à savoir que g est petit, est valable à haute énergie.

Pour le champ scalaire avec $V[\phi] = -\frac{g}{4!}\phi^4$, on a montré que

$$g_R = g - \hbar\frac{3}{32\pi^2}g^2\ln\Lambda^2 + \hbar\,\text{finite} + O(\hbar^2)\,. \tag{5.183}$$

Ceci implique que

$$\beta(g_R) = -\Lambda\frac{d}{d\Lambda}g_R = \hbar\frac{3}{16\pi^2}g_R^2 + O(\hbar^2)\,. \tag{5.184}$$

On est donc dans le cas 1. Ce cas s'applique également à l'électrodynamique quantique, où on trouve

$$e_R = e[1 - \hbar\frac{e^2}{12\pi^2}\ln(\Lambda/\mu)] + O(\hbar^2)\,, \tag{5.185}$$

avec μ un cut-off infrarouge, et donc

$$\beta(e_R) = \hbar\frac{e_R^3}{12\pi^2} + O(\hbar^2). \tag{5.186}$$

Pour les théories de Yang-Mills basée sur le groupe de jauge $SU(N)$, par contre, on peut montrer que

$$g_R = g[1 + \hbar\frac{g^2}{4\pi^2}(\frac{11}{12}N - \frac{n_f}{6})\ln(\Lambda/\mu)] + O(\hbar^2)\,, \tag{5.187}$$

où n_f le nombre de fermions. On en tire que

$$\beta(g_R) = \hbar\frac{g_R^3}{4\pi^2}(\frac{n_f}{6} - \frac{11}{12}N) + O(\hbar^2). \tag{5.188}$$

Pour $n_f \leqslant 5N$, on a $\beta(g_R) < 0$ et on est donc dans le cas 4 de la liberté asymptotique. En particulier, ce cas s'applique pour la chromodynamique quantique, avec $n_f = 6$ et $N = 3$. Ainsi, comme $g_R(\lambda) \to 0$, les quarks se comportent à haute énergie comme des particules libres.

5.5.5 Exercises

5.5.5.1 Callan-Symanzik in momentum space at tree level

Vérifier l'équation de Callan-Symanzik pour la théorie $\frac{g}{4!}\phi^4$ en transformée de Fourier à l'ordre 0 en $\hbar$ et vérifier que les termes en m^2 s'annulent séparément.

5.5.5.2 Dimensional regularisation and renormalization group

Discuter le groupe de renormalisation dans le contexte de la régularisation dimensionnelle en étudiant la dépendance de la théorie renormalisée en le paramètre d'échelle μ. Voir par exemple chapitre 9 de [16] ou page 650, 651 de [9].

5.6 Summary

En physique des particules, on s'intéresse, entre autres, aux éléments de la matrice de diffusion. Les formules de réduction nous permettent de construire ces éléments de matrices à partir des fonctions de Green de la théorie.

Les fonctions de Green se calculent de manière perturbative en utilisant les régles de Feynman. On peut réorganiser la somme sur tous les diagrammes de Feynamn en organisant tous les diagrammes en des parties connexes. Les diagrammes connexes sont constitués de diagrammes en arbre avec vertex remplacés par des vertex propres qui contiennent toutes les boucles.

Les divergences ultraviolettes d'une théorie se trouvent dans les vertex propres et sont absorbées en redéfinissant les constantes de couplages et fonction d'onde du Lagrangien de départ. Cette procédure affecte, entre autres, le comportement d'une théorie sous les changements d'échelle.

5.7 Heat kernel and zeta function regularization

cf. [10], Appendice A.9, [46], [47],[48].

5.7.1 Schwinger proper time

L'intégrale de chemin en Euclidien est donnée par

$$\exp -\frac{1}{\hbar}W[J] = \int \mathcal{D}\phi \ \exp -\frac{1}{\hbar}(S^E_L + J_A\phi^A), \tag{5.189}$$

avec

$$S^E_L = \int d^dx \, [\frac{1}{2}\partial_a\phi\partial^a\phi + \frac{1}{2}m^2\phi^2 + V[\phi]], \tag{5.190}$$

pour le champ scalaire où les indices sont descendus et montés avec δ_{ab} et son inverse. Cette action ressemble donc '' un Hamiltonien à d dimensions avec les moments remplacés par des vitesses. L'approximation semi-classique pour l'action effective devient

$$\Gamma[\phi] = S^E_L[\phi] + \frac{\hbar}{2}\mathrm{Tr}\Big(\ln \frac{\delta^2 S^E_L}{\delta\phi\delta\phi}[\phi] - \ln \frac{\delta^2 S^E_L}{\delta\phi\delta\phi}[0]\Big) + O(\hbar^2), \tag{5.191}$$

Si

$$\boxed{M(x,x')[\phi]} = \frac{\delta^2 S^E_L}{\delta\phi(x)\delta\phi(x')}[\phi]\boxed{= (-\Delta + m^2 + V''[\phi(x)])\delta^d(x,x')}, \tag{5.192}$$

la contribution à une boucle peut s'écrire comme une intégrale sur le temps propre de Schwinger,

$$\boxed{\Gamma^{(1)}[\phi] = -\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\mathrm{Tr}(e^{-\tau M[\phi]} - e^{-\tau M[0]}).} \tag{5.193}$$

On utilise

$$\ln z = -\int_0^\infty \frac{dt}{t}\,(e^{-tz} - e^{-t}),$$

(même développement en série de Taylor autour de 1 et puis on pose $z = \frac{x}{y}$ et on effectue le changement de variable $t \to ty$.

Les divergences à grand moment correspondent à des divergences en $\tau \to 0$ dans la représentation de Schwinger. Une manière de régulariser consiste alors à couper la borne d'intégration inférieure. Pour faciliter la discussion on pose $M[\phi] = H[\phi] + m^2$ et l'action régularisée devient

$$\Gamma^{(1)}_\epsilon[\phi] = -\frac{1}{2}\int_\epsilon^\infty e^{-m^2\tau}\frac{d\tau}{\tau}\,\mathrm{Tr}(e^{-\tau H[\phi]} - e^{-\tau H[0]}). \tag{5.194}$$

5.7.2 Heat kernel

Si on pose de plus $V''[\phi(x)] = U(x)$, $H = -\Delta + U(x)$ est l'Hamiltonien hermitien d'un problème de mécanique quantique à d dimensions. Pour paramétriser les divergences, on a besoin du comportement de $\langle x, e^{-\tau H} x'\rangle \equiv K_H(x, x'; \tau)$ au voisinage de $\tau = 0$. On doit alors résoudre l'équation de la chaleur (reliée à l'équation de Schrödinger si $\tau = it$) pour le noyau $K_H(x, x'; \tau)$,

$$\boxed{-\frac{\partial}{\partial\tau}K_H(x, x'; \tau) = HK_H(x, x'; \tau), \quad K_H(x, x'; 0) = \delta^d(x, x')}. \tag{5.195}$$

Pour le Laplacian, $H = -\Delta$, la solution de cette équation est

$$K_{-\Delta}(x, x'; \tau) = \frac{1}{(4\pi\tau)^{d/2}}\exp -\frac{(x - x')^2}{4\tau} \tag{5.196}$$

Pour l'Hamiltonien $H = -\Delta + U(x)$ on fait l'hypothèse

$$\boxed{K_H(x, x'; \tau) \xrightarrow{\tau\to 0} \frac{1}{(4\pi\tau)^{d/2}}\exp\,(-\frac{(x - x')^2}{4\tau})\Big(\sum_n a_n(x, x')\tau^n\Big)}. \tag{5.197}$$

Les coefficients $a_n(x, x')$ sont appelés coefficients de Seeley et satisfont $a^*_n(x, x') = a_n(x', x)$ à cause de l'hermiticité de H et aussi $a_0(x, x') = 1$ pour satisfaire à la condition intitiale.

On pose aussi $K_H(x, x'; \tau) = \exp -\sigma(x, x'; \tau)$. L'équation de la chaleur est alors équivalente à

$$\frac{\partial\sigma}{\partial\tau} = \Delta\sigma - \frac{\partial\sigma}{\partial x^a}\frac{\partial\sigma}{\partial x_a} + U(x). \tag{5.198}$$

$\sigma(x, x'; \tau)$ peut être développé comme

$$\begin{aligned}\sigma(x, x'; \tau) &= \frac{1}{4\tau}(x - x')^2 + \frac{d}{2}\ln(4\pi\tau) - \ln(1 + \sum_{n\geqslant 1} a_n(x, x')\tau^n)\\ &= \frac{1}{4\tau}(x - x')^2 + \frac{d}{2}\ln(4\pi\tau) + \sum_{m\geqslant 1} b_m(x, x')\tau^m.\end{aligned}$$

Injecté dans (5.198), on trouve

$$b_1 + (x-x')^a \frac{\partial b_1}{\partial x^a} = U(x), \quad 2b_2 + (x-x')^a \frac{\partial b_2}{\partial x^a} = \Delta b_1,$$
$$kb_k + (x-x')^a \frac{\partial b_k}{\partial x^a} = \Delta b_{k-1} - \sum_{l=1}^{k-2} \frac{\partial b_l}{\partial x^a} \frac{\partial b_{k-1-l}}{\partial x_a}, \quad k \geqslant 3.$$

En utilisant $(x-x')^a \dfrac{\partial U(x'+s(x-x'))}{\partial x^a} = s\dfrac{\partial}{\partial s} U(x'+s(x-x'))$, ceci donne

$$b_1(x,x') = \int_0^1 ds\, U(x'+s(x-x')), \quad b_2(x,x') = \int_0^1 ds\, s(1-s)\Delta U(x'+s(x-x')).$$

car

$$b_1 + (x-x')\frac{\partial}{\partial x} b_1 = \int_0^1 ds\, (1+s\frac{\partial}{\partial s})U(x'+s(x-x')) = U(x),$$
$$2b_2 + (x-x')\frac{\partial}{\partial x} b_2 - \Delta b_1 = \int_0^1 ds\, \big[s(1-s)(2+s\frac{\partial}{\partial s}) - s^2\big]\Delta U = 0.$$

car $(1+s\dfrac{\partial}{\partial s})U = \dfrac{\partial}{\partial s}(sU)$ et $\big[2s(1-s)+s^2(1-s)\dfrac{\partial}{\partial s} - s^2\big]\Delta U = \big[2s(1-s)+-2s(1-s)+s^2-s^2\big]\Delta U + \dfrac{\partial}{\partial s}(s^2(1-s)\Delta U)$, $[(s^2(1-s))]_0^1 = 0$ et les autres termes se compensent.

Pour les éléments diagonaux, on injecte la première expansion de $\sigma(x,x';t)$ en les coefficients de Seeley dans (5.198) pour trouver la relation de récurrence

$$\boxed{a_0(x,x) = 1, \quad a_k(x,x) = -\frac{1}{k} H a_{k-1}, \quad k \geqslant 1.} \tag{5.199}$$

En effet, on trouve

$$\frac{k a_k \tau^{k-1}}{1+a_n\tau^n} = \frac{\partial}{\partial x_a} \frac{\frac{\partial a_k}{\partial x^a}\tau^k}{1+a_n\tau^n} + \frac{\frac{\partial a_k}{\partial x_a}\tau^k}{1+a_n\tau^n}\frac{\frac{\partial a_l}{\partial x^a}\tau^l}{1+a_m\tau^m} - U.$$

En particulier,

$$a_1 = -U, \quad a_2 = \frac{1}{2}(-\Delta U + U^2),\ a_3 = \frac{1}{6}(-\Delta^2 U + \Delta U^2 + U\Delta U - U^3), \tag{5.200}$$

En injectant dans (5.194) et en négligeant les termes aux bords, on retrouve alors la forme générale des divergences à une boucle pour le champ scalaire.

5.7.3 1-loop divergences in scalar field theory

Pour $d \leqslant 6$, les divergences à une boucle sont paramétrisées par

$$\begin{aligned}
\Gamma_\epsilon^{(1)}[\phi] &= -\frac{1}{2}\frac{1}{(4\pi)^{d/2}} \int_\epsilon \frac{d\tau\, e^{-m^2\tau}}{\tau^{1+d/2}} \int d^dx \Big[-U\tau + \frac{1}{2}U^2\tau^2 - \frac{1}{6}(U^3 + \frac{\partial U}{\partial x^a}\frac{\partial U}{\partial x_a})\tau^3 + O(\tau^4)\Big] \\
&= \frac{1}{2}\frac{1}{(4\pi)^{d/2}} \Big[-\frac{\epsilon^{1-d/2}}{1-d/2}\int d^dx\, U + \frac{1}{2}\frac{\epsilon^{2-d/2}}{2-d/2}\int d^dx\,(U^2+2m^2U) - \\
&\quad -\frac{1}{6}\frac{\epsilon^{3-d/2}}{3-d/2}\int d^dx\,(U^3 + 3m^2U^2 + 3m^4U + \frac{\partial U}{\partial x^a}\frac{\partial U}{\partial x_a})\Big] + \text{fini},
\end{aligned} \tag{5.201}$$

où on a gardé la contribution de la borne inférieure dans la dernière ligne et $\frac{\epsilon^0}{0} = \ln \epsilon$.

En particulier, pour $d = 6, V = \frac{g}{3!}\phi^3$, et $\epsilon = \frac{1}{\Lambda^2}$,

$$\boxed{\Gamma^{(1)}_{div}[\phi] = \frac{1}{2^7\pi^3}\int d^6x\Big[\frac{\Lambda^4}{2}g\phi - \frac{\Lambda^2}{2}(g^2\phi^2 + 2gm^2\phi)+} \\ + \boxed{\frac{1}{3}\ln\frac{\Lambda}{m}(g^3\phi^3 + 3g^2m^2\phi^2 + 3gm^4\phi + g^2\partial_a\phi\partial^a\phi)\Big]}, \quad (5.202)$$

pour $d = 4, V = \frac{g}{4!}\phi^4$,

$$\boxed{\Gamma^{(1)}_{div}[\phi] = \frac{1}{32\pi^2}\int d^4x\Big[\Lambda^2\frac{g}{2}\phi^2 - \ln\frac{\Lambda}{m}(\frac{g^2}{4}\phi^4 + gm^2\phi^2)\Big]}, \quad (5.203)$$

et pour $d = 2$ quelque soit $V[\phi]$,

$$\boxed{\Gamma^{(1)}_{div}[\phi] = \frac{1}{4\pi}\ln\frac{\Lambda}{m}\int d^2x\, V''[\phi]}. \quad (5.204)$$

5.7.4 Zeta function regularization

Soit λ_κ les valeurs propres réelles de l'opérateur hermitien H et $\varphi_\kappa(x)$ un ensemble orthonormée et complet de vecteurs propres,

$$H\varphi_\kappa(x) = \lambda_\kappa\varphi_\kappa(x), \quad \int d^dx\,\bar{\varphi}^\kappa(x)\varphi_{\kappa'}(x) = \delta^\kappa_{\kappa'}, \quad \bar{\varphi}^\kappa(x)\varphi_\kappa(x') = \delta^d(x, x'). \quad (5.205)$$

Le noyau chaleur est alors donné par

$$K_H(x, x'; \tau) = \langle x, e^{-H\tau}x'\rangle = e^{-\lambda_{(\kappa)}\tau}\bar{\varphi}^\kappa(x)\varphi_\kappa(x'). \quad (5.206)$$

Pour des opérateurs H appropriés (différentiels elliptiques sur une variété compacte, nombre fini de valeurs propres négatives), on définit la fonction zeta par

$$\boxed{\zeta_H(s) \equiv \mathrm{Tr}H^{-s}} = \sum_{\lambda_\kappa\neq 0}\lambda_\kappa^{-s}, \quad (5.207)$$

avec λ_k^{-s} remplacé par $\mathrm{sgn}\,(\lambda_\kappa)|\lambda_\kappa|^{-s}$ pour $\lambda_\kappa < 0$.

En particulier,

$$-\zeta'_H(0) = \sum_{\lambda_\kappa\neq 0}\ln\lambda_k e^{-s\ln\lambda_\kappa}|_{s=0} = \sum_{\lambda_\kappa\neq 0}\ln\lambda_k \equiv \ln\det{}'H, \quad (5.208)$$

où la notation $\det'$ dénote le déterminant après avoir enlevé de l'espace les vecteurs propres de valeur propre zéro.

On va montrer que $\zeta_H(s)$ *converge et est holomorphe pour* $Re(s) > d/2$ *et admet une continuation analytique en une fonction méromorphe dans* $\mathbb{C}$ *avec au plus des pôles simples en* $s = d/2, d/2 - 1, \ldots, d/2 - [\frac{d-1}{2}]$.

Puisque $\{\lambda_\kappa, \lambda_\kappa < 0\}$ est un ensemble fini, $\sum_{\kappa,\lambda_\kappa<0}\mathrm{sgn}\,(\lambda_\kappa)|\lambda_\kappa|^{-s}$ existe pour tout $s \in \mathbb{C}$, on peut se limiter au cas où les $\lambda_\kappa \geqslant 0$. Comme le spectre λ_κ n'est en général pas connu de manière explicite, on établit la représentation suivante:

$$\boxed{\zeta_H(s)} = \frac{1}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}(\sum_\kappa e^{-\lambda_\kappa\tau} - \dim\mathrm{Ker}\,H) \boxed{= \frac{1}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}\mathrm{Tr}(e^{-\tau H} - P)}, \quad (5.209)$$

où P est la projection orthogonale sur $\mathrm{Ker}\, H$.

En effet, à partir de

$$\Gamma(s) = \int_0^\infty d\tau\, \tau^{s-1} e^{-\tau},$$

on trouve par changement de variable $\tau \to \lambda\tau$ que

$$\Gamma(s) = \lambda^s \int_0^\infty d\tau\, \tau^{s-1} e^{-\lambda\tau}, \quad Re(\lambda), Re(s) > 0,$$

et puis on utilise (5.207) pour conclure.

En particulier, on trouve la relation suivante entre la fonction zeta et le noyau chaleur,

$$\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} \Big[\int d^d x\, K_H(x, x; \tau) - \dim \mathrm{Ker}\, H \Big]. \tag{5.210}$$

En séparant l'intégrale en 2 morceaux,

$$\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^1 d\tau\, \tau^{s-1} \mathrm{Tr}(e^{-\tau H} - P) + \frac{1}{\Gamma(s)} \int_1^\infty d\tau\, \tau^{s-1} \mathrm{Tr}(e^{-\tau H} - P), \tag{5.211}$$

la deuxième intégrale est de la forme $(\Gamma(s))^{-1} \int_1^\infty d\tau\, \tau^{s-1} O(e^{-\widetilde{\lambda}\tau})$, avec $\widetilde{\lambda}$ la première valeur propre strictement positive. Elle converge et définit une fonction holomorphe de s car $\Gamma(s)$ ne s'annule jamais. De plus comme, $\frac{1}{\Gamma(s)} = s + O(s^2)$, la deuxième partie donne 0 pour $s = 0$.

Pour la première intégrale on peut utiliser que

$$K_H(x, x; \tau) = \frac{1}{(4\pi)^{d/2}} \sum_n a_n(x, x) \tau^{n-d/2}, \quad \boxed{A_k = \int d^d x \frac{a_k(x, x)}{(4\pi)^{d/2}}} \tag{5.212}$$

et si $N > d/2$,

$$\frac{1}{\Gamma(s)} \int_0^\epsilon d\tau\, \tau^{s-1} \Big(\sum_{k=0}^N A_k \tau^{k-d/2} + O(\tau^{N+1-d/2}) - \dim \mathrm{Ker}\, H \Big) = \tag{5.213}$$

$$= \boxed{\frac{1}{\Gamma(s)} \Big(\sum_{k=0}^N \frac{A_k}{s + k - d/2} - \frac{\dim \mathrm{Ker}\, H}{s}\Big)} + R(s), \tag{5.214}$$

avec $R(s)$ une fonction bornée de s. Si $\mathrm{Re}(s) > d/2$ cette expression converge. Les pôles se trouvent en $s = d/2, d/2 - 1 \ldots, d/2 - [\frac{d-1}{2}]$. Pour $s = 0$, le pôle de la somme se simplifie avec le zéro de $\frac{1}{\Gamma(s)}$ et on trouve

$$\zeta_H(0) = \begin{cases} \int d^d x \frac{a_{d/2}(x,x)}{(4\pi)^{d/2}} - \dim \mathrm{Ker}\, H \text{ si d est pair} \\ -\dim \mathrm{Ker}\, H \text{ si d est impair} \end{cases}. \tag{5.215}$$

Pour

$$Z_H = \int \mathcal{D}\phi \exp -\frac{1}{2} \int d^d x\, \phi(x) H \phi(x), \tag{5.216}$$

on a $\phi(x) = a^\kappa \varphi_\kappa(x)$. Faisons l'hypothèse qu'il n'y a pas de mode zéro. Comme on effectue une transformation orthonormée de Jacobien unité $\mathcal{D}\phi = \prod_\kappa da^\kappa$. L'intégrale de chemin devient

$$Z_H = \prod_\kappa \int da^\kappa e^{-\frac{\lambda_\kappa (a^\kappa)^2}{2}} = \prod_\kappa \sqrt{\frac{2\pi}{\lambda_k}} = \Big(\det(\frac{H}{2\pi})\Big)^{-\frac{1}{2}}. \tag{5.217}$$

On a donc aussi

$$\ln Z_H = \frac{1}{2}\zeta'_{H/2\pi}(0) \tag{5.218}$$

Notons encore que sous une transformations d'échelle, $H \to H/\mu^2$, où μ a la dimension d'une masse si on veut que H' soit sans dimensions, on a

$$\ln Z_{H/2\pi\mu^2} = \frac{1}{2}\zeta'_H(0) + \frac{1}{2}\ln(2\pi\mu^2)\zeta_H(0), \tag{5.219}$$

puisque $\zeta_{H/k^2}(s) = k^{2s}\zeta_H(s)$ et donc $\zeta'_{H/k^2}(s) = k^{2s}\zeta'_H(s) + 2\ln k\,\zeta_H(s)$. En fait, il faudrait comprendre tous les déterminants que l'on veut définir par la fonction ζ dans ce sens car seul des quantités sans dimensions peuvent être continuées analytiquement.

5.7.5 Partition function of massless scalar field

Pour le champ scalaire libre sans masse à 4 dimensions,

$$S[\phi] = -\frac{1}{2}\int d^4x\,\partial_\mu\phi\partial^\mu\phi, \tag{5.220}$$

qui satisfait des conditions aux bords spatiales périodiques dans une boîte rectangulaire de dimensions L^i, on a $H = -\Delta$, $x^4 = \tau$

$$\varphi_\kappa(x) = e^{i\kappa_a x^a}, \quad \kappa^a = \frac{2\pi n^a}{L^{(a)}}, \quad n^a \in \mathbb{Z}, \tag{5.221}$$

avec $L^4 = \beta$. Ceci implique que $\lambda_\kappa = \kappa_a\kappa^a = (\frac{2\pi n_4}{\beta})^2 + k^2$ à la limite continue pour une boîte (spatiale) grande, la densité de valeurs propres pour $n \equiv n^4 \neq 0$ est $\frac{2V}{(2\pi)^3}\int d^3k$ et $\frac{V}{(2\pi)^3}\int d^3k$ pour $n = 0$ en tenant compte de la dégénérescence des valeurs propres. La fonction zeta vaut donc

$$\zeta_{-\Delta}(s) = \frac{4\pi V}{(2\pi)^3}\Big[\int_0^\infty dk\,k^{2-2s} + 2\sum_{n>0}\int_0^\infty dk\,k^2(4\pi^2\beta^{-2}n^2 + k^2)^{-s}\Big]. \tag{5.222}$$

La première intégrale diverge pour k petit et $Re(s) \geqslant 3/2$. Cette divergence infrarouge est régularisée en mettant une borne inférieure à l'intégrale justifiée par le fait que la boîte est grande mais finie. On a alors $\int_\epsilon^\infty dk\,k^{2-2s} = -(3-2s)^{-1}\epsilon^{3-2s}$.

On trouve[2]

$$\boxed{\zeta_{-\Delta}(s) = -\frac{4\pi V}{(2\pi)^3}(3-2s)^{-1}\epsilon^{3-2s}} - \frac{8\pi V}{(2\pi)^3}(2-2s)^{-1}(2\pi\beta^{-1})^{3-2s}\zeta_R(2s-3)2^{2s-4}\frac{\Gamma(s-\frac{1}{2})\Gamma(s-\frac{3}{2})}{\Gamma(2s-2)}.$$

[2] Il y a une coquille dans le calcul de l'intégral de [46] eq. (3.7).

En effet, le deuxième terme est intégré par parties et donne

$$-\frac{8\pi V}{(2\pi)^3}\sum_{n>0}\int_0^\infty dk\,(4\pi^2\beta^{-2}n^2+k^2)^{-s+1}(2-2s)^{-1}.$$

En effectuant le changement de variables $k=2\pi n\beta^{-1}\sinh y$, on trouve

$$-\frac{8\pi V}{(2\pi)^3}(2-2s)^{-1}\sum_{n>0}(2\pi\beta^{-1}n)^{-2s+3}\int_0^\infty dy\,(\cosh y)^{3-2s}.$$

Dans Gradshteyn, formule 3.512, on trouve

$$I(\beta,\nu,a)=\int_0^\infty dx\,\frac{\cosh 2\beta x}{(\cosh ax)^2\nu}=\frac{4^{\nu-1}}{a}B(\nu+\frac{\beta}{a},\nu-\frac{\beta}{a}),\quad \mathrm{Re}\,(\nu\pm\beta>0,a>0,\beta>0);\quad B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

ce qui donne

$$\int_0^\infty dy\,(\cosh y)^{3-2s}=\int_0^\infty dy\,\frac{\cosh y}{(\cosh y)^{2(s-1)}}=I(\frac{1}{2},s-1,1)=2^{2s-4}\frac{\Gamma(s-\frac{1}{2})\Gamma(s-\frac{3}{2})}{\Gamma(2s-2)}.$$

Comme $[\Gamma(2s-2)]^{-1}=4s+O(s^2)$, on trouve $\zeta_{-\Delta}(0)=0$ ce qui implique que la fonction de partition ne dépend pas de l'échelle μ. De plus

$$\zeta'_{-\Delta}(0)=-\pi V\beta^{-3}\zeta_R(-3)\Gamma(-\frac{1}{2})\Gamma(-\frac{3}{2})=\frac{\pi^2}{45}V\beta^{-3}.$$

On trouve donc pour la fonction de partition

$$\ln Z_{-\Delta}=\frac{\pi^2}{90}V\beta^{-3}. \tag{5.223}$$

ce qui donne comme énergie, entropie et pression de radiation

$$E=-\frac{\partial\ln Z_{-\Delta}}{\partial\beta}=\frac{\pi^2}{30}V\beta^{-4},\quad S=\beta E+\ln Z_{-\Delta}=\frac{2\pi^2}{45}V\beta^{-3}, \tag{5.224}$$

$$P=\beta^{-1}\frac{\partial\ln Z_{-\Delta}}{\partial V}=\frac{\pi^2}{90}\beta^{-4}. \tag{5.225}$$

do the discussion in any dimension as in the canonical part, comment on independence of boundary conditions in this way of computing, add the discussion of the Casimir effect as in Hawking's CMP

5.7.6 Exercises

5.7.6.1 Laplacian on the circle

Trouver les vecteurs et valeurs propres du Laplacien Δ sur le cercle. Trouver le noyau chaleur et montrer que

$$\lim_{t\to 0}\frac{\sum_{n\in\mathbb{Z}}e^{-n^2t}-\sqrt{\frac{\pi}{t}}}{t^m}=0,\quad m>0, \tag{5.226}$$

en comparant les noyaux chaleurs du Laplacien sur $\mathbb{R}$ et S^1. Montrer pour le cercle que $\zeta_{-\Delta}(s)=\zeta_R(2s)$ où $\zeta_R(s)$ est la fonction zeta de Riemann (cf. [48] section 1.1).

5.7.7 High temperature expansion of partition function for massive scalar field

to be done, compare Laine section 2.3 with e.g. Zinn-Justin

5.7.8 Bose-Einstein condensation

5.7.9 Scalar field theory and Ising model

Chapter 6

Classical gauge fields

cf. [49]

6.1 Group theory background

Soit $\{t_a\}, a = 1, \dots, d$ une base d'une algèbre de Lie $\mathfrak{g}$ à d dimensions et f_{ab}^c les constantes de structure, $[t_a, t_b] = f_{ab}^c t_c$. Les matrices $(t_a^A)_b^c = f_{ab}^c$ forment une représentation à d dimensions de $\mathfrak{g}$, appelé représentation adjointe. La métrique de Killing est définie par $g_{ab}^K = \mathrm{Tr}(t_a^A t_b^A) = f_{ac}^d f_{bd}^c$.

D'une manière invariante, la représentation adjointe est définie par $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$, $x \mapsto \mathrm{ad}_x$, $\mathrm{ad}_x y = [x, y]$ et la métrique de Killing par $g^K(x, y) = \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$.

Pour une somme direct $\oplus_n \mathfrak{g}_n$ d'algèbres de Lie, on peut choisir une base $\{t_{n\alpha}\}$, avec $\{t_{n\alpha}\}$ pour n fixé une base de $\mathfrak{g}_n$ tel que $[t_{n\alpha}, t_{m\beta}] = f_{n\alpha m\beta}^{l\gamma} t_{l\gamma}$ avec $f_{n\alpha m\beta}^{l\gamma} = \delta_m^l \delta_{mn} f_{\alpha\beta}^{(n)\gamma}$.

- $\mathfrak{g}$ est abélienne si $[\mathfrak{g}, \mathfrak{g}] = 0$
- $\mathfrak{i} \subset \mathfrak{g}$ est un idéal de $\mathfrak{g}$ si $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{g}$
- $\mathfrak{g}$ est simple si elle est non abélienne et n'admet pas d'idéal non trivial (distinct de $0, \mathfrak{g}$)
- une représentation $\rho : \mathfrak{g} \to \mathrm{End}V$ est réductible s'il existe un sous-espace $W \subset V$ non trivial et invariant: W distinct de $0, W$ et $\rho(\mathfrak{g})W \subset W$
- une représentation $\rho : \mathfrak{g} \to \mathrm{End}V$ est complètement réductible si pour tout $W \subset V$ invariant, il existe W' invariant et supplémentaire: $W = W \oplus W'$ avec $\rho(\mathfrak{g})W' \subset W'$
- une représentation $\rho : \mathfrak{g} \to \mathrm{End}V$ est fidèle si ρ est injectif
- la forme de trace de ρ est définie par $\mathrm{Tr}\rho(e_a)\rho(e_b) = k_{ab}$. elle est invariante, $f_{ad}^c k_{cb} + f_{bd}^c k_{ac} = 0$
- dans une algèbre de Lie fini-dimensionnelle simple, tout tenseur symétrique invariant est proportionnelle à la métrique de Killing, et donc $\mathrm{Tr}\rho(t_a)\rho(t_b) = \lambda g_{ab}^K$ avec $\lambda \in \mathbb{R}$ ou $\mathbb{C}$
- une algèbre de Lie fini-dimensionnelle semi-simple $\mathfrak{g}$ est définie par les conditions équivalentes

  1. la métrique de Killing est non-dégénérée
  2. $\mathfrak{g}$ est somme directe d'idéaux simples
  3. $\mathfrak{g}$ n'admet pas d'idéal abélien ($\neq 0$)
  4. toute représentation fini-dimensionnelle de $\mathfrak{g}$ est complètement réductible

- la complexification $\widetilde{\mathfrak{g}}$ d'une algèbre de Lie $\mathfrak{g}$ sur $\mathbb{R}$ est sem-simple ssi $\mathfrak{g}$ est semi-simple
- si $\widetilde{\mathfrak{g}}$ est simple alors $\mathfrak{g}$ est simple, contre-exemple pour la proposition inverse: $\mathfrak{so}(3,1)$
- une algèbre de Lie fini-dimensionnelle réductive $\mathfrak{g}$ est définie par les conditions équivalentes
  1. $\mathfrak{g} = Z_\mathfrak{g} \oplus \mathfrak{g}'$ avec $\mathfrak{g}'$ semi-simple. $Z_\mathfrak{g}$ est l'idéal abélien défini par $[Z_\mathfrak{g}, \mathfrak{g}] = 0$ et $\mathfrak{g}'$ est l'idéal de $\mathfrak{g}$ défini par $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$
  2. la représentation adjointe de $\mathfrak{g}$ est complètement réductible
  3. $\mathfrak{g}$ admet une rerésentation fini-dimensionnelle fidèle avec une forme de trace non dégénérée
  4. $\mathfrak{g}$ admet une rerésentation fini-dimensionnelle fidèle complètement réductible
- une représentation ρ d'une algèbre de Lie réductive $\mathfrak{g}$ est complètement réductible ssi les transformations $\rho(Z_\mathfrak{g})$ sont réductibles
- une algèbre de Lie fini-dimensionnelle $\mathfrak{g}$ sur $\mathbb{R}$ compacte est définie par les conditions équivalentes
  1. $\mathfrak{g}$ admet un produit scalaire $\langle \cdot, \cdot \rangle$ invariant défini négatif, c-à-d les valeurs propres de $g_{ab} = \langle t_a, t_b \rangle$ sont strictement négatives
  2. $\mathfrak{g}$ est somme direct d'un idéal semi-simple compact avec $Z_\mathfrak{g}$; $\mathfrak{g}$ est donc en particulier réductive
  3. il existe une base dans laquelle les constantes de structure sont complètement antisymétriques

Pour $\mathfrak{g}$ compacte, il existe une base adaptée à la décomposition en somme directe avec bases $t_{n\alpha}$ tel que $g_{n\alpha m\beta} = -\frac{1}{g_n^2}\delta_{nm}\delta_{\alpha\beta}$ avec $0 < g_n^2 \in \mathbb{R}$. En utilisant l'invariance, les constantes de structure sont en effet complètement anti-symétriques dans cette base.

6.2 Global symmetries of matter fields

Soit $L_M(y^i, \partial_\mu y^i)$ le Lagrangien pour les champs de matière. Par exemple, $y^i = \phi^a, (\bar{\phi}^a)$, des champs scalaires réels ou complexes, $y^i = \psi^a, \bar{\psi}^a, \xi^\alpha, \bar{\xi}^{\dot{\alpha}}$, des fermions de Dirac ou de Weyl.

On fait l'hypothèse que L_M est invariant sous les transformations

$$\delta_k y^i = -k^a T_a{}^i{}_j y^j \tag{6.1}$$

avec $k = k^a e_a$, k^a des paramètres constants, e_a les générateurs d'une algèbre de Lie $\mathfrak{g}$ sur $\mathbb{R}$ et $T_a{}^i{}_j$ les générateurs d'une représentation matricielle,

$$[e_a, e_b] = f^c_{ab} e_c, \quad f^d_{[ab} f^e_{c]d} = 0, \quad [\delta_{k_1}, \delta_{k_2}] y^i = \delta_{[k_1,k_2]} y^i. \tag{6.2}$$

Cette condition d'invariance s'exprime explicitement par

$$0 = \delta_k L_M = \delta_k y^i \frac{\partial L_M}{\partial y^i} + \partial_\mu \delta_k y^i \frac{\partial L_M}{\partial \partial_\mu y^i} = \delta_k y^i \frac{\delta L_M}{\delta y^i} + \partial_\mu \big(\delta_k y^i \frac{\partial L_M}{\partial \partial_\mu y^i}\big). \tag{6.3}$$

D'après le 1er théorème de Noether, les courants conservés sont

$$j^\mu_a(y, \partial y) = -\delta_{e_a} y^i \frac{\partial L_M}{\partial \partial_\mu y^i} = T_a{}^i{}_j y^j \frac{\partial L_M}{\partial \partial_\mu y^i}, \quad \partial_\mu j^\mu_a \approx 0. \tag{6.4}$$

Par exemple, le Lagrangien

$$L_M(\psi^l,\bar{\psi}^l,\partial_\mu\psi^l) = -\sum_{l=1}^{N}\bar{\psi}^l(\partial\!\!\!/ + m)\psi^l \tag{6.5}$$

est invariant sous les transformations $\psi'^l = (U^{-1}\psi)^l$ avec U une matrice unitaire agissant sur les différents fermions de Dirac,

$$L_M(\psi'^l,\bar{\psi}'^l,\partial_\mu\psi'^l) = L_M(\psi^l,\bar{\psi}^l,\partial_\mu\psi^l). \tag{6.6}$$

Si $U = \mathbf{1}+k^aT_a+O(k^2)$, avec T_a des générateurs des matrices antihermitiennes, on a $\delta_k\psi^l = -k^aT_a{}^l{}_m\psi^m$ et l'invariance sous forme finie implique l'invariance sous forme infinitésimale, $\delta_kL_M = 0$.

6.3 Gauge principle and covariant derivative

La question qu'on se pose alors est : Comment faut-il modifier la théorie pour avoir invariance sous les transformations locales ? On voudrait donc que $k^a \to \epsilon^a(x)$, avec $\epsilon^a(x)$ des fonctions arbitraires sur l'espace-temps.

On a

$$\delta_\epsilon L_M = -\epsilon^a\big(T_a{}^i{}_jy^j\frac{\partial L_M}{\partial y^i} + T_a{}^i{}_j\partial_\mu y^j\frac{\partial L_M}{\partial\partial_\mu y^i}\big) - \partial_\mu\epsilon^aT_a{}^i{}_jy^j\frac{\partial L_M}{\partial\partial_\mu y^i} = -\partial_\mu\epsilon^aj^\mu_a, \tag{6.7}$$

les 2 premiers termes s'annulent en vertu de l'invariance sous transformations globales.

La réponse consiste à introduire de nouveaux champs, les potentiels de jauge $A^a_\mu(x)$, et de définir les dérivées covariantes des champs de matière par

$$D_\mu y^i = \partial_\mu y^i + A^a_\mu T_a{}^i{}_jy^j. \tag{6.8}$$

La loi de transformation des $A^a_\mu(x)$ est alors choisie de manière à ce que le Lagrangien $L_M(y^i,D_\mu y^i)$ soit invariant. C'est le cas si $\delta_\epsilon D_\mu y^i = -\epsilon^aT_a{}^i{}_jy^jD_\mu y^j$.

En effet,

$$\delta_\epsilon L^M(y,D_\mu y) = -\epsilon^aT_a{}^i{}_jy^j\frac{\partial L^M}{\partial y^i}(y,Dy) - \epsilon^aT_a{}^i{}_j\partial_\mu y^j\frac{\partial L^M}{\partial\partial_\mu y^i}(y,Dy) = (\delta_kL_M)|_{k\to\epsilon}(y,Dy) = 0.$$

Ceci fixe la transformations de potentiels de jauge,

$$\delta_\epsilon A^a_\mu = \partial_\mu\epsilon^a + f^a_{bc}A^b_\mu\epsilon^c. \tag{6.9}$$

En effet, on veut que

$$\delta_\epsilon D_\mu y^i = -\epsilon^aT_a{}^i{}_j\partial_\mu y^j - \partial_\mu\epsilon^aT_a{}^i{}_jy^j + \delta_\epsilon A^a_\mu T_a{}^i{}_jy^j - A^a_\mu T_a{}^i{}_j\epsilon^bT_b{}^j{}_ky^k =$$
$$= -\epsilon^aT_a{}^i{}_j(\partial_\mu y^i + A^b_\mu T_b{}^j{}_ky^k).$$

Les termes en $\partial_\mu y^j$ s'annulent et en bougeant le terme en T^2 dans le membre de droite on voit apparaître le commutateur des matrices, ce qui donne le résultat.

6.4 Finite gauge transformations

Supposons le Lagrangien invariant sous une transformation globale $y'^i = (g^{-1})^i{}_jy^j$, $L(y'\partial_\mu y') = L(y,\partial_\mu y)$ avec $g^i{}_j$ la représentation matricielle d'un groupe de Lie G. Si cette transformation devient locale, $(g^{-1})^i{}_j \to (g^{-1})^i{}_j(x)$, la dérivée covariante définie par

$$D_\mu y^i = \partial_\mu y^i + A^a_\mu T^i_{aj}y^j, \tag{6.10}$$

où T^i_{aj} est une représentation matricielle de l'algèbre de Lie $\mathfrak{g}$ associée à G, se transforme comme y^i,

$$(D_\mu y^i)' = (g^{-1})^i{}_j(x) D_\mu y^i, \tag{6.11}$$

et laisse donc le Lagrangien $L(y, D_\mu y)$ invariant ssi $A_\mu = A^a_\mu T^i_{aj}$ se transforme comme

$$A'_\mu = g^{-1}\partial_\mu g + g^{-1} A_\mu g. \tag{6.12}$$

En effet

$$(D_\mu y)' = \partial_\mu y' + A'_\mu y' = \partial_\mu(g^{-1}y) + (g^{-1}\partial_\mu g + g^{-1}A_\mu g)g^{-1}y = g^{-1}D_\mu y.$$

De plus, si $g^i{}_j = \delta^i_j + \epsilon^a T^i_{aj}$, on retrouve les transformations infinitésimales.

6.5 Yang-Mills fields

Pour simplifier les calculs, on introduit une notation condensée: $y = y^i e_i$ dénote un vecteur colonne, $A_\mu = A^a_\mu \delta_a$, $\epsilon = \epsilon^a \delta_a$, avec δ_a le générateur d'une représentation fidèle (matricielle ou abstraite) de $\mathfrak{g}$, qui n'est pas fixée à priori, mais dépend de l'objet sur lequel A_μ agit.

Dans ce cas, la dérivée covariante s'écrit comme $D_\mu = \partial_\mu + A_\mu\cdot$, par exemple $D_\mu y = \partial_\mu y + A^a_\mu T_a y$ et une transformation de jauge infinitésimale comme $\delta_\epsilon = -\epsilon\cdot$, par exemple $\delta_\epsilon y = -\epsilon^a T_a y$ et aussi $\delta_\epsilon D_\mu y = -\epsilon \cdot D_\mu y$ où $D_\mu y$ se transforme dans la même représentation que y.

Dans cette notation, on peut écrire (6.9) comme

$$\delta_\epsilon A_\mu = (\delta_\epsilon A^a_\mu)\delta_a = \partial_\mu \epsilon + [A_\mu, \epsilon] = D_\mu \epsilon \tag{6.13}$$

en sous-entendant que ϵ se transforme dans la représentation adjointe.

Pour le commutateur de 2 dérivées covariantes, on trouve

$$[D_\mu, D_\nu]y = (\partial_\mu + A_\mu\cdot)(\partial_\nu + A_\nu)y - (\mu \leftrightarrow \nu) = (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])y = F_{\mu\nu}\cdot y, \tag{6.14}$$

où le champ de Yang-Mills est

$$F_{\mu\nu} = F^a_{\mu\nu}\delta_a, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu. \tag{6.15}$$

Pour établir la loi de transformations des champs de Yang-Mills, on utilise d'un côté que $\delta_\epsilon[D_\mu, D_\nu]y = -\epsilon\cdot[D_\mu, D_\nu]y$ et de l'autre côté que $\delta_\epsilon(F_{\mu\nu}\cdot y) = (\delta_\epsilon F_{\mu\nu})\cdot y + F_{\mu\nu}\cdot(-\epsilon\cdot y)$, ce qui donne $\delta_\epsilon F_{\mu\nu}\cdot y = -[\epsilon, F_{\mu\nu}]\cdot y$, ce qui donne

$$\delta_\epsilon F_{\mu\nu} = -[\epsilon, F_{\mu\nu}] = -\epsilon\cdot F_{\mu\nu}, \quad \delta_\epsilon F^a_{\mu\nu} = -f^a_{bc}\epsilon^b F^c_{\mu\nu}, \tag{6.16}$$

en sous-entendant que $F_{\mu\nu}$ se transforme dans la représentation adjointe.

Les identités de Bianchi sont des relations entre dérivées covariantes des courbures qui sétablissent à partir des identités $[D_\mu, [D_\nu, D_\rho]]y + (\text{cyclique})(\mu, \nu, \rho) = 0$ pour donner

$$D_\mu F_{\nu\rho} + \text{cyclique } (\mu, \nu, \rho) = 0, \quad \partial_\mu F^a_{\nu\rho} + f^a_{bc} A^b_\mu F^c_{\nu\rho} + \text{cyclique } (\mu, \nu, \rho) = 0. \tag{6.17}$$

Notons encore que (6.16) et (6.17) peuvent aussi directement se démontrer en composantes.

6.6 Yang-Mills Lagrangian

Une conséquence de la construction d'un Lagrangien invariant sous des symétries locales est l'introduction d'interactions:

$$L_M(y,Dy) = L_M(y,\partial y) + L_I(y,\partial y,A), \quad \frac{\partial L_M(y,Dy)}{\partial A^a_\mu} = T^i_{aj} y^j \frac{\partial L_M}{\partial \partial_\mu y}(y,Dy) = j^\mu_a(y,Dy), \tag{6.18}$$

ou encore, $L_M(y,Dy) = L_M(y,\partial y) + A^a_\mu j^\mu_a(y,\partial y) + O(A^2)$. Dans le Lagrangian obtenu par substitution minimale $\partial y \to Dy$, le couplage au potentiel de jauge A^a_μ se fait au courant de Noether de la symétrie globale de départ au premier ordre en les potentiels.

On introduit également un terme cinétique pour les potentiels de jauge. Exigeant que ce terme soit invariant de jauge, on trouve qu'il ne peut dépendre que des courbures et de leurs dérivées covariantes.

En effet, on peut considérer un changement de variables triangulaire entre les potentiels de jauge et leurs dérivées $A^a_\mu, \partial_\nu A^a_\mu, \partial_{\nu_1}\partial_{\nu_2} A^a_\mu, \dots$ et $A^a_\mu, \partial_{(\mu} A^a_{\nu)}, F^a_{\mu\nu}, \partial_{(\nu_1}\partial_{\nu_2} A^a_{\mu)}, D_{(\rho} F^a_{\mu)\nu}, \dots$ où les parenthèses indiquent une symétrisation des indices.

On a alors $\delta_\epsilon A^a_\mu = D_\mu \epsilon^a$, $\delta_\epsilon \partial_{(\mu} A^a_{\nu)} = \partial_{(\mu} D_{\nu)} \epsilon^a$, $\delta_\epsilon \partial_{(\nu_1}\partial_{\nu_2} A^a_{\mu)} = \partial_{(\nu_1}\partial_{\nu_2} D_{\mu)} \epsilon^a$. Le résultat suit car on peut choisir les paramètres de jauge ϵ^a et leurs dérivées de manière indépendante en un point, il en est de même pour $D_\mu \epsilon^a, \partial_{(\nu_1}\partial_{\nu_2} D_{\mu)} \epsilon^a, \dots$ (par un autre changement de variables triangulaire), donc un Lagrangien invariant de jauge ne peut pas dépendre de $A^a_\mu, \partial_{(\mu} A^a_{\nu)}, \partial_{(\nu_1}\partial_{\nu_2} A^a_{\mu)}, \dots$. Notons encore que les transformations de jauge sur les variables restantes sont données par $\delta_\epsilon F^a_{\mu\nu} = -f^a_{bc}\epsilon^b F^a_{\mu\nu}$, $\delta_\epsilon D_{(\rho} F^a_{\mu)\nu} = -f^a_{bc}\epsilon^b D_{(\rho} F^a_{\mu)\nu}, \dots$.

Le Lagrangien le plus général donnant un terme cinétique quadratique et invariant de Lorentz ($L^\uparrow_+$) à 4 dimensions est

$$L_{\rm YM} = m^1_{ab} F^a_{\mu\nu} F^{b\mu\nu} + m^2_{ab} F^a_{\mu\nu} F^b_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}. \tag{6.19}$$

L'invariance de jauge $\delta_\epsilon L_{\rm YM} = 0$ exige alors que

$$f^d_{ca} m^i_{db} + f^d_{cb} m^i_{ad} = 0, \quad i = 1,2. \tag{6.20}$$

Le deuxième terme peut être négligé car c'est une dérivée totale,

$$m^2_{ab} F^a_{\mu\nu} F^b_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = 4 m^2_{ab} \partial_\mu \left(\epsilon^{\mu\nu\rho\sigma} A^a_\nu (\partial_\rho A^b_\sigma + \frac{1}{3} f^b_{cd} A^c_\rho A^d_\sigma) \right), \tag{6.21}$$

qui n'affecte pas les équations d'Euler-Lagrange. On peut également montrer qu'il n'affecte pas la théorie quantique au niveau perturbatif.

On veut que chaque A^a_μ ait un terme cinétique, donc on exige que m^1_{ab} soit non dégénéré. De plus on exige que ce tenseur symétrique invariant soit défini négatif pour que l'Hamiltonien soit borné inférieurement. Ceci implique que $\mathfrak{g}$ est une algèbre de Lie compacte. Dans une base appropriée $\{t_{m\alpha}\}$, on trouve donc que $m^1_{ab} = g_{m\alpha n\beta} = -\frac{1}{g_m^2}\delta_{mn}\delta_{\alpha\beta}$ avec m constantes de couplages différentes g_m associées aux différents facteurs simples et abéliens.

Dans la base $\{t_{m\alpha}\}$, le Lagrangien de Yang-Mills est donné par

$$L_{\rm YM} = \frac{1}{4} {\rm Tr} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{4 g_m^2} F^{m\alpha}_{\mu\nu} F^{\mu\nu}_{m\alpha}, \tag{6.22}$$

en choisissant les générateurs $t_{m\alpha}$ tel que ${\rm Tr}(t_{m\alpha} t_{n\beta}) = -\frac{1}{g_m^2}\delta_{mn}\delta_{\alpha\beta}$.

La redéfinition $A^{m\alpha}_\mu \to g_m A^{m\alpha}_\mu$ permet d'absorber la constante de couplage devant le terme quadratique. Elle apparaît alors devant le terme cubique et quartique. De manière équivalente, on peut prendre $g_{m\alpha n\beta} = -\delta_{mn}\delta_{\alpha\beta}$ si en même temps on prend $f^{(m)\alpha}_{\beta\gamma} \to g_m f^{(m)\alpha}_{\beta\gamma}$ dans le Lagrangian de Yang-Mills et $t_{m\alpha} \to g_m t_{m\alpha}$ dans le Lagrangien des champs de matière.

6.7 Conserved currents for coupled equations

Pour un facteur simple ou abélien, et $L[A,y] = L_{\mathrm{YM}}[A] + L_M[y, Dy]$, en utilisant (6.18) on a

$$\frac{\delta L}{\delta A^\alpha_\mu} = 0 \iff D_\nu F^{\nu\mu}_\alpha + g j^\mu_\alpha(y, Dy) = 0, \tag{6.23}$$

où $F^{\nu\mu}_\alpha$ se transforme dans la représentation co-adjointe. Comme les constantes de structures sont complèment antisymétrique dans cette base,

$$D_\mu D_\nu F^{\nu\mu}_\alpha = \frac{1}{2}[D_\mu, D_\nu] F^{\nu\mu}_\alpha = F^\gamma_{\mu\nu}\delta_\gamma F^{\nu\mu}_\alpha = F^\gamma_{\mu\nu} f^\beta_{\gamma\alpha} F^{\nu\mu}_\beta = 0, \tag{6.24}$$

ce qui implique

$$D_\mu j^\mu_\alpha(y, Dy) \approx 0, \tag{6.25}$$

où $\approx$ veut dire “quand les équations du mouvement sont satisfaites”, c'est-à-dire quand $\frac{\delta L}{\delta A^\alpha_\mu} = 0$. En contractant les équations du mouvement sous la forme $\partial_\nu F^{\nu\mu}_\alpha - g A^\beta_\nu f^\gamma_{\beta\alpha} F^{\nu\mu}_\gamma + g j^\mu_\alpha(y, Dy) = 0$ avec ∂_μ et en définissant $J^\mu_\alpha = j^\mu_\alpha(y, Dy) - A^\beta_\nu f^\gamma_{\beta\alpha} F^{\nu\mu}_\gamma$, on trouve aussi

$$\partial_\mu J^\mu_\alpha \approx 0. \tag{6.26}$$

Le champ de Yang-Mills est sa propre source si l'algèbre de Lie est non-abélienne. Même en l'absence de champs de matière, il y a des interactions et les équations sont non-linéaires.

Notons encore que les équations des champs de matière $\frac{\delta L}{\delta y^i} = \frac{\delta L^M}{\delta y^i}(y, Dy) = 0$ sont obtenues à partir des équations libres en effectuant la substitution mimimale $\partial_\mu y \to D_\mu y$ et que $J^\mu_\alpha = \frac{1}{g}(\frac{\delta L}{\delta A^\alpha_\mu} - \partial_\nu F^{\nu\mu}_\alpha)$ est un courant trivial dans le sens où la charge de Noether associée s'annule sur toute solution si les champs décroissent à l'infini spatial.

6.8 Exercises

6.8.1 Adjoint and coadjoint representation

Montrer que les matrices $(t^A_a)^b{}_c = f^b_{ac}$, $a = 1, \dots, d$ forment une représentation matricielle $d \times d$ de l'algèbre de Lie $\mathfrak{g}$: $[t^A_a, t^A_b] = f^c_{ab} t^A_c$. Cette représentation s'appelle la represénation adjointe.

Même question pour la représentation co-adjointe définie par les matrices $(t^C_a)_b{}^c = -f^c_{ab}$.

Montrer que si v^b se transforme dans la représentation adjointe, $\delta_a v^b = -(t^A)^b{}_c v^c$, alors $v_c = g_{cb} v^b$ avec g_{ab} invariant, se transforme dans la représentation co-adjointe, $\delta_a v_c = -(t^C_a)_c{}^b v_b$.

6.8.2 Gauge covariant variables

Montrer explicitement que si $A^a_\mu, \partial_\nu A^a_\mu, \partial_{\nu_1}\partial_{\nu_2} A^a_\mu$ sont des variables indépendantes, alors A^a_μ, $\partial_{(\mu} A^a_{\nu)}$, $F^a_{\mu\nu}$, $\partial_{(\nu_1}\partial_{\nu_2} A^a_{\mu)}$, $D_{(\rho} F^a_{\mu)\nu}$ le sont aussi.

Même question pour $\epsilon^a, \partial_\mu \epsilon^a, \partial_{\mu_1}\partial_{\mu_2}\epsilon^a, \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\epsilon^a$ et $\epsilon^a, D_\mu\epsilon^a, \partial_{(\mu_1} D_{\mu_2)}\epsilon^a, \partial_{(\mu_1}\partial_{\mu_3} D_{\mu_3)}\epsilon^a$.

6.8.3 Lagrangien de Chern-Simons

Montrer que le Lagrangien de Chern-Simons à 3 dimensions

$$L_{CS} = \epsilon^{\mu\nu\rho} g_{ab} A^a_\mu (\partial_\nu A^b_\rho + \frac{1}{3} f^b_{cd} A^c_\nu A^d_\rho), \tag{6.27}$$

est invariant à une dérivée totale près.

En termes de formes $L_{CS}d^3x = \frac{1}{g}\mathrm{Tr}(AdA + \frac{2}{3}A^3)$, avec $A = A^a_\mu dx^\mu \delta_a$, $d = dx^\mu \partial_\mu$, $\mathrm{Tr}(\delta_a\delta_b) = gg_{ab}$, $F = \frac{1}{2}F^a_{\mu\nu}dx^\mu dx^\nu \delta_a = dA + \frac{1}{2}[A, A] = dA + A^2$. Montrer que $\mathrm{Tr}(FF) = d\mathrm{Tr}(AdA + \frac{2}{3}A^3)$ (ce qui est équivalent à l'équation (6.21)) à toute dimension n.

6.8.4 Lorentz invariance of the Yang-Mills Lagrangian

Montrer que $L_{YM}d^nx$ est invariant sous une transformation de Lorentz de $L^\uparrow_+$ qui agit comme

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad A^a_\mu \to A'^a_\mu = \Lambda_\mu{}^\nu A^a_\nu, \tag{6.28}$$

où les indices sont montés et descendus avec η.

6.8.5 Gauge invariance and coupling constants

Montrer que pour un facteur simple, l'invariance de jauge du Lagrangien de Yang-Mills et des champs de matière requiert que tous les champs de matière se couplent avec la même constante de couplage. Est-ce vrai pour les facteurs abéliens ?

6.8.6 Gauge invariant operator of dimension 5

Montrer qu'une interaction de la forme

$$g_{ij}\bar{\psi}^i[\gamma^\mu, \gamma^\nu]F^\alpha_{\mu\nu}T^j_{\alpha k}\psi^k, \tag{6.29}$$

est invariante de jauge si g_{ij} est un tenseur invariant et vérifier qu'elle est de dimension canonique 5.

6.8.7 Hamiltonian for the Yang-Mills field

Calculer, sur la surface des contraintes, l'Hamiltonien associé à $L_{YM} = -\frac{1}{4g^2}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha$. Montrer que l'énergie est bornée inférieurement.

Chapter 7

Quantum gauge fields

Le chapitre suit de près les chapitres 15 et 17 du [25] (voir aussi [10] et l'introduction de [50]).

On fixe d'abord la jauge en utilisant le formalisme des antichamps appliqué à la théorie Chern-Simons et de Yang-Mills. Le potentiel effectif de la théorie de Chern-Simons (voir p.ex. [51] pour les règles de Feynman) s'annule ce qui implique que la fonction beta est nulle. On montre ensuite que l'indépendance des amplitudes physiques de la fixation de jauge est une conséquence de l'identité de Ward associée à la symétrie BRST et on discute l'équation de Zinn-Justin.

La jauge du champs de fond est introduite. Des cancellations supplémentaires au niveau de l'action effective de la théorie de Chern-Simons sont discutées. Pour la théorie de Yang-Mills, l'invariance de jauge de fond permet de la relier la renormalisation de la constante de couplage à celle du champs de jauge de fond qui peut se calculer de nouveau par le potentiel effectif.

Les fonctions β de l'électrodynamique quantique et de la chromodynamique sont calculées explicitement. L'interprétation physique du résultat est très brièvement discutée.

7.1 Non-invertibility of quadratic kernel and gauge invariance

5.1 Propagateurs et invariance de jauge

Thursday, September 8, 2016 6:45 PM

Ex 1: plusieurs champs scalaires

$$S_0 = -\frac{1}{2}\int d^n x \left[\partial_\mu \phi^i \partial^\mu \phi_i + m^2 \phi^i \phi_i\right] + i\epsilon$$

$$= -\frac{1}{2}\int d^n x \int d^n x' \; \phi^i(x)\, \underbrace{\delta_{ij}\left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'_\mu}\delta^n(x-x') + m^2 - i\varepsilon\right]}_{\mathcal{D}_{ij}(x,x')} \phi^j(x')$$

propagateur

$$\Delta^{ij}(x,x') = (\mathcal{D}^{-1})^{ij}(x,x')$$

$$\int d^n x' \; \mathcal{D}_{ij}(x,x')\, \Delta^{jk}(x',x'') = \delta^n(x-x'')\,\delta_i^k$$

$$\delta^n(x-x') = \frac{1}{(2\pi)^n}\int d^n p \; e^{ip(x-x')}$$

$$\mathcal{D}_{ij}(x,x') = \delta_{ij}\frac{1}{(2\pi)^n}\int d^n p \; (p^2+m^2-i\varepsilon)\, e^{ip(x-x')}$$

$$\tilde{\mathcal{D}}_{ij}(p) = \delta_{ij}\,(p^2+m^2-i\varepsilon)$$

$$\tilde{\Delta}^{ij}(p) = \delta^{ij}\,\frac{1}{p^2+m^2-i\varepsilon}$$

$$\Delta^{ij}(x-x') = \delta^{ij}\frac{1}{(2\pi)^n}\int d^n p \; \frac{1}{p^2+m^2-i\varepsilon}\, e^{ip(x-x')}$$

Ex 2: champ vectoriel massif

$$L_0 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}m^2 A^\mu A_\mu + i\varepsilon$$

$$= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2} m^2 A^\mu A_\mu + i\,\varepsilon$$

$$D_{\rho\sigma}(x,y) = \left(\eta_{\rho\sigma}\frac{\partial^2}{\partial x^\mu \partial y_\mu} - \frac{\partial^2}{\partial x^\sigma \partial y^\rho} + m^2 \eta_{\rho\sigma}\right)\delta^n(x-y) \quad - i\varepsilon$$

$$= \frac{1}{(2\pi)^n}\int d^n p\; e^{ip(x-y)}\left(\eta_{\rho\sigma}(p^2+m^2) - p_\rho p_\sigma - i\varepsilon\right)$$

ε neglect

$$\Delta^{\rho\sigma}(x,y) = \frac{1}{(2\pi)^n}\int d^n p\; e^{ip(x-y)}\left(\frac{\eta^{\rho\sigma} + \frac{p^\rho p^\sigma}{m^2}}{p^2+m^2}\right)$$

Γ

$$\left(\eta_{\rho\sigma}(p^2+m^2) - p_\rho p_\sigma\right)\left(\frac{\eta^{\sigma\tau} + \frac{p^\sigma p^\tau}{m^2}}{p^2+m^2}\right)$$

$$= \frac{1}{p^2+m^2}\left(\delta_\rho^\tau (p^2+m^2) - p_\rho \cancel{p^\tau} + \frac{p_\rho p^\tau}{m^2}(\cancel{p^2}+\cancel{m^2}) - \frac{p^2}{m^2}\cancel{p_\rho} p^\tau\right)$$

$$= \delta_\rho^\tau \quad \lrcorner$$

que se passe-t-il pour $m=0$?

$$D_{\rho\sigma}(x,y) = \int \frac{d^n p}{(2\pi)^n}\; e^{ip(x-y)}\underbrace{\left(\eta_{\rho\sigma}p^2 - p_\rho p_\sigma\right)}_{\tilde{\Delta}_{\rho\sigma}(p)}$$

p^σ vecteur propre de valeur propre 0 :

$$\tilde{A}_{\rho\sigma}(p)\, p^{\sigma} = 0$$

$\Rightarrow$ la partie quadratique est non-inversible à cause de l'invariance de jauge.

En effet, ceci correspond à faire

$$\int d^n y \; \mathcal{D}_\rho{}^{\sigma}(x-y)\, A^g_\sigma(y) = 0 \qquad \text{avec}$$

$$A^g_\sigma(y) = \partial_\sigma \varepsilon(y).$$

$\Rightarrow$ il faut fixer la jauge pour rendre la partie quadratique inversible.

7.2 BRST invariance

$S^{inv}[A,\psi] = \int d^n x \, L^{inv}[A,\psi]$ invariant de jauge

• théorie de Yang-Mills:

$$L^{inv}[A,\psi] = \underbrace{-\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu b} g_{ab}}_{= L^{YM}} + L_M[\psi, D\psi]$$

champs de matière: $\psi^i = (\phi, \xi, \psi)$

scalars, fermions (Weyl, Dirac, ...)

transformations de jauge infinitésimales:

$$\delta_\epsilon A^a_\mu = D_\mu \epsilon^a \,, \quad \delta_\epsilon \psi^i = -\epsilon^a T_a{}^i{}_j \psi^j$$

invariance de jauge du Lagrangien:

$$\delta_\epsilon L^{inv} = 0 \qquad \forall \, \epsilon^a(x)$$

normalisation:

$$A^a_\mu \to g A^a_\mu \,, \quad f^a{}_{bc} \to g f^a{}_{bc} \,, \quad T_a{}^i{}_j \to g T_a{}^i{}_j$$

$L^{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu b} g_{ab}$, normalisation canonique si $g_{ab} = \delta_{ab}$

• théorie de Chern-Simons:

$n=3 \qquad L^{YM} \to L^{CS} = \frac{k}{8\pi} \epsilon^{\mu\nu\rho} \; g_{ab} \, A^a_\mu \left(\partial_\nu A^b_\rho + \frac{1}{3} f^b{}_{cd} A^c_\nu A^d_\rho \right)$

$A = A^a{}_\mu \, dx^\mu \, T_a \quad , \quad \langle T_a , T_b \rangle = g_{ab}$

$$d^3x \, L^{CS} = \frac{k}{4\pi} \langle A , dA + \frac{2}{3} \underset{\frac{1}{3}[A,A]}{\underset{\shortparallel}{A^2}} \rangle = \frac{k}{8\pi} \langle A_\mu , \partial_\nu A_\rho + \frac{1}{3} [A_\nu , A_\rho] \rangle \, dx^\mu dx^\nu dx^\rho$$

graded commutator

normalisation : $\frac{1}{g^2} = \frac{k}{4\pi}$

$A^a_\mu \to g \, A^a{}_\mu \quad , \quad f^a{}_{bc} \to g \, f^a{}_{bc}$

$L^{CS}_{quad} = \frac{1}{2} \epsilon^{\mu\nu\rho} A^a_\mu \partial_\nu A^b_\rho \; g_{ab}$

- introduction de nouveaux champs fermioniques (et bosoniques)

dans la théorie $\qquad \epsilon^a(x) \longrightarrow C^a(x)$

fantôme

NB: scalaire fermionique, pas en accord avec spin-statistique

pas de contradiction car on n'est pas en train de considérer

des représentations unitaires irréductibles du groupe de Poincaré

$\phi^A = (A^a_\mu , y^i , C^a , \bar{C}^a , B^a) \qquad$ champs de la théorie

$\phi^*_A = (A^{*\mu}{}_a , y^*_i , C^*_a , \bar{C}^*_a , B^*_a) \qquad$ antichamps, sources externes

"moments conjugués"

"moments conjugués"

	A_μ^a	ϕ	ξ	ψ	C^a	$\bar{C}^a$	B^a
p	0	0	1	1	1	1	0
gh	0	0	0	0	1	-1	0

p : parité
gh : ghost number

	$A^{*\mu}_a$	ϕ^*	ξ^*	ψ^*	C^*_a	$\bar{C}^*_a$	B^*_a
p	1	1	0	0	0	0	1
gh	-1	-1	-1	-1	-2	0	-1

$p(\phi^*_A) = p(\phi^A) + 1 \pmod 2$

$gh(\phi^*_A) = -gh(\phi^A) - 1$

Anticrochet :

$$(F, G) = \int d^n x \left[\frac{\delta^R F}{\delta \phi^A(x)} \frac{\delta^L G}{\delta \phi^*_A(x)} - \frac{\delta^R F}{\delta \phi^*_A(x)} \frac{\delta^L G}{\delta \phi^A(x)} \right]$$

dérivées gauches et droites:

parité de F

$$\delta\phi^B(x) \frac{\delta^L F}{\delta\phi^B(x)} = \frac{\delta^R F}{\delta\phi^B(x)} \delta\phi^B(x) \iff \frac{\delta^R F}{\delta\phi^B(x)} = (-)^{\varepsilon(B+F)} \frac{\delta^L F}{\delta\phi^B(x)}$$

Propriétés :

(i) $(F, G) = -(-)^{(F+1)(G+1)} (G, F)$ antisymétrie graduée

(ii) $(F, (G, H)) = ((F, G), H) + (-)^{(F+1)(G+1)} (G, (F, H))$

$\iff (F, (G, H)) (-)^{(F+1)(H+1)} + \text{cyclic}(F, G, H) = 0$

identité de Jacobi graduée

identité de Jacobi graduée

(iii) si $(-)^F = 1$, $\frac{1}{2}(F,F) = \int d^4x \, \frac{\delta^R F}{\delta\phi^A(x)} \frac{\delta^L F}{\delta\phi^*_A(x)} = -\int d^4x \, \frac{\delta^R F}{\delta\phi^*_A(x)} \frac{\delta^L F}{\delta\phi^A(x)}$

$(F,F) \not\equiv 0$ mais $(F,(F,F)) \equiv 0$ à cause de (ii).

Action maîtresse et transformations BRST

$$S[\phi^A, \phi^*_A] = S^{inv} + \int d^4x \left[-D_\mu C^a A^{*\mu}_a + C^a T^i_{a j} y^j y^*_i + \frac{1}{2} f^a_{bc} C^b C^c C^*_a - B^a \bar{C}^*_a \right]$$

$$s = (S, \;) \; , \quad s\phi^A(x) = -\frac{\delta^R S}{\delta\phi^*_A(x)} \; , \quad s\phi^*_A(x) = \frac{\delta^R S}{\delta\phi^A(x)} = (-)^A \frac{\delta^L S}{\delta\phi^A(x)}$$

$$\begin{cases} s A^a_\mu = D_\mu C^a \; , \quad s y^i = -C^a T^i_{a j} y^j \; , \quad s C^a = -\frac{1}{2} f^a_{bc} C^b C^c \\ s \bar{C}^a = B^a \; , \quad s B^a = 0 \end{cases}$$

NB: sur A^a_μ, y^i les transformations BRST agissent comme des transformations de jauge avec $\epsilon^a(x) \to C^a(x)$.

$$\begin{cases} \frac{\delta^L S}{\delta A^a_\mu} = \frac{\delta S^{inv}}{\delta A^a_\mu} - f^b_{ac} C^c A^{*\mu}_b \; , \quad \frac{\delta^L S}{\delta y^i} = \frac{\delta^L S^{inv}}{\delta y^i} + (-)^i C^a T^j_{a i} y^*_j \\ \frac{\delta^L S}{\delta \bar{C}^a} = 0 \; , \quad \frac{\delta^L S}{\delta B^a} = -\bar{C}^*_a \; , \quad \frac{\delta^L S}{\delta C^a} = \underbrace{D_\mu A^{*\mu}_a}_{?} + T^i_{a j} y^j y^*_i + C^*_c f^c_{ab} C^b \end{cases}$$

représentation coadjointe: $D_\mu v_a = \partial_\mu v_a - f^b_{ca} A^c_\mu v_b$

NB: $gh(S) = 0$, $gh(s) = 1$

$$S = S^{inv} - \int d^4x \; s\phi^A \, \phi^*_A$$

On a $\boxed{\frac{1}{2}(S,S)=0} \iff sS=0$ invariance BRST de l'action maîtresse

équation maîtresse

<u>Dém</u> :

$$\frac{1}{2}(S,S) = \int d^4x \left(\frac{\delta^R S}{\delta\phi^*_A(x)}\right)\frac{\delta^L S}{\delta\phi^A(x)}$$

$$\underset{"}{s\phi^A(x)}$$

$$= \int d^4x \Big[\partial_\mu C^a \Big(\underbrace{\frac{\delta S^{inv}}{\delta A^a_\mu}} - \underbrace{f^b{}_{ac} C^c A^{*\mu}_b} \Big)$$

$$- C^a T^i{}_{aj} y^j \Big(\underbrace{\frac{\delta S^{inv}}{\delta y^i}} + \underbrace{(-)^i C^b T_b{}^k{}_i y^*_k} \Big)$$

$$- \frac{1}{2} f^a{}_{bc} C^b C^c \Big(\underbrace{\partial_\mu A^{*\mu}_a - f^d{}_{ea} A^e_\mu A^{*\mu}_d} + \underbrace{T^i{}_{aj} y^j y^*_i} + \underbrace{C^*_c f^c{}_{ae} C^e} \Big) \Big]$$

$\sqcup = 0$ invariance de jauge S^{inv}

$$\sqcup = -\partial_\mu C^a f^b{}_{ac} C^c A^{*\mu}_b - f^a{}_{\cancel{de}} A^{\cancel{d}\,e}_\mu C^{\cancel{e}\,b} f^{\cancel{b}\,d}{}_{ac} C^c A^{*\mu}_{\cancel{b}\,d}$$

$$\underbrace{- \frac{1}{2} f^{\cancel{a}\,b}{}_{\cancel{b}c} C^{\cancel{b}\,a} C^c \partial_\mu A^{*\mu}_{\cancel{a}\,b}}_{-\partial_\mu\left(\frac{1}{2} f^b{}_{ac} C^a C^c A^{*\mu}_b\right)} \quad \underbrace{+ \frac{1}{2} f^a{}_{bc} C^b C^c f^d{}_{ea} A^e_\mu A^{*\mu}_d}_{\frac{1}{2} C^b C^c \left(f^d{}_{ea} f^a{}_{bc} - f^d{}_{ac} f^a{}_{eb} + f^d{}_{ab} f^a{}_{ec} \right) A^e_\mu A^{*\mu}_d}$$

$$+ f^d{}_{ca} f^a{}_{eb} + f^d{}_{ba} f^a{}_{ce}$$

$$\shortparallel$$

$$0 \text{ Jacobi pour } f's.$$

terme de bord

$$\int d^4x \; \underbrace{\qquad} = 0$$

$$\sqcup : (-)^i = (-)^j$$

$$\underbrace{-c^a c^b \left[T^k_{[b|i} T_{a]}{}^i{}_j\, y^j y^*_k + \frac{1}{2} f^c{}_{ab} T_c{}^k{}_j\, y^j y^*_k \right]}_{=0}$$

$$\left(\frac{1}{2}[T_b, T_a]\right)^k{}_j = \frac{1}{2} f^c{}_{ba} T_c{}^k{}_j$$

(les T forment une représentation de l'algèbre de jauge)

$$\sqcup = -\frac{1}{2} C^*_c\, f^c{}_{a[e}\, f^a{}_{bc]}\, c^e c^b c^c = 0$$ (Jacobi pour les f's) $\square$.

Corollaire : $s^2 = 0$

En effet : $s^2 H = (S, (S, H)) = ((S, S), H) - (S, (S, H))$

$$\Rightarrow s^2 H = \frac{1}{2}((S,S), H) = 0$$

7.3 Gauge fixation and propagators

Digression : transformation canonique et fonction génératrice de $2^{\text{ème}}$ espèce $F_2(q, P)$

$(q^a, p_b) \longleftrightarrow (Q^a, P_b)$

$$\begin{cases} p_a dq^a = P_a dQ^a + d\left(F_2(q,P) - P_a Q^a\right) \\ F_2(q,P) = P_a q^a + \psi(q) \end{cases}$$

$$p_a dq^a = P_a \cancel{d}Q^a + P_a dq^a + \underbrace{dP_a q^a} + \frac{\partial \psi}{\partial q^a} dq^a - P_a \cancel{d}Q^a - \underbrace{dP_a Q^a}$$

$$\Leftrightarrow \begin{cases} q^a = Q^a \\ p_a = P_a + \dfrac{\partial \psi}{\partial q^a} \end{cases}$$

Propriété : si $\tilde{G}(Q,P) = G(q(Q,P), p(Q,P))$

alors $\{\tilde{G}_1, \tilde{G}_2\}_{Q,P} = \{G_1, G_2\}_{q,p}$

Choisissons $\psi(\phi)$, $gh(\psi) = -1$

$$\psi = \int d^4x\ \bar{C}^a (\partial_\mu A^{\mu b} - \frac{\xi}{2} B^b)\ g_{ab}$$

ξ : paramètre $\Rightarrow$ classe de fixations de jauge.

transformation anticanonique :

$$\int d^4x\ \delta\phi^A(x)\ \phi^*_A(x) = \int d^4x\ \delta\tilde{\phi}^A(x)\ \tilde{\phi}^*_A(x) + \delta F$$

$$F = \int d^4x \left[\phi^A(x) \tilde{\phi}^*_A(x) - \tilde{\phi}^A(x) \tilde{\phi}^*_A(x)\right] + \Psi[\phi]$$

$$\Rightarrow \begin{cases} \phi^A = \tilde{\phi}^A \\ \phi^*_A = \tilde{\phi}^*_A + \dfrac{\delta^L \psi}{\delta \phi^A} \end{cases}$$

$$S[\phi, \phi^*] = S^{inv}[\phi] - \int d^4x \; s\phi^A(x)\, \phi^*_A(x) \longrightarrow \tilde{S}[\hat{\phi}, \tilde{\phi}^*] = S_\psi[\hat{\phi}, \tilde{\phi}^*]$$

$$S_\psi[\hat{\phi}, \tilde{\phi}^*] = S[\phi, \tilde{\phi}^* + \frac{\delta^L \psi}{\delta \phi}] = S^{inv}[\phi] - \int d^4x \; s\phi^A(x)\, \tilde{\phi}^*_A(x) - s\Psi$$

$$-s\Psi = \int d^4x \; [-\partial^\mu \bar{C}^a D_\mu C^b - B^a \partial^\mu A^b_\mu + \frac{\xi}{2} B^a B^b]\, g_{ab}$$

transformation anticanonique:

$$\frac{1}{2}(S_\psi, S_\psi)_{\phi, \tilde{\phi}^*} = 0 \iff s_\psi S_\psi[\phi, \tilde{\phi}^*] = 0$$

$s_\psi = (S_\psi, \;)_{\phi, \tilde{\phi}^*}$, $s_\psi \phi^A = (S_\psi, \phi^A)_{\phi, \tilde{\phi}^*} = s\phi^A$ inchangé

$$s_\psi \tilde{\phi}^*_A = (S_\psi, \tilde{\phi}^*_A) = \frac{\delta^L(S^{inv} - s\Psi - \int d^4x \; s\phi^B(x)\, \tilde{\phi}^*_B(x))}{\delta \phi^A}$$ changé

2 options : B^a : champs auxiliaires, (i) conserver, (ii) éliminer

(ii) $$\frac{\delta S_\psi}{\delta B^a} = 0 \iff B_a = \frac{1}{\xi}(\partial^\mu A_{\mu a} + \tilde{\bar{C}}^*_a)$$

éliminer par leurs équations du mouvement $\iff$ intégration gaussienne dans l'intégrale de chemin

$$S'_\psi[\phi, \tilde{\phi}^*] = S^{inv} + \int d^4x \,[-\partial^\mu \bar{C}^a D_\mu C^b - \frac{1}{2\xi}(\partial^\mu A^a_\mu + \tilde{\bar{C}}^{*a})(\partial^\nu A^b_\nu + \tilde{\bar{C}}^{*b})]\, g_{ab}$$
$$- \int d^4x \; [s A^a_\mu \tilde{A}^{*\mu}_a + s C^a \tilde{C}^*_a + s y^i \tilde{y}^*_i]$$

NB: $$\begin{cases} s'_\psi \bar{C}^a = -\dfrac{\delta^R S'_\psi}{\delta \tilde{\bar{C}}^*_a} = \dfrac{1}{\xi}(\partial^\mu A^a_\mu + \tilde{\bar{C}}^*_a) \\ s'_\psi \tilde{\bar{C}}^*_a = \dfrac{\delta^R S'_\psi}{\delta \bar{C}^a} = -\partial^\mu D_\mu C^a \end{cases}$$ $\quad s'^2_\psi = 0$

$\Rightarrow$ si on élimine B^a, l'action maîtresse devient quadratique en les $\tilde{\bar{C}}^*_a$

si on met alors $\tilde{\phi}^* = 0$, $s'^2_\psi \bar{C}^a \approx 0$, on-shell pour $\bar{C}^a$.

• propagateurs théorie de Yang-Mills, option (ii)

$$S'^{2}_{f}[\phi,0] = \int d^4x \left[-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu b} - \partial^\nu A^{\mu b}) - \frac{1}{2\xi}\partial^\mu A^a_\mu \partial^\nu A^b_\nu \right.$$
$$\left. - \partial^\mu \bar{C}^a \partial_\mu C^b \right] g_{ab}$$

$$= -\frac{1}{2}\phi^A_{(B)} \mathcal{D}^{(B)}_{AB} \phi^B_{(B)} - \bar{\phi}^\alpha_{(F)} \mathcal{D}^{(F)}_{\alpha\beta} \phi^\beta_{(F)}$$

$$= -\frac{1}{2}\int d^4x \int d^4y \, \mathcal{D}^{\mu\ \nu}_{a\ b}(x,y) A^a_\mu(x) A^b_\nu(y) - \int d^4x \int d^4y \, \bar{C}^a(x) \mathcal{D}^G_{ab}(x,y) C^b(y)$$

$$\mathcal{D}^{\mu\nu}_{ab}(x,y) = \left[\eta^{\mu\nu} \frac{\partial^2}{\partial x^\lambda \partial y_\lambda} \delta^4(x-y) - \left(1-\frac{1}{\xi}\right) \frac{\partial^2}{\partial x_\mu \partial y_\nu} \delta^4(x-y) \right] g_{ab} + \epsilon \text{ termes}$$

$$= (2\pi)^{-4} \int d^4p \left[\eta^{\mu\nu}\, p^2 - i\epsilon - \left(1-\frac{1}{\xi}\right) p_\mu p_\nu \right] e^{ip\cdot(x-y)} g_{ab}$$

$$(\mathcal{D}^{-1})^{ab}_{\mu\nu}(x,y) = (2\pi)^{-4} \int d^4p \left[\eta_{\mu\nu} + (\xi-1)\frac{p_\mu p_\nu}{p^2} \right] \frac{1}{p^2 - i\epsilon} e^{ip\cdot(x-y)} g^{ab}$$

$$\begin{cases} \text{jauge de Landau : } \xi = 0 \\ \text{jauge de Feynman : } \xi = 1 \rightarrow \text{propagateurs simples} \end{cases}$$

$$\mathcal{D}^G_{ab}(x,y) = \frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial y_\lambda} \delta^4(x-y) g_{ab}$$

$$\mathcal{D}^{-1\,ab}_{G}(x,y) = \frac{1}{(2\pi)^4} \int d^4p \frac{g^{ab}}{p^2 - i\epsilon}$$

option (i) : (neglect $i\epsilon$ terms)

$$\mathcal{D}^{\mu\nu}_{ab}(x,y) = g_{ab}\left[\eta^{\mu\nu} \frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial y_\lambda} \delta^4(x-y) - \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu} \delta^4(x-y) \right]$$

$$D^{\mu\nu}_{ab}(x,y) = g_{ab}\left[\eta^{\mu\nu}\frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial y_\lambda}\delta^4(x-y) - \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\delta^4(x-y)\right]$$

$$= \frac{1}{(2\pi)^4}\, g_{ab}\int d^4p\; e^{ip(x-y)}\left[\eta^{\mu\nu}p^2 - p^\mu p^\nu\right]$$

$$D_{ab}{}^{\nu}(x,y) = -g_{ab}\frac{\partial}{\partial y_\nu}\delta^4(x-y) \qquad D^{\mu}_{a}{}_{b}(x,y) = -g_{ab}\,\partial^\mu_x\,\delta^4(x-y)$$

$$= \frac{1}{(2\pi)^4}\int d^4p\,(g_{ab}\,i p^\nu)\,e^{ip\cdot(x-y)} \qquad = \frac{1}{(2\pi)^4}\int d^4p\,(-g_{ab}\,i p^\mu)\,e^{ip\cdot(x-y)}$$

$$D_{ab}(x,y) = -\xi\, g_{ab}\,\delta^4(x-y) = \frac{1}{(2\pi)^4}\int d^4p\,(-\xi g_{ab})\,e^{ip\cdot(x-y)}$$

Matrice :

$$g_{ab}\left(\begin{array}{c|c} \eta^{\mu\nu}p^2 - p^\mu p^\nu & -i p^\mu \\ \hline i p^\nu & -\xi \end{array}\right)$$

Matrice inverse :

$$g^{bc}\left(\begin{array}{c|c} \dfrac{\eta_{\nu\rho} + (\xi-1)\dfrac{p_\nu p_\rho}{p^2}}{p^2} & -i\dfrac{p_\nu}{p^2} \\ \hline i\dfrac{p_\rho}{p^2} & 0 \end{array}\right)$$

propagateurs théorie de Chern-Simons, option (i)

normalisation canonique.

partie ghosts et B^a inchangée

$$S_2^{inv}[A] = \frac{1}{2} g_{ab} \int d^3x \, \epsilon^{\mu\nu\rho} A_\mu^a \partial_\nu A_\rho^b$$

$$= -\frac{1}{2} \int d^3x \int d^4y \; A_\mu^a(x) \, D_{ab}^{\mu\nu}(x,y) \, A_\nu^b(y)$$

$$D_{ab}^{\mu\nu}(x,y) = \epsilon^{\mu\rho\nu} \frac{\partial}{\partial y^\rho} \delta^3(x-y) \, g_{ab}$$

$$= \frac{1}{(2\pi)^3} \int d^3p \, e^{ip\cdot(x-y)} \left(-i\epsilon^{\mu\rho\nu} p_\rho\right)$$

Matrice :

$$g_{ab} \left(\begin{array}{c|c} i\,\epsilon^{\mu\nu\rho} p_\rho & -i\,p^\mu \\ \hline i\,p^\nu & -\xi \end{array} \right)$$

Matrice inverse :

$$g^{bc} \left(\begin{array}{c|c} \dfrac{(-)^s i\epsilon_{\nu\lambda\sigma} p^\sigma + \dfrac{p_\nu p_\lambda \xi}{p^2}}{p^2} & -i\dfrac{p_\nu}{p^2} \\ \hline i\dfrac{p_\lambda}{p^2} & 0 \end{array} \right)$$

s : signature de η $\qquad \epsilon^{\lambda\mu\nu} \epsilon_{\lambda\alpha\beta} = (-)^s \left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha \right)$

Starting point: quadratic form of the bosonic part of the gauge fixed action

$$S_B^2 = \int d^4x \left[-\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu b} - \partial^\nu A^{\mu b}) - B^a \partial_\mu A^{\mu b} + \frac{\xi}{2} B^a B^b \right] g_{ab}$$

$$= \int d^4x \left(-\frac{1}{2} \partial_\mu A_\nu^a \partial^\mu A^{b\nu} + \frac{1}{2} \partial^\mu A_\mu^a \partial^\nu A_\mu^b - B^a \partial_\mu A^{\mu b} + \frac{\xi}{2} B^a B^b \right] g_{ab} \qquad (X)$$

$$= -\frac{1}{2} \phi^A_{(B)} \mathcal{D}^{(B)}_{AB} \phi^B = -\frac{1}{2} \int d^4x \int d^4y \; \phi^i_B(x) \mathcal{D}_{ij}(x,y) \phi^j_B(y)$$

$$\phi^i(y) = \begin{pmatrix} A_\nu^b(y) \\ B^b(y) \end{pmatrix} \qquad \mathcal{D}_{ij}(x,y) = \begin{pmatrix} \mathcal{D}^{\mu\nu}_{ab}(x,y) & \mathcal{D}^\nu_{ab}(x,y) \\ \mathcal{D}^\mu_{ab}(x,y) & \mathcal{D}_{ab}(x,y) \end{pmatrix}$$

$$-\frac{1}{2} \int d^4x \, d^4y \; (A_\mu^a(x), B^a(x)) \begin{pmatrix} \mathcal{D}^{\mu\nu}_{ab}(x,y) & \mathcal{D}^\mu_{ab}(x,y) \\ \mathcal{D}^\nu_{ab}(x,y) & \mathcal{D}_{ab}(x,y) \end{pmatrix} \begin{pmatrix} A_\nu^b(y) \\ B^b(y) \end{pmatrix} \overset{?}{=} (X)$$

$$\mathcal{D}^{\mu\nu}_{ab}(x,y) = g_{ab} \left[\eta^{\mu\nu} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y_\alpha} - \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \right] \delta^4(x,y) \; , \quad \mathcal{D}^\mu_{ab}(x,y) = -g_{ab} \frac{\partial}{\partial x_\mu} \delta^4(x,y)$$

$$\mathcal{D}^\nu_{ab}(x,y) = -g_{ab} \frac{\partial}{\partial y_\nu} \delta^4(x,y) \qquad \mathcal{D}_{ab}(x,y) = -\xi \, g_{ab} \, \delta^4(x,y)$$

$$= -\frac{1}{2} \int d^4x \, d^4y \; (A_\mu^a(x), B^a(x)) \, g_{ab} \begin{pmatrix} \left(\eta^{\mu\nu} \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y_\alpha} - \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\right) \delta^4(x,y) A_\nu^b(y) - \frac{\partial}{\partial x^\mu} \delta^4(x,y) B^b(y) \\ -\frac{\partial}{\partial y^\nu} \delta^4(x,y) A_\nu^b - \xi \, \delta^4(x,y) \, B^b(y) \end{pmatrix}$$

$$-\frac{\partial}{\partial x_\nu}$$

$$= -\frac{1}{2} \int d^4x \int d^4y \left\{ A_\mu^a(x) \left[\eta^{\mu\nu} \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y_\alpha} - \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu} \right] \delta^4(x,y) A_\nu^b(y) - A_\mu^a(x) \frac{\partial}{\partial x_\mu} \delta^4(x,y) B^b(y) \right.$$
$$\left. + B^a(x) \left(-\frac{\partial}{\partial y^\nu} \delta^4(x,y)\right) A_\nu^b(y) - \xi B^a(x) \delta^4(x,y) B^b(y) \right\}$$

$$= -\frac{1}{2} \int d^4x \int d^4y \left\{ \frac{\partial}{\partial x^\alpha} A_\mu^a(x) \frac{\partial}{\partial y_\alpha} \eta^{\mu\nu} A_\nu^b(y) - \frac{\partial}{\partial x_\mu} A_\mu^a(x) \frac{\partial}{\partial y_\nu} A_\nu^b(y) + \frac{\partial}{\partial x_\mu} A_\mu^a(x) B^b(y) + B^a(x) \frac{\partial}{\partial y_\nu} A_\nu^b(y) - \xi B^a(x) B^b(y) \right\} \delta^4(x,y) \, g_{ab}$$

$$= -\frac{1}{2} \int d^4x \; g_{ab} \left\{ \partial_\alpha A_\mu^a(x) \partial^\alpha A^{b\mu} - \partial^\mu A_\mu^a(x) \partial^\nu A_\nu^b(x) + 2 \partial^\mu A_\mu^a(x) B^b(x) - \xi B^a(x) B^b(x) \right\} g_{ab} \overset{!}{=} (X)$$

7.4 Black body partition function from path integral

L'action du champ électromagnétique libre est donnée par

$$S[A_\mu] = \int d^4x(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}). \tag{7.1}$$

L'action fixée de jauge est

$$S[A_\mu, C, \bar{C}, B] = \int d^4x(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \partial_\mu\bar{C}\partial^\mu C + B\partial_\mu A^\mu + \frac{\xi}{2}B^2). \tag{7.2}$$

Dans la jauge de Feynman $\xi = 1$, ceci donne après l'élimination du champs auxiliaire B,

$$S[A_\mu, C, \bar{C}] = \int d^4x(-\frac{1}{2}\partial_\alpha A_\mu \partial^\alpha A^\mu + \partial_\mu\bar{C}\partial^\mu C). \tag{7.3}$$

Dans ce cas, on trouve pour la fonction de partition

$$Z_{em} = \Big(\det(\frac{-\Delta}{2\pi\mu^2})\Big)^{-2}\Big(\det(\frac{-\Delta}{2\pi\mu^2})\Big), \tag{7.4}$$

qui est le même résultat que pour 4 copies d'un champ scalaire et une intégrale fermionique complexe. Pour $\ln Z_{em}$ on trouve donc le même résultat que pour 2 champs scalaires réels,

$$\boxed{\ln Z_{em} = \frac{\pi^2}{45}V\beta^{-3}}, \tag{7.5}$$

et aussi

$$E = -\frac{\partial \ln Z_{em}}{\partial\beta} = \frac{\pi^2}{15}V\beta^{-4}, \quad S = \beta E + \ln Z_{em} = \frac{4\pi^2}{45}V\beta^{-3}, \tag{7.6}$$

$$P = \beta^{-1}\frac{\partial \ln Z_{em}}{\partial V} = \frac{\pi^2}{45}\beta^{-4}. \tag{7.7}$$

NB: L'argument qui mène à l'équation (7.4) ne tient pas compte du signe $+$ inhabituel dans l'exponentielle pour le terme quadratique en A_0, qui n'est donc pas vraiment une intégrale gaussienne. Les détails de la cancellation de la contribution des photons longitudinaux et temporels et des fantômes en termes d'oscillateurs à travers le "quartet mecanisme" au niveau de l'intégrale de chemin sont discutés dans [11].

7.5 Vanishing of Chern-Simons beta function

5.4 Annulation de la fonction beta en théorie de Chern-Simons

Wednesday, November 22, 2017 13:06

Pour avoir accès directement à la contribution divergeante à g on peut aussi calculer le potentiel effectif, comme pour le champ scalaire. On calcule alors $\Gamma[\bar{A}]$ avec $\bar{A}$ constant dans la jauge ordinaire, mais avec $\xi=0$.

$$\Gamma[\bar{A}] = -V_{eff}[\bar{A}] \int d^3x = -V_{eff}[\bar{A}] (2\pi)^3 \delta^3(0)$$

Pour simplifier, prenons $g_{ab} = \delta_{ab}$.

On introduit $P_\mu{}^a{}_b = \delta^a_b p_\mu + i \bar{a}_\mu{}^a{}_b$, $(\bar{a}_\mu)^a{}_b = \bar{A}^c_\mu f_c{}^a{}_b$

et $(p\cdot\bar{a})\delta = p^\rho \bar{a}_\rho{}^a{}_b \delta^\mu{}_\nu$, $p\bar{a} = p^\mu (\bar{a}_\nu)^a{}_b$

On peut adapter les formules (5.3) à (5.7) au cas présent :

$$V^{(1)}_{eff}[\bar{A}] = \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2i} \operatorname{tr} \ln \tilde{\mathcal{D}}^{-1}_{bos}(p)\, \tilde{\mathcal{D}}^{\bar{A}}_{bos}(p) - \frac{1}{i} \operatorname{tr} \tilde{\mathcal{D}}^{-1}_{gh}(p)\, \tilde{\mathcal{D}}^{\bar{A}}_{gh}(p)\right]$$

- $\tilde{\mathcal{D}}^{\bar{A}}_{bos}(p)$, $\tilde{\mathcal{D}}^{\bar{A}}_{gh}$ se calculent à partir $\left.\frac{\delta S_\psi[\phi, \bar{\Phi}^c=0]}{\delta\phi^A \delta\phi^B}\right|_{A=\bar{A}}$

- tr : trace matricielle sur les indices μ,ν et a,b

$$\tilde{\mathcal{D}}^{-1}_{bos} = \delta^a{}_b \left(\begin{array}{c|c} \dfrac{i\,(-)^s \epsilon_{\nu\lambda\sigma\rho}\, p^\sigma}{p^2} & -i\dfrac{p^\nu}{p^2} \\ \hline i\dfrac{p_\lambda}{p^2} & 0 \end{array} \right)$$

$$\tilde{\mathcal{D}}^{\bar{A}}_{bos} = \left(\begin{array}{c|c} i\,\epsilon^{\lambda\tau\rho} P_\rho & -i\delta_\rho{}^\lambda \\ \hline i\delta_\rho{}^\tau & 0 \end{array} \right)$$

$$\Rightarrow \quad \tilde{\mathcal{D}}^{-1}_{bos}\, \tilde{\mathcal{D}}^{\bar{A}}_{bos} = \left(\begin{array}{c|c} \delta\delta + i\dfrac{p\cdot\bar{a}}{p^2}\delta - i\dfrac{p\bar{a}}{p^2} & 0 \\ \hline -i\dfrac{p_\lambda}{p^2}\epsilon^{\lambda\tau\rho}\bar{a}_\rho & \delta \end{array} \right)$$

(triangulaire)

$$\tilde{\mathcal{D}}^{-1}_{gh} = \delta^{ab}\frac{1}{p^2}\,, \qquad \tilde{\mathcal{D}}^{\bar{A}}_{gh} = \delta_{ab}\, p^2 + i\, p\cdot\bar{a}$$

$$\Rightarrow \quad \tilde{\mathcal{D}}^{-1}_{gh}\, \tilde{\mathcal{D}}^{-1\bar{A}}_{gh} = \delta + i\, p\cdot\bar{a}$$

$$V_{eff}[\bar{A}] = \frac{1}{2i}\,\mathrm{tr}\,\ln \tilde{\mathcal{D}}^{-1}_{bos}\, \tilde{\mathcal{D}}^{\bar{A}}_{bos} - \frac{1}{i}\,\mathrm{tr}\,\ln \tilde{\mathcal{D}}^{-1}_{gh}\, \tilde{\mathcal{D}}^{\bar{A}}_{gh}$$

$$= \frac{1}{2i}\,\mathrm{tr} \sum_{n=1} \frac{(-)^{n-1}}{n} \left(\begin{array}{c|c} \dfrac{i(p\cdot\bar{a})\delta - i\,p\bar{a}}{p^2} & 0 \\ \hline 0 & 0 \end{array} \right)^n - \frac{1}{i}\,\mathrm{tr} \sum_{n=1} \frac{(-)^{n-1}}{n} \left(\frac{i(p\cdot\bar{a})}{p^2} \right)^n$$

$$= \sum_{n=1} \frac{(-1)^{n-1}}{n} \left(i^{n-1}\right) p^{-2n} \left[\underbrace{\frac{1}{2}\,\mathrm{tr}\left[(p\cdot\bar{a}\,\delta - p\bar{a})^n - (p\cdot\bar{a})^n \right]}_{(*)} \right]$$

$$= \sum_{n=1} \frac{1}{n} (i \ldots) \, p \left[\underbrace{\tfrac{1}{2} \operatorname{tr} \left[(p\cdot\bar{a}\,\delta - p\bar{a})^{n} - (p\cdot\bar{a})^{n} \right]}_{(x)} \right]$$

$$(p\cdot\bar{a}\,\delta - p\bar{a})^2 = (p\cdot\bar{a})^2\delta - \cancel{(p\cdot\bar{a})\,p\bar{a}} - p\bar{a}\,(p\cdot\bar{a}) + \cancel{p\bar{a}\cdot p\bar{a}}$$

$$= (p\cdot\bar{a})^2\delta - p\bar{a}(p\cdot\bar{a})$$

$$(p\cdot\bar{a}\,\delta - p\bar{a})^3 = \left((p\cdot\bar{a})^2\delta - p\bar{a}\; p\cdot\bar{a}\right)(p\cdot\bar{a}\,\delta - p\bar{a})$$

$$= (p\cdot\bar{a})^3\delta - \cancel{(p\cdot\bar{a})^2 p\bar{a}} - p\bar{a}(p\cdot\bar{a})^2 + \cancel{p\bar{a}\; p\cdot\bar{a}\; p\bar{a}}$$

$$= (p\cdot\bar{a})^3\delta - p\bar{a}(p\cdot\bar{a})^2$$

$$\vdots$$

$$(p\cdot\bar{a}\,\delta - p\bar{a})^n = (p\cdot\bar{a})^n\delta - p\bar{a}(p\cdot\bar{a})^{n-1}$$

$$(x) = \operatorname{tr}_G \left\{ \frac{1}{2}\left[3\,(p\cdot\bar{a})^n - (p\cdot\bar{a})^n \right] - (p\cdot\bar{a})^n \right\} = \boxed{0}$$

Le potentiel effectif s'annule à une boucle. De nouveau il y a cancellation entre les contributions des bosons de jauge et des fantômes.

La constante de couplage g n'est pas renormalisée perturbativement.

La fonction β s'annule.

Des cancellations similaires se font entre bosons et fermions "physiques" lors de la renormalisations de théorie supersymétriques.

7.6 Gauge independence and Zinn-Justin equation

En fixant la jauge pour pouvoir faire du calcul perturbatif, la symétrie de jauge de l'action de départ a été remplacée par la symétrie BRST de l'action fixée de jauge. Il s'agit d'une supersymétrie rigide dans le sens qu'elle échange des bosons avec des fermions (non-physiques). 2 questions importantes sont :

(i) Comment les résultats dépendent du choix arbitraire de fixation de jauge ?

(ii) Exploiter les identités de Ward associées à l'invariance BRST.

(i) Les résultats physiques ne dépendent pas du choix de fixation de jauge :

$$\langle \text{Vac, out} | T \prod_i \hat{O}^i(x_i) | \text{Vac, in} \rangle_\Psi = \int \mathcal{D}\phi \prod_i O^i(x_i)\, e^{i/\hbar\, S_\Psi[\phi, \tilde{\phi}^* = 0]}$$

$$S_\Psi[\phi, \tilde{\phi}^* = 0] = S^{inv} - s\Psi \quad , \quad s_\Psi S_\Psi[\phi, \tilde{\phi}^* = 0] = s S_\Psi[\phi, \tilde{\phi}^* = 0] = 0$$

Supposons $s \prod_i \hat{O}^i(x_i) = 0$, les opérateurs sont donc invariants de jauge s'ils ne dépendent que des champs de départs A^a_μ, ψ^i.

$$\langle \text{Vac, out} | T \prod_i \hat{O}^i(x_i) | \text{Vac, in} \rangle_{\Psi + \delta\Psi} - \langle \text{Vac, out} | T \prod_i \hat{O}^i(x_i) | \text{Vac, in} \rangle_\Psi$$

$$= -\frac{i}{\hbar} \int \mathcal{D}\phi \; s\,\delta\Psi \prod_i O^i(x_i)\, e^{\frac{i}{\hbar} S_\Psi[\phi, \tilde{\phi}^* = 0]}$$

$$= -\frac{i}{\hbar} \int d^4x \int \mathcal{D}\phi \; s\phi^A(x) \frac{\delta}{\delta \phi^A(x)} (\delta\Psi) \prod_i O^i(x_i)\, e^{i/\hbar\, S_\Psi[\phi, \tilde{\phi}^* = 0]}$$

[illegible] $e^{i/\hbar\, S_\Psi[\phi, \tilde{\phi}^* = 0]}$ [illegible]

$$= -\frac{i}{\hbar}\int d^4x \int \mathcal{D}\phi \underbrace{\frac{\delta}{\delta\phi^A(x)}}\Big[s\phi^A(x)\, \delta\Psi \prod_i O^i(x_i)\, e^{i/\hbar\, S_\Psi[\phi,\tilde{\phi}^*=0]}\Big] \quad \cancel{(-)^{A(A+1)}}$$

$$+ \frac{i}{\hbar}\int d^4x \int \mathcal{D}\phi \Big[\underbrace{\frac{\delta}{\delta\phi^A(x)}\big(s\phi^A(x)\big)}\, \delta\Psi \prod_i O^i(x^i)\, e^{i/\hbar\, S_\Psi[\phi,\tilde{\phi}^*=0]}$$

$$- \delta\Psi\; s\underbrace{\Big(\prod_i O^i(x_i)\, e^{i/\hbar\, S_\Psi[\phi,\tilde{\phi}^*=0]}\Big)}\Big]$$

$\underbrace{}$ $= 0$: invariance par translation de l'intégrale de chemin

$\underbrace{}$ $= 0$: invariance BRST de $\prod_i O^i(x_i)$ & $S_\Psi[\phi,\tilde{\phi}^*=0]$

$$\frac{\delta\big(s A^a_\mu(x)\big)}{\delta A^a_\mu(x)} = \delta(0)\, f^a{}_{ac}\, C^c\,, \qquad \frac{\delta\big(s C^a(x)\big)}{\delta C^a(x)} = -\delta(0)\, f^a{}_{ac}\, C^c$$

$$\frac{\delta\big(s \bar{C}^a(x)\big)}{\delta \bar{C}^a(x)} = 0\,, \qquad \frac{\delta\big(s B^a(x)\big)}{\delta B^a(x)} = 0\,, \qquad \frac{\delta\big(s y^i(x)\big)}{\delta y^i(x)} = -\delta(0)\, C^a T_a{}^i{}_i$$

$= 0$ pour groupes compactes et représentations sans traces.

$= 0$ $\square$

Considérons

$$Z[J,\tilde{\phi}^*] = \int \mathcal{D}\phi\; e^{i/\hbar\left[S_\Psi[\phi,\tilde{\phi}^*] + \int d^4x\, J_A(x)\,\phi^A(x)\right]}$$

$$S_\Psi[\phi,\tilde{\phi}^*] = S^{inv}[\phi] - s\Psi - \int d^4x\; s\phi^A(x)\,\tilde{\phi}^*_A(x)$$

Les identités de Ward pour la symétrie BRST s'appellent "identités de Slavnov-Taylor". Puisque

$$0 = \frac{1}{2}(S_\Psi, S_\Psi)_{\phi,\tilde{\phi}^*} = -\int d^4x\, \frac{\delta^R S_\Psi}{\delta\tilde{\phi}^*_A(x)}\, \frac{\delta^L S_\Psi}{\delta\phi^A(x)} = \int d^4x\; s\phi^A\, \frac{\delta^L S_\Psi}{\delta\phi^A(x)}$$

on peut les établir en faisant le changement de variables

$$\phi^A \to \phi^A + \epsilon\, s\phi^A$$

dans l'intégrale de chemin.

A partir de

$$\int \mathcal{D}\phi \int d^4x \, \frac{\partial^L}{\delta\phi^A(x)} \left[s\phi^A(x)\, e^{i/\hbar\,[S_\phi + J\phi]} \right] = 0$$

on trouve

$$\int \mathcal{D}\phi \int d^4x \, (-)^A J_A(x)\, s\phi^A(x)\; e^{i/\hbar\,[S_\phi + J\phi]} = 0$$

$$\frac{1}{Z[J,\tilde{\phi}^*]} : \quad \int d^4x \, (-)^A J_A(x) \, \langle s\phi^A \rangle^{J,\tilde{\phi}^*} = 0$$

Puisque $s\phi^A(x)$ est non-linéaire en les champs quantiques, les transformations BRST seront affectés par le processus de renormalisation, " opérateurs composites ". Les sources pour contrôler cela sont précisement les $\tilde{\phi}^*_A(x)$.

Si $W[J,\tilde{\phi}^*] = \frac{\hbar}{i} \ln \frac{Z[J,\tilde{\phi}^*]}{Z[0,0]}$, on a

$$\langle s\phi^A \rangle^{J,\tilde{\phi}^*} = - \frac{\delta^R W}{\delta \tilde{\phi}^*_A(x)} \quad \text{et} \quad - \int d^4x \, (-)^A J_A(x) \frac{\delta^R W}{\delta \tilde{\phi}^*_A(x)} = 0 \quad (x)$$

La transformée de Legendre s'effectue maintenant en présence des sources externes $\tilde{\phi}^*_A$.

$$\phi^A_{J,\tilde{\phi}^*}(x) = \frac{\partial^L W[J,\tilde{\phi}^*]}{\delta J_A(x)} \iff J_A(x) = J_A^{\phi,\tilde{\phi}^*}$$

$$\Gamma[\phi,\tilde{\phi}^*] = \Big[W[J,\tilde{\phi}^*] - J\cdot\phi \Big] \Big|_{J = J^{\phi,\tilde{\phi}^*}}$$

$$\frac{\delta^R \Gamma}{\delta \tilde{\phi}^*_A(x)} = \frac{\delta^R W}{\delta \tilde{\phi}^*_A(x)}\Big| + \int d^4y \, \frac{\delta^R W}{\delta J_B(y)}\Big| \cancel{\frac{\delta^R J_B^{\phi,\tilde{\phi}^*}(y)}{\delta \tilde{\phi}^*_A(x)} - \int d^4y \, \frac{\delta^R J_B^{\phi,\tilde{\phi}^*}}{\delta \tilde{\phi}^*_A(x)}}\, \phi^B(y) \, (-)^{(A+1)B}$$

$$\underbrace{\frac{\delta^R\Gamma}{\delta\tilde{\phi}^*_A(x)} = \frac{\delta^R W}{\delta\tilde{\phi}^*_A(x)}\Bigg|}_{(i)} + \int d^4y \, \underbrace{\cancel{\frac{\delta^R W}{\delta J_B(y)}}}_{\substack{\shortparallel \\ \phi^B(y)\,(-)^B}}\Bigg| \cancel{\frac{\delta^R J_B^{\phi,\tilde{\phi}^*}(y)}{\delta\tilde{\phi}^*_A(x)}} - \int d^4y \, \cancel{\frac{\delta^R J_B^{\phi,\tilde{\phi}^*}}{\delta\tilde{\phi}^*_A(x)}} \, \phi^B(y) \, (-)^{(A+1)B}$$

Comme précédemment pour des transformées de Legendre, on a aussi

$$\underbrace{J_A^{\phi,\tilde{\phi}^*}(x) = -\frac{\delta^R\Gamma[\phi,\tilde{\phi}^*]}{\delta\phi^A(x)}}\quad (ii)$$

En exprimant (x) en termes de $\phi, \tilde{\phi}^*$, on trouve en utilisant (i) et (ii)

$$0 = \int d^4x \, (-)^A \frac{\delta^R\Gamma}{\delta\phi^A(x)} \frac{\delta^R\Gamma}{\delta\tilde{\phi}^*_A(x)} = \int d^4x \, (-)^{\cancel{A}+(\cancel{A}+1)} \frac{\delta^R\Gamma}{\delta\phi^A(x)} \frac{\delta^L\Gamma}{\delta\tilde{\phi}^*_A(x)}$$

$$\Leftrightarrow \quad \boxed{\frac{1}{2}(\Gamma,\Gamma)_{\phi,\tilde{\phi}^*} = 0} \qquad \text{Equation de Zinn-Justin}$$

Cette équation est clé pour démontrer la renormalisabilité de la théorie de Yang-Mills. C'est la version quantique de

$$\frac{1}{2}(S_\Psi, S_\Psi)_{\phi,\tilde{\phi}^*} = 0$$

qui encode l'invariance de jauge de l'action de départ au niveau de l'action fixée de jauge.

(i") We now turn to the case of generic gauge theories in the presence of sources. In this case, $S_\Psi = S(\phi, \widetilde{\phi}^* + \frac{\delta\Psi}{\delta\phi})$.

Let us first show that expectation values of BRST exact operators (S_Ψ, Y) vanish. Indeed,

$$\begin{aligned}\int \mathcal{D}\phi (S_\Psi, Y) e^{\frac{i}{\hbar}S_\Psi} &= \int \mathcal{D}\phi [\frac{\delta^R S_\Psi}{\delta\phi^A}\frac{\delta^L Y}{\delta\widetilde{\phi}^*_A} - \frac{\delta^R S_\Psi}{\delta\widetilde{\phi}^*_A}\frac{\delta^L Y}{\delta\phi^A}] e^{\frac{i}{\hbar}S_\Psi} \\ &= \int \mathcal{D}\phi \Big((-)^A \frac{\hbar}{i}\frac{\delta^L}{\delta\phi^A}[e^{\frac{i}{\hbar}S_\Psi}]\frac{\delta^L Y}{\delta\widetilde{\phi}^*_A} + (-)^{|Y|(A+1)}\frac{\delta^L Y}{\delta\phi^A} s_\Psi\phi^A e^{\frac{i}{\hbar}S_\Psi}\Big).\end{aligned} \tag{7.8}$$

The first part can be written as a total derivative up to contact terms, and thus vanishes. In the second part, after integrating by parts and neglecting contact terms, one remains with

$$(-)^{|Y|+1}\frac{i}{\hbar}\int \mathcal{D}\phi\, Y s_\Psi \phi^A \frac{\delta^L S_\psi}{\delta\phi^A} e^{\frac{i}{\hbar}S_\Psi} \tag{7.9}$$

which vanishes when taking into account antifield dependent BRST invariance of the gauge fixed action in the form

$$s_\Psi \phi^A \frac{\delta^L S_\Psi}{\delta\phi^A} = \frac{1}{2}(S_\Psi, S_\Psi) = 0, \tag{7.10}$$

It follows in particular that expectation values of operators $X[\phi, \widetilde{\phi}^*]$ that are BRST closed,

$$(S_\Psi, X) = 0, \tag{7.11}$$

do not depend on the choice of gauge fixing. This sometimes goes under the name of "Fradkin-Vilkovisky theorem".

Indeed, if Ψ and $\Psi + \delta\Psi$ are two different gauge-fixing fermions,

$$\begin{aligned}\int \mathcal{D}\phi \left(e^{\frac{i}{\hbar}S_{\Psi+\delta\Psi}} - e^{\frac{i}{\hbar}S_\Psi}\right) X &= \frac{i}{\hbar}\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}S_\Psi}\frac{\delta^R S_\Psi}{\delta\widetilde{\Phi}^*_A}\frac{\delta^L\delta\Psi}{\delta\Phi^A} X + O((\delta\Psi)^2) \\ &= -\frac{i}{\hbar}\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}S_\Psi}(S_\Psi, \delta\Psi) X + O((\delta\Psi)^2).\end{aligned} \tag{7.12}$$

One may then move X inside the antibracket since it is BRST closed. The term of first order in $\delta\Psi$ thus vanishes on account of the previous result.

Remark: These are formal arguments that hold at tree level. They may be violated by $\hbar$-correction when taking renormalization into account. Some of these $\hbar$-corrections are captured in an elegant way by the so-called quantum BV formalism. However, this formalism remains formal as well unless due care is devoted to renormalization.

More details on the Batalin-Vilkovisky formalism for generic gauge theories can be found in the reviews [11, 52, 53].

7.7 Background field gauge

À partir de l'action fixée de jauge $S_\Psi[\phi,\tilde{\phi}^*] = S[\phi, \tilde{\phi}^* + \frac{\delta\Psi}{\delta\phi}]$ on peut calculer $\Gamma[\phi,\tilde{\phi}^*]$. Il y aura des divergences et il faudra montrer qu'on peut les absorber par des redéfinitions des constantes de couplage et des champs.

En présence du champ de fond, on a

$\tilde{\Gamma}[\phi, \tilde{\phi}, \tilde{\phi}^*] = \Gamma[\phi+\tilde{\phi}, \tilde{\phi}^*]$ et $\tilde{\Gamma}[0,\tilde{\phi},\tilde{\phi}^*] = \Gamma[\tilde{\phi},\tilde{\phi}^*]$

où $\tilde{\Gamma}[\phi,\tilde{\phi},\tilde{\phi}^*]$ est l'action effective calculée avec

$$S_\Psi[\phi+\tilde{\phi}, \tilde{\phi}^*] = S[\phi+\tilde{\phi}, \tilde{\phi}^* + \frac{\delta\Psi[\phi+\tilde{\phi}]}{\delta\phi}]$$

On avait choisi précédemment $\Psi = \int d^4x\ \bar{C}^a(\partial_\mu A^{\mu b} - \frac{\xi}{2}B^b)g_{ab}$

Maintenant on va choisir un Ψ^B qui dépend explicitement des champs de fond:

$$\Psi_B[\phi-\tilde{\phi},\tilde{\phi}] = \int d^4x\ (\bar{C}^a - \tilde{\bar{C}}^a)[\tilde{D}_\mu(A^{\mu b}-\tilde{A}^{\mu b}) - \frac{\xi}{2}(B^b - \tilde{B}^b)]g_{ab}$$

où $\tilde{D}_\mu k^a = \partial_\mu k^a + f^a_{bc}\tilde{A}^b_\mu k^c$

$\tilde{\Gamma}_B[\phi,\tilde{\phi},\tilde{\phi}^*]$ est calculée avec

$$S_B[\phi,\tilde{\phi},\tilde{\phi}^*] = S[\phi+\tilde{\phi}, \tilde{\phi}^* + \frac{\delta\Psi^B[\phi,\tilde{\phi}]}{\delta\phi}] = S^{inv}[\phi+\tilde{\phi}] - \int d^4x\ s\phi^A[\phi+\tilde{\phi}](\tilde{\phi}^*_A + \frac{\delta^L\Psi^B[\phi,\tilde{\phi}]}{\delta\phi^A})$$

où $\Psi^B[\phi,\tilde{\phi}] = \int d^4x\ \bar{C}^a[\tilde{D}_\mu A^{\mu b} - \frac{\xi}{2}B^b]g_{ab}$

On trouve explicitement,

$$S_B[\phi,\tilde\phi,\tilde\phi^*] = S^{inv}[\phi+\tilde\phi] - \int d^4x \Big\{ \mathcal{D}_\mu^{A+\tilde A}(C+\tilde C)^a \big[\tilde A^{*\mu} - \tilde{\mathcal{D}}^\mu \bar C\big]^b g_{ab}$$

$$+ (B+\tilde B)^a \big[\tilde{\bar C}^* + \tilde{\mathcal{D}}_\mu A^\mu - \tfrac{\xi}{2} B\big]^b g_{ab}$$

$$- (C+\tilde C)^a \, T_a{}^i{}_j \, (\psi+\tilde\psi)^j \, \tilde\psi^*_i$$

$$- \tfrac{1}{2} [C+\tilde C, C+\tilde C]^a \, \tilde C^*_a \Big\}$$

- Si on annule les champs de fond, on retrouve l'action fixée de jauge précédemment $\Rightarrow$ les propagateurs des champs quantiques restent inchangés. (On pourrait interpoler entre la jauge précédente et celle-ci en définissant

$$\tilde{\mathcal{D}}_\mu^{\xi'} h^a = \partial_\mu h^a + \xi' f^a{}_{bc} \tilde A_\mu^b h^c ;$$

$\xi' = 0$: jauges précédentes ; $\xi' = 1$ jauge du champ de fond).

- L'action $S_B[\phi,\tilde\phi,\tilde\phi^*]$ a une nouvelle invariance (due au choix de Ψ^B).

Considérons $\epsilon = \epsilon^a(x) T_a$.

$$\left.\begin{aligned} \delta_\epsilon \tilde A &= \tilde{\mathcal{D}}_\mu \epsilon \\ \delta_\epsilon A_\mu &= [A_\mu, \epsilon] \end{aligned}\right\} \Rightarrow \delta_\epsilon (A_\mu + \tilde A_\mu) = \mathcal{D}_\mu^{A+\tilde A} \epsilon$$

$$\left.\begin{aligned} \delta_\epsilon \tilde\psi &= -\epsilon T \tilde\psi \\ \delta_\epsilon \psi &= -\epsilon T \psi \end{aligned}\right\} \Rightarrow \delta_\epsilon (\psi+\tilde\psi) = -\epsilon\, T(\psi+\tilde\psi)$$

$$\Rightarrow \delta_\epsilon S^{inv}[\phi+\tilde\phi] = 0$$

$$\delta_\epsilon \tilde{\bar C} = [\tilde{\bar C}, \epsilon] \qquad \delta_\epsilon \tilde C = [\tilde C, \epsilon] \qquad \delta_\epsilon \tilde B = [\tilde B, \epsilon]$$

$$\delta_\epsilon \tilde{\bar{C}} = [\tilde{\bar{C}}, \epsilon], \quad \delta_\epsilon \tilde{C} = [\tilde{C}, \epsilon], \quad \delta_\epsilon \tilde{B} = [\tilde{B}, \epsilon]$$

$$\delta_\epsilon \bar{C} = [\bar{C}, \epsilon], \quad \delta_\epsilon C = [C, \epsilon], \quad \delta_\epsilon B = [B, \epsilon]$$

$$\Rightarrow \delta_\epsilon (\tilde{D}_\mu A^\mu) = \delta_\epsilon \left(\partial_\mu A^\mu + [\tilde{A}_\mu, A^\mu]\right) = \partial_\mu\left([A^\mu, \epsilon]\right) + [\partial_\mu \epsilon + [\tilde{A}_\mu, \epsilon], A^\mu]$$
$$+ [\tilde{A}_\mu, [A_\mu, \epsilon]] = [D_\mu^{\tilde{A}} A^\mu, \epsilon]$$

$$\delta_\epsilon D_\mu^{A+\tilde{A}} (C+\tilde{C}) = \delta_\epsilon \left(\partial_\mu (C+\tilde{C}) + [A+\tilde{A}, C+\tilde{C}]\right)$$
$$= \partial_\mu \left([C+\tilde{C}, \epsilon]\right) + [\partial_\mu \epsilon + [A+\tilde{A}, \epsilon], C+\tilde{C}] + [A+\tilde{A}, [C+\tilde{C}, \epsilon]]$$
$$= [D_\mu^{A+\tilde{A}} C+\tilde{C}, \epsilon]$$

$\Rightarrow$ Tous les termes supplémentaires de $S_B[\varphi, \tilde{\varphi}, \tilde{\varphi}^*]$ (au-delà de $S^{inv}[\varphi+\tilde{\varphi}]$) qui ne contiennent pas les antichamps sont de la forme $M^a N^b g_{ab}$ où M^a, N^b se transforment dans la représentation adjointe. Ils sont alors invariants à cause de l'invariance de g_{ab}.

(Dans le cas où $g_{ab} = \mathrm{Tr}\, T_a T_b$, $M = M^a T_a$, $N = N^b T_b$

$$\delta_\epsilon \mathrm{Tr}\, MN = \mathrm{Tr}\left([M, \epsilon] N\right) + \mathrm{Tr}\left(M, [N, \epsilon]\right)$$
$$= \mathrm{Tr}\left(\cancel{M\epsilon N} - \epsilon \cancel{MN} + M \cancel{N\epsilon} - \cancel{M\epsilon N}\right) = 0,)$$

Pour les antichamps, on définit

$$\delta_\epsilon \tilde{A}^{*\mu}_a = -g f^b{}_{ac} \tilde{A}^{*\mu}_b \epsilon^c, \quad \delta_\epsilon \tilde{C}^*_a = g f^b{}_{ac} \tilde{C}^*_b \epsilon^c, \quad \delta_\epsilon \tilde{y}^*_i = \epsilon^c g T_c{}^j{}_i \tilde{y}^*_j$$

(représentation coadjointe) (représentation cogrédiente à la représentation T)

Au final : $\boxed{\delta_\epsilon \quad S_B[\phi, \tilde{\phi}, \tilde{\phi}^*] = 0}$

La symétrie "de jauge de fond" est linéaire en les champs quantiques et les antichamps

Les identités de Ward s'écrivent donc

$$\delta_\epsilon \tilde{\Gamma}_B[\phi, \tilde{\phi}, \tilde{\phi}^*] = 0 \iff \delta_\epsilon \phi \frac{\delta \tilde{\Gamma}_B}{\delta \phi} + \delta_\epsilon \tilde{\phi} \frac{\delta \tilde{\Gamma}_B}{\delta \tilde{\phi}} + \delta_\epsilon \tilde{\phi}^* \frac{\delta \tilde{\Gamma}_B}{\delta \tilde{\phi}^*} = 0$$

En annulant ϕ et $\tilde{\phi}^*$ et en utilisant que

$$\tilde{\Gamma}_B[0, \tilde{\phi}, 0] = \Gamma_B[\tilde{\phi}, 0]$$

on trouve :

l'action effective $\Gamma_B[\tilde{\phi}, 0]$ basée sur le fermion de fixation de jauge Ψ^B est invariante sous une transformation de jauge standard de $\tilde{\phi}$.

Rappel : dans le cas normalisé, $W[0] = 0$, on avait montré que $\Gamma[\phi] = S[\phi] - \frac{\hbar}{2i} \mathrm{Tr} \ln(\delta^A_C + D^{-1AB} V''_{BC}[\phi]) + O(\hbar^2)$ (2.37)

Pour cela, on avait commencé avec une action classique $S[\phi]$ quadratique, et imposé $\langle \phi^A \rangle^\circ = \frac{\delta W}{\delta \phi^A}\Big|_{J=0} = 0$.

Si on est pas dans ce cas, on peut reprendre à partir de (2.31) et on trouve au lieu de (2.37)

$$\boxed{\Gamma[\phi] = \mathcal{N} + S[\phi] - \frac{\hbar}{2i} \operatorname{Tr} \ln \frac{\delta^2 S}{\delta\phi^A_{(B)}\delta\phi^B_{(B)}} + \frac{\hbar}{i} \operatorname{Tr} \ln \frac{\delta^2 S}{\delta\phi^\alpha_{(F)}\delta\phi^\beta_{(F)}} + O(\hbar^2)}$$

avec $\mathcal{N}$ une constante indépendante des champs et le

$-\frac{1}{2} \to 1$ pour les champs fermioniques fermioniques complexes.

($(\mathrm{Det})^{-1/2} \to \mathrm{Det}$ dans l'integrale gaussienne)

Dans la jauge du champ de fond, on trouve

$$\Gamma_B[\tilde{\phi}, 0] = \tilde{\Gamma}_B[0, \tilde{\phi}, 0] =$$

$$= \mathcal{N} + S_B[0, \tilde{\phi}, 0]$$

$$- \frac{\hbar}{2i} \operatorname{Tr} \ln \frac{\delta^2 S_B}{\delta \phi^A_{(B)} \delta\phi^B_{(B)}} [0, \tilde{\phi}, 0] + \frac{\hbar}{i} \operatorname{Tr} \ln \frac{\delta^2 S_B}{\delta \phi^\alpha_{(F)}\delta\phi^\beta_{(F)}} [0, \hat{\phi}, 0]$$

$$+ O(\hbar^2)$$

Il faut donc trouver la partie quadratique en les champs quantiques de

$$S^{inv}[\phi + \tilde{\phi}] - \int d^n x \left\{ D_\mu^{A+\tilde{A}} (C+\tilde{C})^a (-\tilde{D}^\mu \bar{C}^b) + (B+\tilde{B})^a (\tilde{D}_\mu A^{\mu b} - \frac{\xi}{2} B^b) \right\} g_{ab}$$

Pour la partie fixation de jauge, on trouve en mettant $\tilde{C}^a = 0 = \tilde{B}^a$

$$\int d^n x \left\{ - \tilde{D}^\mu \bar{C}^a \tilde{D}_\mu C^b - B^a \tilde{D}_\mu A^{\mu b} + \frac{\xi}{2} B^a B^b \right\} g_{ab}$$

ce qui est la même chose que l'on avait trouvé précedemment, avec la substitution $\partial_\mu \to \tilde{D}_\mu$.

7.8 Effective action in Chern-Simons theory

5.7 Action effective à une boucle en théorie de Chern-Simons

Monday, November 20, 2017 4:13 PM

Supposons un groupe de Lie compact avec $g_{ab} = \delta_{ab}$ en Euclidien:

$$Z[J] = \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar}(S[\phi]+J\phi)} = e^{-\frac{1}{\hbar}W[J]}$$

La partie quadratique en les champs quantiques de

$$\int \tfrac{1}{2} \langle A+\tilde{A}, d(A+\tilde{A}) + \tfrac{2}{3}(A^2 + A\tilde{A} + \tilde{A}A + \tilde{A}^2) \rangle$$

$$= \int \left\{ \tfrac{1}{2} \langle \tilde{A}, d\tilde{A} + \tfrac{2}{3}\tilde{A}^3 \rangle + \langle A, \underbrace{d\tilde{A} + \tilde{A}^2}_{\tilde{F}} \rangle + \boxed{\tfrac{1}{2}\langle A, \tilde{D} A \rangle} + \tfrac{1}{2}\langle A, \tfrac{2}{3}A^3 \rangle \right\}$$

$\tilde{F}$ (=0 si la connection $\tilde{A}$ est plate)

$\boxed{}$: $\frac{1}{2}\left(\langle A, dA\rangle + \frac{2}{3}\langle A, A\tilde{A} + \tilde{A}A\rangle + \frac{2}{3}\langle \tilde{A}, A^2\rangle\right) = \frac{1}{2}\langle A, dA + \{\tilde{A}, A\}\rangle$

$\Rightarrow$ la nouvelle partie quadratique est donc

$$\mathcal{D}^{\tilde{A}}_{bos} : \left(\begin{array}{c|c} \epsilon^{\mu\nu\rho} \tilde{D}^y_\rho \, \delta^3(x-y) & -\tilde{D}^\mu_x \, \delta^3(x-y) \\ \hline -\tilde{D}^\nu_y \, \delta^3(x-y) & -\xi \, \delta^3(x-y) \end{array} \right) \delta_{ab}$$

Prenons $\xi = 0$ et supposons $\tilde{F}_{\mu\nu} = 0$.

$$\mathcal{D}^{\tilde{A}}_{gh} : \quad \tilde{D}^\mu_x \tilde{D}^y_\mu \, \delta^3(x,y) \, \delta_{ab}$$

On a $\tilde{\Gamma}^{(1)}_{\tilde{B}}[0, \tilde{\phi}, 0] \Big|_{\tilde{c}=0=\tilde{B}} = -\frac{1}{2} \mathrm{Tr} \ln \mathcal{D}^{\tilde{A}}_{bos} + \mathrm{Tr} \ln \mathcal{D}^{\tilde{A}}_{gh}$

<u>mais</u>: $\mathcal{D}^{\tilde{A}}_{bos} \, \mathcal{D}^{\tilde{A}}_{bos}$

$$= \int d^3y \left(\begin{array}{c|c} -\epsilon^{\mu}{}_{\nu}{}^{\rho}\tilde{D}^{x}_{\rho}\,\delta^3(x-y) & -\tilde{D}^{\mu}_{x}\,\delta^3(x-y) \\ \hline +\tilde{D}^{x}_{\nu}\,\delta^3(x-y) & 0 \end{array}\right)\left(\begin{array}{c|c} \epsilon^{\nu}{}_{\lambda}{}^{\sigma}\tilde{D}^{z}_{\sigma}\,\delta^3(y-z) & +\tilde{D}^{\nu}_{z}\,\delta^3(y-z) \\ \hline -\tilde{D}^{z}_{\lambda}\,\delta^3(y-z) & 0 \end{array}\right)\delta^{a}_{c}$$

$$= \left(\begin{array}{c|c} \delta^{\mu}_{\lambda}\tilde{D}^{x}_{\rho}\tilde{D}^{\rho}_{z}\,\delta^3(x-z) - \tilde{D}^{x}_{\lambda}\tilde{D}^{\tilde{A},\mu}_{z}\cancel{\delta^3(x-z)} + \tilde{D}^{\mu}_{x}\cancel{\tilde{D}^{z}_{\lambda}}\,\delta^3(x-z) & 0 \\ \hline 0 & \tilde{D}^{x}_{\rho}\tilde{D}^{\rho}_{z}\,\delta^3(x-z) \end{array}\right)\delta^{a}_{c}$$

$$-\tilde{D}^{x}_{\lambda}\tilde{D}^{\mu}_{z}\,\delta^3(x-z) + \tilde{D}^{\mu}_{x}\tilde{D}^{z}_{\lambda}\,\delta^3(x-z) = \tilde{D}^{x}_{\lambda}\tilde{D}^{\mu}_{x}\,\delta^3(x-z) - \tilde{D}^{\mu}_{x}\tilde{D}^{x}_{\lambda}\,\delta^3(x-z)$$

$$= \tilde{F}_{\lambda}{}^{\mu}\,\delta^3(x-z) = 0$$

$\det M = e^{\mathrm{Tr}\ln M} \Rightarrow \ln\det M = \mathrm{Tr}\ln M$

On peut donc écrire $-\frac{1}{2}\mathrm{Tr}\ln \mathcal{D}^{\tilde{A}}_{bos} = -\frac{1}{4}\mathrm{Tr}\ln \mathcal{D}^{\tilde{A}\,2}_{bos} = -\frac{1}{4}\ln\det(\mathcal{D}^{\tilde{A}}_{bos})^2$

$$= -\frac{1}{4}\ln\det\left(\tilde{D}^{x}_{\rho}\tilde{D}^{\rho}_{z}\,\delta^3(x-z)\delta^{a}_{c}\right)^4 = -\ln\det\tilde{D}^{x}_{\rho}\tilde{D}^{\rho}_{z}\,\delta^3(x-z)\delta^{a}_{c} = -\mathrm{Tr}\ln\tilde{D}^{x}_{\rho}\tilde{D}^{\rho}_{z}\,\delta^3(x-z)\delta^{a}_{c}$$

et $\quad \mathrm{Tr}\ln\mathcal{D}^{\tilde{A}}_{gh} = \mathrm{Tr}\ln\tilde{D}^{x}_{\rho}\tilde{D}^{\rho}_{z}\,\delta^3(x-z)\delta^{a}_{c}$

Il y a donc cancellation entre les contributions à une boucle des bosons de jauge et des fantômes de fixation de jauge. Cette cancellation est valable à la fois pour la partie divergente et la partie finie.

7.9 Structure of Yang-Mills divergences in background field gauge

Pour les divergences à une boucle de la théorie de Yang-Mills dans la jauge du champ de fond, il faut extraire la partie divergeante de $\Gamma^{(1)}_B[\tilde{\phi}] = \tilde{\Gamma}^{(1)}_B[0,\tilde{\phi},0]$

Or $\tilde{\Gamma}^{(1)}_B[0,\tilde{\phi},0] = -\frac{1}{2i}\,\mathrm{Tr}\ln \left.\frac{\delta^2 S_B}{\delta\phi^A_{(B)}\delta\phi^B_{(B)}}\right|_{0,\tilde{\phi},0} + \frac{1}{i}\mathrm{Tr}\ln \left.\frac{\delta^2 S_B}{\delta\phi^\alpha_{(F)}\delta\phi^\beta_{(F)}}\right|_{0,\tilde{\phi},0}$

On peut montrer (discussion après (3.30)) que les divergences à une boucle sont

1) les intégrales de polynômes en les champs

$\tilde{\phi}^A = \tilde{A}^a_\mu, \tilde{\psi}^\ell, \tilde{\bar{\psi}}^\ell, \tilde{B}^a, \tilde{\bar{C}}^a, \tilde{C}^a$ et leurs dérivées

2) la dimension canonique des intégrands est ≤ 4

si $[\tilde{A}^a_\mu] = 1$, $[\tilde{\psi}^\ell] = [\tilde{\bar{\psi}}^\ell] = \frac{3}{2}$, $[\partial_\mu] = 1$, $[\tilde{C}^a] = 0$, $[\tilde{\bar{C}}^a] = 2 = [\tilde{B}^a]$

et idem pour les ϕ^A.

[on ne considère que des fermions $\mathcal{L}_M = -\bar{\psi}(\partial\!\!\!/ + m)\psi$

$\mathcal{L}_{inv} = -\frac{1}{4} F^a_{\mu\nu}F^{\mu\nu}_a - \sum_{\ell=1}^{N} \bar{\psi}^\ell(D\!\!\!\!/ + m)\psi^\ell$

(3) $S_B[\phi,\tilde{\phi},0]$ est de nombre de fantôme zéro, comme ceci peut être implémenté comme une symétrie linéaire, il en est

de même pour $\tilde{\Gamma}_B[\phi,\tilde{\phi},0]$ et $\tilde{\Gamma}_B[0,\tilde{\phi},0]$

(4) $S_B[\phi,\tilde{\phi},0]$ est invariant de Lorentz, donc aussi $\tilde{\Gamma}_B[\phi,\tilde{\phi},0]$, $\tilde{\Gamma}_B[0,\tilde{\phi},0]$

(5) $S_B[\phi,\tilde{\phi},0]$ est invariant sous les transformations de jauge de fond $\delta_f \tilde{\phi}^\alpha, \delta_f \phi^\alpha$, donc aussi $\tilde{\Gamma}_B[\phi,\tilde{\phi},0]$ et $\tilde{\Gamma}_B[0,\tilde{\phi},0]$

transformation linéaire et homogène

6) On fait l'hypothèse que le groupe est simple et que les fermions appartiennent à une représentation irréductible ($g_{ab} = \delta_{ab}$)

7) $\frac{\delta^2 S_B}{\delta\phi\,\delta\phi}$ ne fait pas intervenir $\tilde{B}$ ni $\tilde{C}$. (on peut le vérifier sur l'expression explicite) $\Rightarrow$ il n'apparaissent pas non plus dans $\tilde{\Gamma}_B^{(1)}[0,\tilde{\phi},0]_{div}$. Puisque le nombre de fantômes $=0$, $\bar{C}$ ne peut pas apparaître non plus. On reste donc avec les champs de fond $\tilde{A}^a_\mu, \tilde{\bar{\psi}}^\ell, \tilde{\psi}^\ell$.

—

La seule fonctionnelle qui satisfait toutes ces contraintes est

$$\tilde{\Gamma}_B^{(1)}[0,\tilde{\phi},0]_{div} = \int d^4x \left[-\frac{1}{4} L_A \tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu}_a - L_\psi \tilde{\bar{\psi}}\gamma^\mu \tilde{D}_\mu \tilde{\psi} - m L_m \tilde{\bar{\psi}}\tilde{\psi}\right]$$

avec L_A, L_ψ, L_m des constantes sans dimension.

Puisque $S_B[0,\tilde{\phi},0] = \int d^4x \left[-\frac{1}{4} \tilde{F}^a_{\mu\nu} \tilde{F}_a^{\mu\nu} - \tilde{\bar{\psi}} \gamma^\mu D^{\hat{A}}_\mu \tilde{\psi} - m \tilde{\bar{\psi}} \tilde{\psi} \right]$,

on peut définir des champs de fonds renormalisés

$$\tilde{A}^{aR}_\mu = \sqrt{1+\hbar L_A}\,, \quad \tilde{\psi}^{R\ell} = \sqrt{1+\hbar L_\psi}\, \tilde{\psi}^\ell .$$

On a alors $S_B[0,\tilde{\phi},0] + \hbar \tilde{\Gamma}^{(1)}_B[0,\tilde{\phi},0]_{div}$

$$= \int d^4x \left[-\frac{1}{4} \tilde{F}^{aR}_{\mu\nu} \tilde{F}^{\mu\nu R}_a - \tilde{\bar{\psi}}^R \gamma^\mu D^{\hat{A},R}_\mu \tilde{\psi}^R - m^R \tilde{\bar{\psi}}^R \psi^R \right] + O(\hbar^2)$$

avec $m_R = \dfrac{m(1+\hbar L_m)}{(1+L_\psi)}$

$$\tilde{F}^{aR}_{\mu\nu} = \partial_\mu \tilde{A}^{aR}_\nu - \partial_\nu \tilde{A}^{aR}_\mu + g^R f^a{}_{bc} \tilde{A}^{bR}_\mu \tilde{A}^{cR}_\nu$$

avec $\boxed{g^R = \dfrac{g}{\sqrt{1+\hbar L_A}}}$

$$D^{\hat{A},R}_\mu \tilde{\psi}^R = \partial_\mu \tilde{\psi}^R + g_R \tilde{A}^{Ra}_\mu T_a \tilde{\psi}^R .$$

Dans la jauge du champ de fond, la renormalisation de la constante de couplage est déterminée par celle du champ de fond $\tilde{A}^a_\mu$.

7.10 Yang-Mills coupling constant at 1 loop

5.9 Constante de couplage à une boucle

Wednesday, November 22, 2017 21:15

Pour avoir accès à L_A, on peut calculer $\tilde{\Gamma}^{(1)}_B[0,\tilde{\Phi},0]$ avec $\tilde{A}^a_\mu$ constant et en annulant $\tilde{\psi}, \tilde{\bar{\psi}}, \tilde{B}, \tilde{C}$. Dans ce cas $\tilde{\Gamma}^{(1)}_B[0,\tilde{\Phi},0]_{div}$ se réduit à $\int d^4x \, -\frac{1}{4} Tr[\tilde{A}_\mu, \tilde{A}_\nu][\tilde{A}^\mu, \tilde{A}^\nu] L_A$

Or la partie quadratique en les champs quantiques, après avoir intégré sur B^a, et en utilisant

$$\partial_\mu(A_\nu+\tilde{A}_\nu) - \partial_\nu(A_\mu+\tilde{A}_\mu) + [A_\mu+\tilde{A}_\mu, A_\nu+\tilde{A}_\nu] = \tilde{F}_{\mu\nu} + \tilde{D}_\mu A_\nu - \tilde{D}_\nu A_\mu + [A_\mu, A_\nu]$$

$$S^2_B[\phi,\tilde{\Phi},0] = \int d^4x \, Tr\Big[-\frac{1}{4}(\tilde{D}_\mu A_\nu - \tilde{D}_\nu A_\mu)(\tilde{D}^\mu A^\nu - \tilde{D}^\nu A^\mu) - \frac{1}{2}\tilde{F}_{\mu\nu}[A^\mu, A^\nu]$$

$$- \bar{\psi}(\tilde{\not{D}}+m)\psi - \frac{1}{2\xi}(\tilde{D}_\mu A^\mu)^2 - \tilde{D}^\mu \bar{C}\, \tilde{D}_\mu C\Big]$$

$$= -\frac{1}{2}\int d^4x d^4y \, A^a_\mu(x)\, \mathcal{D}^{A\mu\nu}_{ab}(x,y)\, A^b_\nu(y) - \int d^4x d^4y \, \bar{\psi}^l(x)\, \mathcal{D}^\psi_{lk}(x,y)\, \psi^k(y)$$

$$- \int d^4x d^4y \, \bar{C}^a(x)\, \mathcal{D}^{gh}_{ab}(x,y)\, C^b(y) \quad ;$$

$$\tilde{a}^a_{\mu b} = g f^a{}_{cb} \tilde{A}^c_\mu$$

$$\mathcal{D}^{A\mu\nu}_{ab}(x,y) = \Big\{ \eta^{\mu\nu}\Big[\delta^d_a (-)\frac{\partial}{\partial x^\lambda} + \tilde{a}^d_{\lambda a}(x)\Big]\Big[\delta_{db}(-)\frac{\partial}{\partial y_\lambda} + \tilde{a}^\lambda_{db}(y)\Big]$$

$$- \Big[\delta^d_a (-)\frac{\partial}{\partial x_\nu} + \tilde{a}^d_{\nu a}(x)\Big]\Big[\delta_{db}(-)\frac{\partial}{\partial y_\mu} + \tilde{a}^\mu_{db}(y)\Big]$$

$$+ \tilde{F}^{\mu\nu}{}_c(x)\, g f^c{}_{ab} +$$

$$+ \frac{1}{\xi}\Big[\delta^d_a (-)\frac{\partial}{\partial x_\mu} + \tilde{a}^{\mu d}{}_a(x)\Big]\Big[\delta_{db}(-)\frac{\partial}{\partial y_\nu} + \tilde{a}^\nu{}_{db}\Big]\Big\} \delta^4(x,y)$$

$$\mathcal{D}^\psi_{lk}(x,y) = \Big(-\gamma^\mu \frac{\partial}{\partial y^\mu} + \tilde{\not{A}}^a(y)\, g T_a + m\Big)_{lk} \delta^4(x,y)$$

$$\mathcal{D}^{gh}_{ab}(x,y) = [\delta^d_a(\cdot)\frac{\partial}{\partial x^\lambda} + \tilde{a}^{\ d}_{\lambda\ a}(x)][\delta_{de}(\cdot)\frac{\partial}{\partial y_\lambda} + \tilde{a}^{\lambda}_{\ de}(y)]\ \delta^4(x,y)$$

avec $\tilde{A}^a_\mu$ constant (on ne met pas de bar supplémentaire)

on passe facilement en transformée de Fourier :

$$\mathcal{D}^{\cdot\cdot}_{\cdot\cdot}(x,y) = \frac{1}{(2\pi)^4}\int d^4p\, e^{ip(x-y)}\, \tilde{\mathcal{D}}^{\cdot\cdot}_{\cdot\cdot}(p)$$

$$\tilde{\mathcal{D}}^{g}_{ab}(p) = (-ip_\lambda\delta^d_a + \tilde{a}_{\lambda\ a}^{\ d})(ip^\lambda\delta_{de} + \tilde{a}^\lambda_{\ de})$$

$$\tilde{\mathcal{D}}^{\psi}_{ke}(p) = (i\not{p} + \tilde{\not{A}}^a g T_a + m\,\delta)_{ke}$$

$$\tilde{\mathcal{D}}^{A\,\mu\nu}_{ab}(p) = \eta^{\mu\nu}[-ip_\lambda\delta^d_a + \tilde{a}_{\lambda\ a}^{\ d}][ip^\lambda\delta_{de} + \tilde{a}^\lambda_{\ de}]$$
$$- [-ip^\nu\delta^d_a + \tilde{a}^{\nu\ d}_{\ a}][ip^\mu\delta_{de} + \tilde{a}^\mu_{\ de}]$$
$$+ \tilde{F}^{\mu\nu}_{\ \ c}\, g f^c_{\ ab}$$
$$+ \frac{1}{\xi}[-ip^\mu\delta^d_a + \tilde{a}^{\mu\ d}_{\ a}][ip^\nu\delta_{de} + \tilde{a}^\nu_{\ de}]$$

En TF le ln des matrices infinies devient alors le ln

des matrices habituelles, la trace $\int d^4x\,(\ )_{y=x}$ donne

$\int d^4x = (2\pi)^4\delta(0)$ et on trouve

$$\tilde{\Gamma}^{(1)}_B[0, \tilde{A}^a_{\mu\ cte}, 0] = \frac{1}{i}\delta(0)\int d^4p\left[-\frac{1}{2}\operatorname{tr}\ln\tilde{\mathcal{D}}^A(p) + \operatorname{tr}\ln\tilde{\mathcal{D}}^\psi(p) + \operatorname{tr}\ln\tilde{\mathcal{D}}^{gh}(p)\right]$$

Pour avoir accès à C_A, il faut isoler le terme quartique de en $\tilde{A}$.

$$f(\tilde{A})\Big|_{\tilde{A}^n} = \frac{1}{n!}\frac{\partial^n}{\partial(\lambda)^n} f(\lambda\tilde{A})\Big|_{\lambda=0}$$

si $\tilde{\mathcal{D}}(\lambda\tilde{A}) = \tilde{\mathcal{D}}_0 + \tilde{\mathcal{D}}_1 + \tilde{\mathcal{D}}_2$, $\tilde{\mathcal{D}}_i$ contenant iA_i , alors $\tilde{\mathcal{D}}'''(\lambda\tilde{A}) = 0$.

$$\left(\ln \tilde{\mathcal{D}}(\tilde{A})\right)\Big|_{\tilde{A}^4} = \frac{1}{4!}\frac{\partial^4}{(\partial\lambda)^4}\left[\ln \tilde{\mathcal{D}}(\lambda\tilde{A})\right] = \frac{1}{4!}\frac{\partial^3}{\partial\lambda^3}\left[\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\right]$$

$$= \frac{1}{4!}\frac{\partial^2}{(\partial\lambda)^2}\left[-\left(\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\right)^2 + \tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}''(\lambda\tilde{A})\right]$$

$$= \frac{1}{4!}\frac{\partial}{\partial\lambda}\Big[-2\,\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\left[(-)\left[\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\right]^2 + \tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}''(\lambda\tilde{A})\right]$$

$$- \tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\,\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}''(\lambda\tilde{A})\Big]$$

$$= \frac{1}{4!}\frac{\partial}{\partial\lambda}\left[2\left(\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\right)^3 - 3\,\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\,\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}''(\lambda\tilde{A})\right]\Big|_{\lambda=0}$$

$$= \frac{1}{4!}\Big[6\left(\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\right)^2\left[-\left(\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\right)^2 + \tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}''(\lambda\tilde{A})\right]$$

$$+3\left[\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\right]^2\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}''(\lambda\tilde{A})$$

$$-3\left(\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}''(\lambda\tilde{A})\right)^2 + 3\left[\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}'(\lambda\tilde{A})\right]^2\tilde{\mathcal{D}}^{-1}(\lambda\tilde{A})\,\tilde{\mathcal{D}}''(\lambda\tilde{A})\Big]\Big|_{\lambda=0}$$

$$= -\frac{1}{4}\left(\tilde{\mathcal{D}}_0^{-1}\,\tilde{\mathcal{D}}_1\right)^4 + \left(\tilde{\mathcal{D}}_0^{-1}\,\tilde{\mathcal{D}}_1\right)^2\tilde{\mathcal{D}}_0^{-1}\,\tilde{\mathcal{D}}_2 - \frac{1}{2}\left(\tilde{\mathcal{D}}_0^{-1}\,\tilde{\mathcal{D}}_2\right)^2 \qquad \Big|\ \text{NB: } \tilde{\mathcal{D}}''(\lambda A)\Big|_{\lambda=0} = 2\tilde{\mathcal{D}}_2$$

Prenons $\xi = 1$.
$$\tilde{\mathcal{D}}_0^{-1\,A\,ab}{}_{\mu\nu} = \delta^{ab}\,\eta_{\mu\nu}\,\frac{1}{p^2 - i\epsilon}$$

$$\tilde{\mathcal{D}}_0^{-1\,\psi\,kl} = \left(i\not{p} + m\right)^{-1}{}_{kl}$$

$$\tilde{\mathcal{D}}_0^{-1\,gh\,ab} = \delta^{ab}\,\frac{1}{p^2 - i\epsilon}$$

1) Contribution des bosons de jauge

$$\tilde{\mathcal{D}}_1{}^{A\,\mu\nu}{}_{ab} = -2\,i\,\eta^{\mu\nu}\,p\cdot\tilde{a}_{ab}$$

$$\tilde{\mathcal{D}}_2{}^{A\,\mu\nu}{}_{ab} = \left[\eta^{\mu\nu}\,\tilde{a}_\lambda^T\,\tilde{a}^\lambda - \tilde{a}^{\nu T}\,\tilde{a}^\mu + \tilde{a}^{\mu T}\,\tilde{a}^\nu\right]_{ab} + \tilde{F}^{\mu\nu}{}_c\,g f^c{}_{ab}$$

$\tilde{a}^T \cdot \tilde{a}$

$\tilde{a}^T = -\tilde{a}$

$$= [\eta^{\mu\nu}(-\tilde{a}_d\tilde{a}^d - \underbrace{[\tilde{a}^\mu, \tilde{a}^\nu]})]_{ab} + \tilde{F}^{\mu\nu}{}_c\, g f^c{}_{ab}$$

$$= -g f_{adc}\, \tilde{A}^{d\mu} g f^c{}_{fb} \tilde{A}^{f\nu} - (\mu \leftrightarrow \nu)\,)$$

$$= -g^2 f^c{}_{df} f_{acb} \tilde{A}^{d\mu}\tilde{A}^{f\nu}$$

$$= g^2 f^c{}_{ab} f_{cdf} \tilde{A}^{d\mu}\tilde{A}^{f\nu} = g f^c{}_{ab} \tilde{F}^{\mu\nu}{}_c$$

$$\Rightarrow ([\tilde{a}_\mu, \tilde{a}_\nu])^a{}_b = g f^a{}_{cb} \tilde{F}^c_{\mu\nu} \quad \text{for } \tilde{A} \text{ cte}$$

$$\Rightarrow \tilde{D}_2^{A\mu\nu}{}_{ab} = -\eta^{\mu\nu}[\tilde{a}^d\tilde{a}_d]_{ab} + 2\tilde{F}^{\mu\nu}{}_c\, g f^c{}_{ab}, \quad \tilde{A} = \underline{\text{cte}}.$$

$$\int d^4p \; tr\,[\tilde{D}_0^{-1A} D_2^A]^2 = \int d^4p \frac{4}{(p^2-i\epsilon)^2}\left\{ tr_G\,[\tilde{a}^d\tilde{a}_d\,\tilde{a}^\eta\tilde{a}_\eta] + \tilde{F}^{\mu\nu c}\tilde{F}^d_{\mu\nu} f_{cab} f_d{}^{ab} g^2 \right\}$$

$(4 = tr\,\delta^\mu_\nu)$ $\quad (tr_G f^c{}_{ab} = 0)$

$$= 4J\left[tr_G\, \tilde{a}^d\tilde{a}_d\tilde{a}^\eta\tilde{a}_\eta + \tilde{F}^{\mu\nu c}\tilde{F}^d_{\mu\nu} f_{cab} f_{dab}\, g^2 \right]$$

$$\boxed{J = \int d^4p \frac{1}{(p^2-i\epsilon)^2}}$$

$$\int d^4p \; tr\,[\tilde{D}_0^{-1A}\tilde{D}_1^A]^2\, \tilde{D}_0^{-1A}\tilde{D}_2^A = \int d^4p \frac{1}{(p^2-i\epsilon)^2 (p^2-i\epsilon)} tr_G\,[+4\cdot 4\,(p\cdot\tilde{a})^2(\tilde{a}^\mu\tilde{a}_\mu)]$$

mais $\int d^4p \, p_\mu p_\nu f(p^2) = \frac{1}{4}\eta_{\mu\nu}\int d^4p\, p^2 f(p^2)$

$$= 4J\, tr_G\,[\tilde{a}^d\tilde{a}_d\tilde{a}^\eta\tilde{a}_\eta]$$

$$\int d^4p\, [D_0^{-1A} D_1^A]^4 = \int d^4p \frac{4\cdot 4^2}{(p^2-i\epsilon)^4} tr_G\,(p\cdot\tilde{a})^4$$

mais

$$\int d^4p\ f(p^2)\, p_\alpha p_\mu p_\nu p_\sigma = \frac{1}{24}\left[\eta_{\alpha\mu}\eta_{\nu\sigma}+\eta_{\alpha\nu}\eta_{\mu\sigma}+\eta_{\alpha\sigma}\eta_{\mu\nu}\right]\int d^4p\,(p^2)^2 f(p^2)$$

$$\frac{4\cdot 4^2}{24} = \frac{8}{3}$$

$$= \frac{8}{3}\int \mathrm{tr}_G\left[2\,\tilde{a}^\alpha\tilde{a}_\alpha\tilde{a}^\eta\tilde{a}_\eta + \tilde{a}^\alpha\tilde{a}^\eta\tilde{a}_\alpha\tilde{a}_\eta\right]$$

On a donc

$$\boxed{\int d^4p\ \mathrm{tr}\ \ln \mathcal{D}^A(\tilde{A})\Big|_{\tilde{A}^4}} = \Big[\underbrace{-\frac{1}{4}\frac{16}{3} - \frac{1}{2}\cdot 4 + 4}_{=\frac{12-4-6}{3} = \frac{2}{3}}\Big]\int \mathrm{tr}_G\, \tilde{a}^\alpha\tilde{a}_\alpha\tilde{a}^\eta\tilde{a}_\eta$$

$$-2\int \tilde{F}^c_{\mu\nu}\tilde{F}^{\mu\nu d}g^2 f_{cab}f_d{}^{ab} - \frac{2}{3}\int \mathrm{tr}_G\,\tilde{a}^\alpha\tilde{a}^\eta\tilde{a}_\alpha\tilde{a}_\eta$$

$$= \frac{2}{3}\int \mathrm{tr}_G\big(\underbrace{\tilde{a}^\alpha\tilde{a}_\alpha\tilde{a}^\eta\tilde{a}_\eta - \tilde{a}^\alpha\tilde{a}^\eta\tilde{a}_\alpha\tilde{a}_\eta}_{\tilde{a}^\alpha\,[\tilde{a}_\alpha,\tilde{a}^\eta]\,\tilde{a}_\eta}\big) - 2\int \tilde{F}^c_{\mu\nu}\tilde{F}^{\mu\nu d} f_{cab}f_d{}^{ab} g^2$$

$$g\, f_{cab}\,\tilde{F}^c{}_{\alpha\eta}\,\tilde{a}^\eta\tilde{a}^\alpha$$

$$g\,\frac{1}{2} f_{cab}\,\tilde{F}^c{}_{\alpha\eta}\,[\tilde{a}^\eta,\tilde{a}^\alpha]$$

$$= g^2\,\frac{1}{2} f_{cab}\,\tilde{F}^c{}_{\alpha\eta}\, f_{dab}\,\tilde{F}^{d\alpha\eta}$$

$$\boxed{= -\frac{5}{3}\int g^2 f_{cab}f_d{}^{ab}\,\tilde{F}^c_{\mu\nu}\tilde{F}^{\mu\nu d}}$$

2) Contribution des fantômes

$$\tilde{\mathcal{D}}^{gh}_{ab}(p) = (-i p_\alpha \delta^d_a + \tilde{a}^d{}_{\alpha a})(i p^\alpha \delta_{db} + \tilde{a}^\alpha{}_{db})$$

$$\tilde{D}_{0\,ab}^{gh}(p) = \delta_{ab}(p^2 - i\epsilon) \quad , \quad \tilde{D}_0^{gh\,-1} = \delta_{ab}\frac{1}{p^2-i\epsilon}$$

$$\tilde{D}_{1\,ab}^{gh}(p) = -2i(p\cdot\tilde{a})_{ab} \quad , \quad \tilde{D}_{2\,ab}^{gh} = \tilde{a}^{d}{}_{da}\,\tilde{a}^{d}{}_{db} = -[\tilde{a}_d\,\tilde{a}^d]_{ab}$$

$$\int d^4p\ \mathrm{tr}\,[\tilde{D}_0^{-1\,gh}\tilde{D}_1^{gh}]^4 = \int d^4p\,\frac{1}{(p^2-i\epsilon)^4}\,2^4\ \mathrm{tr}\,(p_d\,\tilde{a}^d\,p_\mu\,\tilde{a}^\mu\,p_\nu\,\tilde{a}^\nu\,p_\sigma\,\tilde{a}^\sigma)$$

$$= \frac{16}{24}\int d^4p\,\frac{1}{(p^2-i\epsilon)^2}\,\mathrm{tr}\,[2\,\tilde{a}^d\tilde{a}_d\,\tilde{a}^\eta\tilde{a}_\eta + \tilde{a}^d\tilde{a}^\eta\tilde{a}_d\tilde{a}_\eta]$$

$$= \frac{2}{3}\,J\ \mathrm{tr}\,[2\,\tilde{a}^d\tilde{a}_d\tilde{a}^\eta\tilde{a}_\eta + \tilde{a}^d\tilde{a}^\eta\tilde{a}_d\tilde{a}_\eta]$$

$$\int d^4p\ \mathrm{tr}\,[\tilde{D}_0^{-1\,gh}\tilde{D}_2^{gh}]^2 = \int d^4p\,\frac{1}{(p^2-i\epsilon)^2}\ \mathrm{tr}\ \tilde{a}^d\tilde{a}_d\,\tilde{a}^\eta\tilde{a}_\eta$$

$$= J\ \mathrm{tr}\ \tilde{a}^d\tilde{a}_d\tilde{a}^\eta\tilde{a}_\eta$$

$$\int d^4p\ \mathrm{tr}\,[\tilde{D}_0^{-1\,gh}\tilde{D}_1^{gh}]^2[\tilde{D}_0^{-1\,gh}\tilde{D}_2^{gh}]$$

$$= \int d^4p\,\frac{1}{(p^2-i\epsilon)^3}\ \mathrm{tr}\,[(-2i)^2\,p_d\,\tilde{a}^d\,p_\mu\,\tilde{a}^\mu\,(-)\tilde{a}^\eta\tilde{a}_\eta]$$

$$= J\ \mathrm{tr}\ \tilde{a}^d\tilde{a}_d\,\tilde{a}^\eta\tilde{a}_\eta$$

$$\boxed{\int d^4p\ \mathrm{tr}\ \ln\tilde{D}^{gh}\Big|_{A^4}} = J\Big(-\frac{1}{6}\,\mathrm{tr}\,[2\,\tilde{a}^d\tilde{a}_d\,\tilde{a}^\eta\tilde{a}_\eta + \tilde{a}^d\tilde{a}_\eta\tilde{a}^d\tilde{a}_\eta]$$

$$-\frac{1}{2}\,\mathrm{tr}\,[\tilde{a}^d\tilde{a}_d\tilde{a}^\eta\tilde{a}_\eta]$$

$$-\frac{2}{6}-\frac{3}{6}+\frac{6}{6}=\frac{1}{6}$$

$$+\,\mathrm{tr}\,[\tilde{a}^d\tilde{a}_d\,\tilde{a}^\eta\tilde{a}_\eta]\Big)$$

$$= \frac{1}{6}\,J\ \mathrm{tr}\,[\tilde{a}^d\tilde{a}_d\tilde{a}^\eta\tilde{a}_\eta - \tilde{a}^d\tilde{a}^\eta\tilde{a}_d\tilde{a}_\eta]$$

$$= \frac{1}{6}\,J\ \mathrm{tr}\,[\tilde{a}^d[\tilde{a}_d,\tilde{a}_\eta]\tilde{a}^\eta] = -\frac{1}{12}\,J\,\mathrm{tr}\,[\tilde{a}_d,\tilde{a}_\eta][\tilde{a}^d,a^\eta]$$

$$= -\frac{1}{12}\,J\,g\,f^{a}{}_{cb}\,\tilde{F}^{c}{}_{d\eta}\,g\,f^{b}{}_{da}\,\tilde{F}^{dd\eta} = \boxed{\frac{1}{12}\,J\,g^2 f_{cab}\,f_d{}^{ab}\,\tilde{F}^{c}_{d\eta}\,\tilde{F}^{dd\eta}}$$

3) Contribution des champs de matière

$$\tilde{\mathcal{D}}^{\psi} = (i\not{p} + \not{A}^a T_{ag} + m\delta)_{ae} + i\epsilon$$

$$\frac{1}{i\not{p}+m-i\epsilon} = \frac{i\not{p}-m}{-p^2-m^2+i\epsilon} = \frac{-i\not{p}+m}{p^2+m^2-i\epsilon} = (\tilde{\mathcal{D}}^{\psi})^{-1}_0(p)$$

$$\left[\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1} \qquad \mathrm{tr}\,\gamma^\mu\gamma^\nu = \tfrac{1}{2}\mathrm{tr}\{\gamma^\mu,\gamma^\nu\} = 4\eta^{\mu\nu} \qquad \not{a}\gamma^\mu\gamma_\mu = \not{a}\cdot 4\cdot\mathbb{1} \right]$$

$$\tilde{\mathcal{D}}_1^{\psi}(p) = \tilde{\not{A}}^a T_{ag}\,, \qquad \tilde{\mathcal{D}}_2^{\psi} = 0$$

$$\int d^4p\ \mathrm{tr}\ \ln \tilde{\mathcal{D}}^{\psi}\Big|_{\tilde{A}^4} = -\frac{1}{4}\int d^4p \frac{1}{(p^2+m^2-i\epsilon)^4}\ \mathrm{tr}\left[(-i\not{p}+m)\not{A}^a T_{ag}\right]^4$$

On s'intéresse à la partie divergente

pour $m=0$: $\dfrac{k^3 k^4}{k^8} \sim \dfrac{1}{k}$ $\Rightarrow$ divergence logarithmique

$$\frac{\partial}{\partial m}\frac{1}{(i\not{p}+m)^4} = -4\frac{1}{(i\not{p}+m)^5} \Rightarrow \frac{k^3}{k^5} \quad \text{convergent.}$$

$\Rightarrow$ il suffit de prendre $m=0$ si on s'intéresse à la partie divergente

$$\boxed{\int d^4p\ \mathrm{tr}\ \tilde{\mathcal{D}}^{\psi}_{m=0}\Big|_{\tilde{A}^4} = -\frac{g^4}{4}\int d^4p\frac{1}{(p^2-i\epsilon)^4}\ \mathrm{tr}(\not{p}\not{A}^a T_a)^4}$$

$$= -\frac{g^4}{4}\int d^4p\frac{1}{(p^2-i\epsilon)^4}\ \mathrm{tr}\left[(T_aT_bT_cT_d)\tilde{A}^a_\mu\tilde{A}^b_\nu\tilde{A}^c_\lambda\tilde{A}^d_\rho\ \not{p}\gamma^\mu\not{p}\gamma^\nu\not{p}\gamma^\lambda\not{p}\gamma^\rho\right]$$

$$= -\frac{g^4}{96}\int \mathrm{tr}\ T_aT_bT_cT_d\ \tilde{A}^a_\mu\tilde{A}^b_\nu\tilde{A}^c_\lambda\tilde{A}^d_\rho \qquad = [\ \ ]$$

$$\cdot\ \mathrm{tr}\left[2\underbrace{\gamma^\sigma\gamma^\mu}_{}\underbrace{\gamma_\sigma}_{}\gamma^\nu\gamma^\eta\gamma^\lambda\gamma_\eta\gamma^\rho + \underbrace{\gamma^\sigma\gamma^\mu}_{}\gamma^\eta\gamma^\nu\gamma_\sigma\gamma^\lambda\gamma_\eta\gamma^\rho\right]$$

$$(2\{\gamma^\sigma,\gamma^\mu\} - 2\gamma^\mu\gamma^\sigma)\gamma_\sigma$$

$$\rightarrow \mathrm{tr}\left[4\gamma^\mu\gamma^\nu\gamma^\eta\gamma^\lambda\gamma_\eta\gamma^\rho - \underset{4}{\cancel{8}}\gamma^\mu\gamma^\nu\gamma^\eta\gamma^\lambda\gamma_\eta\gamma^\rho + 2\gamma^\eta\gamma^\nu\gamma^\mu\gamma^\lambda\gamma_\eta\gamma^\rho\right.$$

$$- \gamma^\mu\gamma^\sigma\gamma^\eta\gamma^\nu\gamma_\sigma\gamma^\alpha\gamma_\eta\gamma^\rho]$$

$$\to \mathrm{tr}[-4\gamma^\mu\gamma^\nu\gamma^\eta\gamma^\alpha\gamma_\eta\gamma^\rho + 2\gamma^\nu\gamma^\mu\gamma^\alpha\gamma_\eta\gamma^\rho\gamma^\eta - 2\gamma^\mu\gamma^\sigma\gamma_\sigma\gamma^\alpha\gamma^\nu\gamma^\rho$$
$$+ \gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\eta\gamma_\sigma\gamma^\alpha\gamma_\eta\gamma^\rho]$$

$$\to \mathrm{tr}[8\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\rho - 4\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\rho - 8\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\rho + 2\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\eta\gamma_\eta\gamma^\rho$$
$$- \gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\eta\gamma^\alpha\gamma_\sigma\gamma_\eta\gamma^\rho]$$

$$\to \mathrm{tr}[8\underline{\gamma^\mu\gamma^\nu\gamma^\alpha}\gamma^\rho - 4\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\rho - 8\cancel{\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\rho} + 8\cancel{\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\rho}$$
$$-2\gamma^\mu\gamma_\eta\gamma^\nu\gamma^\eta\gamma^\alpha\gamma^\rho + \gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\eta\gamma^\alpha\gamma_\eta\gamma_\sigma\gamma^\rho]$$
$$(-4+8) \qquad (+2-4)$$

$$\to \mathrm{tr}[8\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\rho - 4\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\rho + 4\cancel{\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\rho} - 2\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\alpha\gamma_\sigma\gamma^\rho]$$

$$\to -4\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\rho + 2\gamma^\mu\gamma^\sigma\gamma^\nu\gamma_\sigma\gamma^\alpha\gamma^\rho$$

$$\to -4\cancel{\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\rho}$$

$$\to \mathrm{tr}[8\cancel{\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\rho} - 4\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\rho - 4\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\rho]$$

$$\to -8\gamma^\nu\gamma^\rho\eta^{\mu\alpha} + 4\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\rho \qquad -8\eta^{\mu\alpha}\gamma^\nu\gamma^\rho + 4\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho$$

$$\to 8\gamma^\nu\gamma^\alpha\eta^{\mu\rho} - 4\gamma^\nu\cancel{\gamma^\alpha\gamma^\rho}\gamma^\mu \qquad 8\gamma^\mu\gamma^\nu\eta^{\alpha\rho} - 4\gamma^\mu\cancel{\gamma^\nu\gamma^\alpha}\gamma^\rho$$

$$\to -32\eta^{\nu\rho}\eta^{\mu\alpha} - 32\eta^{\mu\alpha}\eta^{\nu\rho} + 32\eta^{\nu\alpha}\eta^{\mu\rho} + 32\eta^{\mu\nu}\eta^{\alpha\rho}$$

$$\frac{64}{96} = \frac{8}{12} = \frac{2}{3}, \qquad \tilde{A}_\mu = \tilde{A}^a_\mu T_a$$

$$= -\frac{g^2}{96}\int \mathrm{tr}(T_a T_b T_c T_d)\tilde{A}^a_\mu \tilde{A}^b_\nu \tilde{A}^c_\alpha \tilde{A}^d_\rho [-64\eta^{\mu\alpha}\eta^{\nu\rho} + 32\eta^{\mu\rho}\eta^{\nu\alpha} + 32\eta^{\mu\nu}\eta^{\alpha\rho}]$$

$$= \frac{2}{3}g^2\int \mathrm{tr}[\tilde{A}^\alpha\tilde{A}^\eta\tilde{A}_\alpha\tilde{A}_\eta - \tilde{A}^\alpha\tilde{A}_\alpha\tilde{A}^\eta\tilde{A}_\eta] = \frac{g^2}{3}\int \mathrm{tr}[\tilde{A}^\alpha,\tilde{A}^\eta][\tilde{A}_\alpha,\tilde{A}_\eta]$$

$$= \frac{g^2}{3} \int \tilde{F}^{a\,\mu\nu} \tilde{F}^{b}_{\mu\nu} \operatorname{tr} T_a T_b$$

En sommant les contributions des potentiels de jauge avec facteur $-\frac{1}{2}$, des fantômes et des champs de matière, on trouve :

$$\tilde{\Gamma}^{(1)}_{A4}[0, \underbrace{\tilde{A}_\mu}_{\text{cte}}, 0]_{div} = \frac{\int g^2}{i(2\pi)^4} \underbrace{\int d^4x}_{\delta^4(0)} \tilde{F}^{c}_{\mu\nu} \tilde{F}^{\mu\nu d} \left\{ f_{cab} f_d{}^{ab} \left(\frac{5}{6} + \frac{1}{12}\right) + \frac{1}{3} \operatorname{tr} T_c T_d \right\}$$

· dans le cas d'un groupe simple, il n'y a qu'un seul tenseur symétrique à 2 indices invariant, proportionnel à la métrique de Killing qui peut être diagonalisée, dans cette base

$$f^a{}_{cb} f^b{}_{da} = tr(ad(t_c)\, ad(t_d)) = -C_1 \delta_{cd} \quad \text{avec } C_1 \text{ un}$$

entier qui caractérise le groupe simple ; donc

$$f_{cab} f_d{}^{ab} = C_1 \delta_{cd}$$

de même, $\operatorname{tr} T_c T_d = -C_2 \delta_{cd}$ pour une représentation irréductible des champs de matière, avec C_2 qui caractérise la représentation

Ex : pour $SU(2)$: $\quad \sigma_a \sigma_b = \delta_{ab} \mathbb{1} + i \epsilon_{ab}{}^c \sigma_c$

$\mathrm{tr}\, \tau_a \tau_b = 2\delta_{ab}$ $\quad t_a = -\frac{i}{2}\tau_a$, $\quad \boxed{[t_a, t_b] = -\frac{1}{4}(2i)\, \epsilon_{ab}{}^{c}\tau_c = \epsilon_{ab}{}^{c} t_c}$

$\mathrm{tr}\, t_a t_b = -\frac{1}{2}\delta_{ab}$

$\epsilon_{cab}\,\epsilon_d{}^{ab} = 2\,\delta_{cd} \Rightarrow C_1 = 2$

pour les champs de matière dans la représentation fondamentale $C_2 = \frac{1}{2}$

Plus généralement pour $SU(N)$, avec n_f fermions dans la représentation fondamentale, $\quad C_1 = N$, $\quad C_2 = \frac{n_f}{2}$.

$$\Rightarrow \tilde{\Gamma}^{(1)}_{A^4}[0, \tilde{A}_{cte}, 0]_{div} = \delta^4(0)\, \frac{1}{i} \mathcal{J} g^2\, \tilde{F}^a_{\mu\nu}\, \tilde{F}^{\mu\nu}_a \left[\frac{11}{12} C_1 - \frac{1}{3} C_2\right]$$

descendu avec δ_{ab}

$$L_A = \frac{4i g^2}{(2\pi)^4}\, \mathcal{J} \left(\frac{11}{12} C_1 - \frac{1}{3} C_2\right)$$

$$\mathcal{J} = i \int_0^\infty dK\, 2\pi^2\, \frac{K^3}{K^4}$$

Euclidien intégrale sur les angles

→ cut-off en 0 (infrarouge) et en ∞ (ultraviolet) :

$$\mathcal{J} = i \int_\mu^\Lambda dK\, 2\pi^2\, \frac{1}{K} = i\, 2\pi^2 \ln\frac{\Lambda}{\mu}$$

$$L_A = -\frac{g^2}{2\pi^2}\left(\frac{11}{12} C_1 - \frac{1}{3} C_2\right) \ln\frac{\Lambda}{\mu}$$

$$\Rightarrow g_R = g\left[1 - \hbar\, \frac{g^2}{2\pi^2}\left(\frac{11}{12} C_1 - \frac{1}{3} C_2\right) \ln\frac{\Lambda}{\mu}\right]^{-1/2}$$

$$\boxed{g_R = g\left(1+\hbar\,\frac{g^2}{4\pi^2}\left(\frac{11}{12}C_1-\frac{1}{3}C_2\right)\ln\frac{\Lambda}{\mu}\right)+O(\hbar^2)}$$

NB: En régularisation dimensionelle :

$$J = i\int_0^\infty \frac{2\pi^2\, q^{d-1}\, dq}{(q^2+\mu^2)^2}$$

où $d\to 4$ est la dimension complexe
et μ est un cut-off infra rouge

si $\mathrm{Re}(d)<0$ et $\mu^2>0$

$$J = i\pi^2\left(\frac{d}{2}-1\right)\mu^{d-4}\,\pi \Big/ \sin\left[\left(\frac{d}{2}-2\right)\pi\right]$$

$$\mu^{d-4} = e^{(d-4)\ln\mu} = 1+(d-4)\ln\mu+\ldots$$

la prolongation analytique donne

$$J = -2i\pi^2\left[\frac{1}{d-4}+\ln\mu+\mu\text{ independent}\right].$$

7.11 QED and QCD beta functions

5.10 Fonctions beta en QED et QCD

Monday, November 27, 2017 19:56

La constante de couplage renormalisée est donnée par

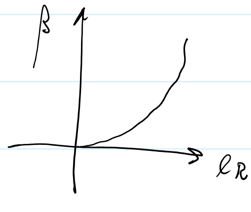
$$g_R = g\left(1 + \hbar \frac{g^2}{4\pi^2}\left(\frac{11}{12} C_1 - \frac{1}{3} C_2\right) \ln \frac{\Lambda}{\mu}\right) + O(\hbar^2)$$

- En électrodynamique quantique $C_1 = 0$ (groupe de jauge abélien)

$C_2 = 1$, 1 type de fermions = électrons

$$\Rightarrow \quad e_R = e\left(1 - \hbar \frac{e^2}{12\pi^2} \ln \frac{\Lambda}{\mu}\right) + O(\hbar^2)$$

Les corrections radiatives affaiblissent la charge électrique renormalisée par rapport à la charge électrique nue. Le nuage d'électrons virtuels font écran.



La fonction $\beta(e_R) = -\Lambda \frac{\partial}{\partial \Lambda} e_R(g, \Lambda) = \hbar \frac{e^3_R}{12\pi^2} + O(\hbar^2) > 0$

La constante de couplage renormalisée ↗ à haute énergie, la théorie des perturbations n'est plus valable ("pôle de Landau").

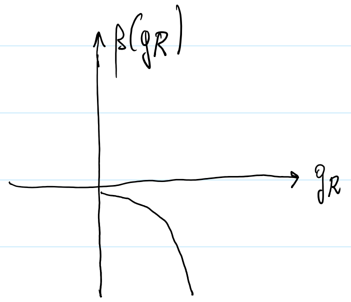
- Pour les théories de jauge non-abéliennes, les corrections radiatives renforcent la constante de couplage renormalisée par rapport à la constante de couplage nue si $C_2 < \frac{11}{4} C_1$ (anti-écrantage par les potentiels de jauge et les fantômes).

Pour $SU(N)$, avec n_f fermions dans la représentation fondamentale, c'est le cas si $\frac{n_f}{2} < \frac{11}{4} N \iff n_f \leq 5N$.

C'est le cas en QCD : $N=3$ $n_f = 6$ (quarks u, c, t, d, s, b).

La fonction

$$\beta(g_R) = -\Lambda \frac{\partial}{\partial \Lambda} g_R(g, \Lambda) = \frac{g_R^3}{4\pi^2} \underbrace{\left(\frac{n_f}{6} - \frac{11}{12} N \right)}_{<0 \text{ en QCD}} + O(\hbar^2)$$



Les quarks deviennent faiblement couplés à haute énergie, on parle de "liberté asymptotique".

7.12 Hilbert space structure

7.12.1 Generalities

7.12.1.1 Constrained Hamiltonian systems

7.12.1.2 Classical Hamiltonian BRST formalism

7.12.1.3 From Hamiltonian to Lagrangian path integral

7.12.1.4 Operator quantization

7.12.2 Dirac-Fock quantization

7.12.2.1 Operator formalism

7.12.2.2 Path integral implementation

7.12.3 Application to Yang-Mills and Chern-Simons theories

7.12.3.1 Hamiltonian formulation

7.12.3.2 Mode expansions

7.12.3.3 Partition functions

Chapter 8

Chiral anomalies

cf. [25], chapitre 22, [47] chapitre 5.

Les anomalies apparaissent quand les symétries de la théorie ne peuvent être maintenues au niveau quantique due à la renormalisation. En effet, la procédure de régularisation peut violer la symétrie et des traces de cette violation peuvent subsister même après avoir enlevé le régulateur à la fin du calcul. On a déjà vu un exemple avec l'équation de Callan-Symanzik où le comportement de la théorie sous une dilatation est affecté par la renormalisation.

8.1 Transformation chirale

Considérons, en Euclidien, $x^a = (x^i, ix^0)$, les fonctions de Green que l'on obtient en théorie de Yang-Mills en intégrant sur les fermions,

$$Z[A_\mu] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{\int d^4x_E \, -\bar{\psi}\not{D}\psi} = \mathrm{Det}\, \not{D} \,, \tag{8.1}$$

Les matrices $\gamma_a = (\gamma_i, i\gamma^0)$ sont hermitiennes et les générateurs T_a antihermitiens. Pour rappel, $\bar{\psi} = \psi^\dagger \gamma_4$, $\gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4$ satisfait $\{\gamma_5, \gamma^a\} = 0$, $\gamma_5^2 = 1$, $\mathrm{Tr}\gamma_5 = 0$, $\gamma_5^\dagger = \gamma_5$.

On considère un changement de variables, $\psi(x) \to U(x)\psi(x)$. La mesure se transforme comme

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to \mathcal{D}\bar{\psi}\mathcal{D}\psi(\mathrm{Det}\,\bar{\mathcal{U}}\mathrm{Det}\,\mathcal{U})^{-1}, \tag{8.2}$$

$$\mathcal{U}_{xn,ym} = U(x)_{nm}\delta^4(x,y), \quad \bar{\mathcal{U}}_{xn,ym} = (\gamma_4 U(x)^\dagger \gamma_4)_{nm}\delta^4(x-y) \tag{8.3}$$

et les indices n, m couvrent à la fois les indices de saveurs et les indices spinoriels. Pour une transformation unitaire non chirale,

$$U(x) = \exp \beta t, \tag{8.4}$$

avec t une matrice antihermitienne dans l'espace des saveurs et β un paramètre arbitraire, on trouve

$$\mathcal{U}\bar{\mathcal{U}} = \mathbf{1} \tag{8.5}$$

et la mesure est invariante. Pour une transformation chirale,

$$U(x) = \exp \gamma_5 \alpha t, \tag{8.6}$$

avec, de nouveau, t une matrice antihermitienne dans l'espace des saveurs et α un paramètre arbitraire, on trouve

$$\bar{\mathcal{U}} = \mathcal{U}. \tag{8.7}$$

La mesure n'est donc pas invariante sous la transformation chirale mais

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to \mathcal{D}\bar{\psi}\mathcal{D}\psi(\mathrm{Det}\,\mathcal{U})^{-2}. \tag{8.8}$$

Pour une transformation infinitésimale de paramètre α, on a

$$[\mathcal{U} - \mathbf{1}]_{nx,my} = \alpha[\gamma_5 t]_{nm}\delta^4(x-y). \tag{8.9}$$

Utilisant $\mathrm{Det} M = \exp Tr \ln M$ et $\ln(1+x) = x + O(x^2)$, on trouve

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \int d^4x_E\, \alpha(x)\mathcal{A}^E(x), \tag{8.10}$$

$$\mathcal{A}^E(x) = -2\mathrm{tr}(\gamma_5 t)\delta^4(x-x), \tag{8.11}$$

la trace se faisant à la fois dans l'espace spinoriel et dans l'espace interne.

Dans l'intégrale de chemin on a $\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp -\frac{1}{\hbar}S_L^E$, et la non invariance de la mesure et une action invariante est équivalent à une mesure invariante avec un Lagrangien non invariant se transformant comme $\mathcal{L}^E \to \mathcal{L}^E - \alpha\hbar\mathcal{A}^E(x)$.

8.2 Calcul par fonction zeta

Pour calculer $\mathcal{A}^E(x)$ il faut régulariser car $\mathcal{A}^E(x) = 0 \times \infty$ car la trace spinorielle vaut zéro. Une manière de calculer $\mathcal{A}^E(x)$ est de calculer le Jacobien en utilisant la définition régularisée des déterminants donnée par la fonction zeta. Tout d'abord, pour l'opérateur antihermitien $\not{D}$, on définit

$$\mathrm{Det}\,\not{D} = \exp(-\frac{1}{2}\zeta'_{-\not{D}^2}(0)), \tag{8.12}$$

ce qui se justifie par $(\mathrm{Det}\,\not{D})(\mathrm{Det}\,\not{D})^* = \mathrm{Det}\,(\not{D}\,\not{D}^\dagger) = \exp(-\zeta'_{-\not{D}^2}(0))$.

Si $J_{\not{D}}[\alpha]$ est le Jacobien de la transformation $\psi'(x) = U(x)\psi(x)$, $U(x) = \exp\gamma_5\alpha(x)t$, $\bar{\psi}'(x) = \bar{\psi}U(x)$, on a à partir de (8.1),

$$\mathrm{Det}\,\not{D} = \int \mathcal{D}\bar{\psi}'\mathcal{D}\psi' \exp\int d^4x\, -\bar{\psi}'\not{D}\,\psi' \tag{8.13}$$

$$= J_{\not{D}}[\alpha]\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\int d^4x\, -\bar{\psi}U(x)\not{D}\,U(x)\psi = J_{\not{D}}[\alpha]\mathrm{Det}\,(\mathcal{U}\not{D}\,\mathcal{U}), \tag{8.14}$$

et donc

$$J_{D}[\alpha] = \frac{\text{Det}\, \not{D}}{\text{Det}\,(\mathcal{U}\not{D}\,\mathcal{U})} = \frac{\text{Det}\, \not{D}}{\text{Det}\,(\not{D} + \{\alpha\gamma_5 t, \not{D}\})} = \exp \int d^4x_E\, \alpha \mathcal{A}^E(x), \tag{8.15}$$

pour α infinitésimal.

On a donc

$$\ln J_{D}[\alpha] = \ln(\text{Det}\, \not{D}) - \ln(\text{Det}\,(\not{D} + \{\alpha\gamma_5 t, \not{D}\})) = -\frac{d}{ds}\Big|_{s=0} \frac{1}{2}(\zeta_{-\not{D}^2}(s) - \zeta_{-\not{D}^2-\bar{\delta}\not{D}^2}(s)), \tag{8.16}$$

$$\bar{\delta}\not{D}^2 = \{\alpha\gamma_5 t, \not{D}^2\} + 2\not{D}\,\alpha\gamma_5 t\not{D}\,, \tag{8.17}$$

et encore,

$$\ln J_{D}[\alpha] = -\frac{d}{ds}\Big|_{s=0} \frac{1}{2}\left[\text{Tr}(-\not{D}^2)^{-s} - \text{Tr}(-\not{D}^2 - \bar{\delta}\not{D}^2)^{-s}.\right. \tag{8.18}$$

Puisque $\text{Tr}(A+\delta A)^{-s} = \text{Tr}A^{-s} - s\text{Tr}A^{-s-1}\delta A$, on trouve en utilisant la cyclicité de la trace,

$$\ln J_{D}[\alpha] = -\frac{d}{ds}\Big|_{s=0} 2s\text{Tr}(-\not{D}^2)^{-s}\alpha\gamma_5 t. \tag{8.19}$$

Notons aussi que

$$J_{-\not{D}^2}[\alpha] \equiv \frac{\text{Det}\ -\not{D}^2}{\text{Det}\,(\mathcal{U}(-\not{D}^2)\mathcal{U})} = J_{D}[\alpha]. \tag{8.20}$$

En effet, $\delta(-\not{D}^2) = \{\alpha\gamma_5 t, (-\not{D}^2)\}$ et

$$\ln J_{-\not{D}^2}[\alpha] = -\frac{d}{ds}\Big|_{s=0}(\zeta_{-\not{D}^2}(s) - \zeta_{-\not{D}^2-\delta\not{D}^2}(s)) = -\frac{d}{ds}\Big|_{s=0} 2s\text{Tr}(-\not{D}^2)^{-s}\alpha\gamma_5 t.$$

L'opérateur $-\not{D}^2$ est hermitien. On a donc un ensemble complet et orthonormée de vecteurs propres $\phi_k(x)$ avec valeurs propres λ_k réelles, voir (5.205). Comme dans (5.210), on trouve

$$\begin{aligned}\text{Tr}(-\not{D}^2)^{-s}\alpha\gamma_5 t &= \int d^4x_E\, \lambda_{(k)}^{-s}\text{tr}\left[\bar{\phi}^k(x)\alpha\gamma_5 t\phi_k(x)\right] \\ &= \frac{1}{\Gamma(s)}\int_0^\infty d\tau\, \tau^{s-1}\int d^4x_E\, \text{tr}\left[K_{-\not{D}^2}(x,x;\tau)\alpha\gamma_5 t\right].\end{aligned} \tag{8.21}$$

Ici la trace tr porte sur les indices spinorielles et les indices internes. Tout comme pour la fonction zeta discutée précédemment, cette fonction peut être prolongée en une fonction régulière en $s=0$ et a au plus des pôles simples en $s=1,2$. En effet, on coupe de nouveau l'intégrale en deux morceaux, de 0 à 1 et de 1 à ∞. Le deuxième morceau ne contribue pas car l'intégrale converge et $\frac{1}{\Gamma(s)} = s + O(s^2)$ et pour le premier morceau on utilise l'expansion du noyau chaleur. On a donc

$$\int d^4x_E\, \alpha \mathcal{A}^E(x) = \ln J_{-\not{D}^2}[\alpha] = -\frac{2}{\Gamma(s)} \int_0^1 d\tau\, \tau^{s-1} \int d^4x_E\, \mathrm{tr}\left[K_{-\not{D}^2}(x,x;\tau)\alpha\gamma_5 t\right]\Big|_{s=0}$$
$$= -2 \int d^4x_E\, \mathrm{tr}\, \alpha\gamma_5 t \frac{a_2(x,x)}{(4\pi)^2}, \qquad (8.22)$$

car $\mathrm{Ker}\, \not{D}^2 = 0$. A partir de la relation de récurrence pour les coefficicents de Seeley diagonaux (5.199) et en utilisant (8.15), on trouve

$$a_2(x,x) = \frac{1}{2}\not{D}^4, \quad \mathcal{A}^E(x) = -\frac{1}{(4\pi)^2}\mathrm{tr}[\gamma_5 t \not{D}_x^4]. \qquad (8.23)$$

On a

$$\not{D}_x^2 = \frac{1}{2}\{\gamma^a D_a^x, \gamma^b D_b^x\} = \frac{1}{2}\gamma^a\{D_a^x, \gamma^b D_b^x\} - \frac{1}{2}[\gamma^a, \gamma^b D_b^x] D_a^x =$$
$$\frac{1}{2}\gamma^a\gamma^b\{D_a^x, D_b^x\} - \frac{1}{2}[\gamma^a,\gamma^b]D_b^x D_a^x = \frac{1}{4}\{D_a^x, D_b^x\}\{\gamma^a,\gamma^b\} + \frac{1}{4}[D_a^x, D_b^x][\gamma^a,\gamma^b]$$
$$= D_x^2 + \frac{1}{4}F_{ab}^\alpha t_\alpha[\gamma^a,\gamma^b]. \qquad (8.24)$$

Le seul terme qui contribue de $\not{D}_x^4$ est celui qui fait intervenir le produit de 4 matrices de Dirac et

$$\mathrm{tr}_D\big(\gamma_5[\gamma_a,\gamma_b][\gamma_c,\gamma_d]\big) = 16\epsilon_{abcd}. \qquad (8.25)$$

On trouve donc finalement

$$\mathcal{A}^E(x) = -\frac{1}{(4\pi)^2}\mathrm{tr}_I\,(t_\alpha t_\beta t)\epsilon^{abcd}F_{ab}^\alpha F_{cd}^\beta. \qquad (8.26)$$

En termes de formes, avec $d^4x_E = dx^1 \wedge \cdots \wedge dx^4$, $F^E = \frac{1}{2}F_{ab}dx^a \wedge dx^b$, on trouve

$$\delta(-\frac{1}{\hbar}\mathcal{L}^E d^4x_E) = \alpha\mathcal{A}^E(x)d^4x_E = -\frac{1}{(2\pi)^2}\mathrm{tr}_I(\alpha t F^E \wedge F^E). \qquad (8.27)$$

Le Jacobien en Minkowskien pour l'anomalie se calcule en passant en Euclidien et est donc le même. On a donc

$$\boxed{\delta(\frac{i}{\hbar}\mathcal{L}d^4x) = -\frac{1}{(2\pi)^2}\mathrm{tr}_I(\alpha t F \wedge F)}. \qquad (8.28)$$

8.3 Divergence anomale du courant axial

Une autre manière d'écrire le résultat est en termes du courant de Noether j_5^μ associée à la symétrie axiale globale de l'action $S = \int d^4x\,\mathcal{L}$, $\mathcal{L} = -\bar{\psi}\not{D}\,\psi$ de départ. En effet, pour t antihermitien commutant avec les générateurs t_α, $\delta\psi = \alpha t\gamma_5\psi$, $\delta\bar{\psi} = \bar{\psi}\gamma_5 t\alpha$ implique $\delta\mathcal{L} = 0$ et donc, en vertu du théorème de Noether,

$$j_5^\mu = \bar{\psi}\gamma^\mu\alpha t\gamma_5\psi, \quad \partial_\mu j_5^\mu = \alpha t\gamma_5\psi\frac{\delta\mathcal{L}}{\delta\psi} + \bar{\psi}\gamma_5\alpha t\frac{\delta\mathcal{L}}{\delta\bar{\psi}}. \tag{8.29}$$

En vertu de la version locale du théorème de Noether (cf. sous-section **3.5.4**) et en tenant compte de la non-invariance du Lagrangien, on trouve alors

$$\frac{\partial}{\partial x^\mu}\langle j_5^\mu(x)\rangle^A d^4x = \frac{\hbar}{i}\langle\delta(\frac{i}{\hbar}\mathcal{L}d^4x)\rangle^A = \langle\frac{i\hbar}{(2\pi)^2}\mathrm{tr}_I(\alpha tF\wedge F)\rangle^A, \tag{8.30}$$

$$\langle\mathcal{O}(x)\rangle^A = \frac{\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{O}(x)e^{\frac{i}{\hbar}S}}{\int\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{\frac{i}{\hbar}S}}. \tag{8.31}$$

Notons que si la représentation est irréductible alors $t = i\mathbf{1}$. Dans ce cas $\mathrm{tr}_I\,(t_\alpha t_\beta t) = -iC_2 g_{\alpha\beta}$. Dans le cas plus général, on suppose que $\mathrm{tr}_I\,(t_\alpha t_\beta t) = -iNg_{\alpha\beta}$ et on peut définir un courant non invariant de jauge, le courant de Chern-Simons,

$$G^\mu = 2\epsilon^{\mu\nu\rho\sigma}g_{\alpha\beta}\big[A_\nu^\alpha\partial_\rho A_\sigma^\beta + \frac{1}{3}f^\alpha_{\gamma\delta}A_\nu^\beta A_\rho^\gamma A_\sigma^\delta\big], \quad \partial_\mu G^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}g_{\alpha\beta}F^\alpha_{\mu\nu}F^\beta_{\rho\sigma}, \tag{8.32}$$

avec $\epsilon^{0123} = 1$. Ceci permet d'écrire (8.30) comme une loi de conservation,

$$\boxed{\partial_\mu K^\mu = 0, \quad K^\mu(x) = \langle\bar{\psi}\gamma^\mu t\gamma_5\psi - \frac{\hbar N}{8\pi^2}G^\mu(x)\rangle^A}. \tag{8.33}$$

8.4 Phénoménologie

En se limitant aux quarks légers u et d que l'on suppose pour faciliter les calculs sans masse, le Lagrangien de la chromodynamique quantique pure est donné par

$$L_{quark,gluon} = -\bar{u}\not{D}^G u - \bar{d}\not{D}^G d, \tag{8.34}$$

$$D_\mu^G = \mathbf{1}\partial_\mu + g_s G_\mu^\alpha t_\alpha. \tag{8.35}$$

Ici g_s est la constante de couplage forte, G_μ^α sont les champs de jauge correspondant aux 8 gluons et t_α des générateurs antihermitiennes sans trace de la représentation fondamentale de l'algèbre $su(3)$. On peut les choisir comme $t_\alpha = \frac{-i}{2}\lambda_\alpha$ avec λ_α les matrices de Gell-Mann, ce qui implique $\mathrm{tr}\,(t_\alpha t_\beta) = -\frac{1}{2}\delta_{\alpha\beta}$. En termes des matrices de Pauli σ_i,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8.36}$$

on a

$$\lambda_i = \begin{pmatrix} \sigma_i & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}. \tag{8.37}$$

On choisit les générateurs antihermitiennes sans trace t_i de la représentation fondamentale de $su(2)$ comme $t_i = \frac{-i}{2}\sigma_i$ de sorte que $\mathrm{tr}\,(t_i t_j) = -\frac{1}{2}\delta_{ij}$, $[t_i, t_j] = \epsilon_{ij}{}^k t_k$. Si

$$q^A = \begin{pmatrix} u \\ d \end{pmatrix}, \tag{8.38}$$

le Lagrangian (8.34) est invariant sous les transformations globales $SU(2) \times SU(2)$,

$$\delta q = -\theta^i_V t_i q - \theta^i_A t_i \gamma_5 q = -\theta^i_+ t_{+,i} q - \theta^i_- t_{-,i} q, \tag{8.39}$$

avec $t_{\pm,i} = t_i P_\pm$, $P_\pm = \frac{1}{2}(\mathbf{1} \pm \gamma_5)$ et $\theta^i_{V,A} = \frac{1}{2}(\theta^i_+ \pm \theta^i_-)$. Aucune de ces symétries n'est anomale: pour les symétries vectorielles il n'y a pas de γ_5 et pour les symétries axiales, $\mathrm{tr}\,(t_\alpha t_\beta t_i) = 0$ car les t_α et les t_i agissent dans des espaces différents et $\mathrm{tr}\, t_i = 0$.

Lorsqu'on considère le couplage des quarks au secteur électro-faible, et en particuliers aux photons, la situation change. Le Lagrangien est

$$L_{quark,\gamma} = -\bar{q} \not{D}^A q, \quad \not{D}^A = \mathbf{1} d_\mu + eQA_\mu, \quad Q = -ie \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}. \tag{8.40}$$

Cette fois-ci, en tenant compte du fait qu'il y a trois couleurs, i.e., que la taille des colonnes u et d est 3, on a $\mathrm{tr}_I\,(Q^2 t_i) = (-e^2)(-\frac{i}{2})\delta^3_i((\frac{2}{3})^2 - (\frac{1}{3})^2) \times 3 = i\frac{1}{2}\delta^3_i e^2$.

Si on choisit la symétrie $\delta u = i\gamma_5 u$, $\delta d = -i\gamma_5 d$ et donc $\alpha = -\theta^3_A = -2$, on a en vertu de (8.28),

$$\boxed{\delta L_{quark,\gamma} = \frac{\hbar e^2}{16\pi^2} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}}. \tag{8.41}$$

à faire: processus $\pi^0 \to 2\gamma$ dans le cadre des théories effectives

8.5 Théorème de l'indice d'Atiyah-Singer

L'opérateur $i\not{D} = i(\partial_a + A^\alpha_a t_\alpha)\gamma^a$ est hermitien dans l'espace Euclidien. Divisons les spineurs en leurs composantes de chiralité positive et négative,

$$S_\pm \ni \psi_\pm(x) = P_\pm \psi(x), \quad \gamma_5 \psi_\pm(x) = \pm \psi^\pm(x). \tag{8.42}$$

et introduisons les opérateurs de Weyl

$$i\not{D}_\pm = i\not{D}\, P_\pm : S_\pm \to S_\mp, \quad (i\not{D}_\pm)^\dagger = i\not{D}_\mp. \tag{8.43}$$

Les Laplaciens hermitiens associées sont définies par

$$\Delta_\pm = (i\not{D}_\pm)^\dagger i\not{D}_\pm : S_\pm \to S_\pm. \tag{8.44}$$

On a donc 2 ensembles complets et orthonormées de vecteurs propres $\varphi_{\pm,\kappa}(x)$ avec valeurs propres $\lambda_{\pm,\kappa}$ réelles, voir (5.205).

Les valeurs propres non-nulles λ_κ de Δ_+ et Δ_- sont identiques. Il en est de même de la dégénérescence des espaces propes associées $S_{\pm,\kappa}$,

$$\dim S_{+,\kappa} = \dim S_{-,\kappa}, \quad \lambda_\kappa \neq 0. \tag{8.45}$$

En effet, si $\Delta_+\varphi_{+,\kappa}(x) = \lambda_\kappa\varphi_{+,\kappa}(x)$, avec $\lambda_\kappa \neq 0$, alors en définissant $\chi_{-,\kappa}(x) = i\not{D}_+\varphi_{+,\kappa}(x)$, on a $\Delta_-\chi_{-,\kappa}(x) = (i\not{D}_-)^\dagger i\not{D}_- i\not{D}_+\varphi_{+,\kappa}(x) = (i\not{D}_-)^\dagger\lambda_\kappa\varphi_{+,\kappa}(x) = \lambda_\kappa\chi_{-,\kappa}(x)$ et vice-versa. L'application $i\not{D}_+$ qui associe à $\varphi_{+,\kappa}$ le vecteur $\chi_{-,\kappa}(x)$ est injective: si $\chi_{-,\kappa}(x) = 0$ alors $0 = (i\not{D}_+)^\dagger i\not{D}_+\varphi_{+,\kappa}(x) = \lambda_\kappa\varphi_{+,\kappa}(x)$ ce qui implique que $\varphi_{+,\kappa}(x) = 0$. L'application inverse consiste à prendre $\frac{1}{\lambda_\kappa}\chi_{-,\kappa}$ et à y appliquer $i\not{D}_-$ ce qui donne $\varphi_{+,\kappa}(x)$. Comme cette application est également injective, l'application de départ est bijective.

Les modes zéro de $i\not{D}$ sont également divisés en modes de chiralité positive $S_{+,0}$ et négative $S_{-,0}$,

$$S_{\pm,0} \ni \varphi_{\pm,0}(x) = P_\pm\varphi_0(x), \quad i\not{D}\,\varphi_{\pm,0}(x) = 0, \quad \gamma_5\varphi_{\pm,0}(x) = \pm\varphi_{\pm,0}(x). \tag{8.46}$$

Notons par $\varphi_{+,0u}(x)$, $u = 1,\ldots,n_+$, un ensemble complet et orthonormée de vecteurs de $S_{+,0}$ et par $\varphi_{-,0,v}(x)$, $v = 1,\ldots n_-$, un ensemble complet et orthonormée de vecteurs de $S_{0,-}$.

L'indice de l'opérateur de Weyl $\not{D}_+$ est défini par

$$\begin{aligned} \mathrm{index}\, i\not{D}_+ &= \dim \mathrm{Ker}\, i\not{D}_+ - \dim \mathrm{Ker}\,(i\not{D}_+)^\dagger = \dim \mathrm{Ker}\, i\not{D}_+ - \dim \mathrm{Ker}\, i\not{D}_- \\ &= n_+ - n_-. \end{aligned} \tag{8.47}$$

On a

$$\mathrm{Ker}\,\Delta_\pm = \mathrm{Ker}\, i\not{D}_\pm. \tag{8.48}$$

En effet, si $i\not{D}_\pm\varphi(x) = 0$ alors $\Delta_\pm\varphi(x) = 0$. Si $\Delta_+\varphi(x) = 0$ alors $0 = \int d^4x\,\bar\varphi(x)\Delta_+\varphi(x) = \int d^4x\,\left[\overline{(i\not{D}_+)\varphi(x)}\right]i\not{D}_+\varphi(x)$ et donc $i\not{D}_+\varphi(x) = 0$.

On a alors

$$\mathrm{index}\, i\not{D}_+ = \mathrm{Tr}_{S_+}\, e^{-\tau\Delta_+} - \mathrm{Tr}_{S_-}\, e^{-\tau\Delta_-}, \quad \forall\tau > 0. \tag{8.49}$$

En effet, si

$$\mathrm{Tr}_{S_+}\, e^{-\tau\Delta_+} - \mathrm{Tr}_{S_-}\, e^{-\tau\Delta_-} = e^{-\tau\lambda_{(\kappa)}}\Big[\int d^4x\, \bar\varphi_+^\kappa(x)\varphi_{+,\kappa}(x) - \int d^4x\, \bar\chi_-^\kappa(x)\chi_{-,\kappa}(x)\Big]$$
$$= \sum_\kappa e^{-\tau\lambda_{(\kappa)}}\big[\dim S_{+,\kappa} - \dim S_{-,\kappa}\big].$$

En vertu de (8.45), il y a cancellation pour tous les termes de la somme mise à part les modes zéro, ce qui donne le résultat.
On a encore

$$\mathrm{index}\, i\not{D}_+ = \mathrm{Tr}_{S_+}\, e^{-\tau(i\not{D}_+)^\dagger i\not{D}_+} - \mathrm{Tr}_{S_-}\, e^{-\tau(i\not{D}_-)^\dagger i\not{D}_-}$$
$$= \mathrm{Tr}_{S_+}\, e^{-\tau(-\not{D}^2)}P_+ - \mathrm{Tr}_{S_-}\, e^{-\tau(-\not{D}^2)}P_- = \mathrm{Tr}_S e^{-\tau(-\not{D}^2)}\gamma_5.$$

Or cette dernière expression est reliée à (8.21). En calculant directement les éements diagonaux du noyau chaleur pour petit τ et en utilisant la discussion sur les traces en dessous de (8.21) en tenant compte de (8.26) on trouve le résultat final:

Le théorème de l'indice d'Atiyah-Singer permet de calculer l'indice de l'opérateur de Weyl à partir d'une fonctionnelle des champs de jauge,

$$\boxed{\mathrm{index}\, i\not{D}_+} = n_+ - n_- = \frac{1}{32\pi^2}\epsilon^{abcd}\mathrm{tr}_I\,(t_\alpha t_\beta)\int d^4x\, F^\alpha_{ab}F^\beta_{cd}\boxed{= \frac{1}{2}\int_{\mathbb{R}^4}\mathrm{tr}_I(\frac{F}{2\pi})^2}. \tag{8.50}$$

Ceci montre aussi que la fonctionnelle ne peut pas varier de manière continue quand les champs de jauge varient, mais uniquement par des entiers. On peut montrer quelle est déterminée par la topologie des champs de jauge.

8.6 Calcul direct des anomalies

cf. [25]

Posons $\hbar = 1$ et considérons un Lagrangien du type

$$\mathcal{L} = -\bar\Psi\widehat{D}\Psi, \quad \widehat{D} = \not\partial + \not{A}^\alpha P_+ = \not\partial\, P_- + \not{D}_+, \quad \not{D} = \not\partial + \not{A}^\alpha T_\alpha, \tag{8.51}$$

avec T_α antihermitien. Ceci signifie que le propagateur est le propagateur usuel pour les fermions de Dirac, mais que seulement les fermions de chiralité positive $\chi = P_+\Psi$ couplent aux champs de jauge.

Si on a des fermions de Dirac ψ, on définit

$$\chi = \begin{pmatrix} P_+\psi \\ (\mathcal{C}P_-\psi)^* \end{pmatrix} = \begin{pmatrix} P_+\psi \\ P_+\mathcal{C}\psi^* \end{pmatrix}, \tag{8.52}$$

avec $\mathcal{C}\gamma_\mu^T\mathcal{C}^{-1} = \gamma_\mu$ pour que toutes composantes de χ soit de chiralité gauche et appartiennent à la représentation $\Gamma^{1/2,0}$ du groupe de Lorentz. Pour une transformation de jauge ordinaire des fermions de

Dirac, on a

$$\delta\psi = \theta^\alpha (P_+ t^L_\alpha + P_- t^R_\alpha)\psi, \tag{8.53}$$

$$\delta\chi = \theta^\alpha T_\alpha \chi, \quad T_\alpha = \begin{pmatrix} t^L_\alpha & 0 \\ 0 & -(t^R_\alpha)^T \end{pmatrix}. \tag{8.54}$$

Si

$$j^\mu_\alpha(x) = \frac{\delta(S)}{\delta A^\alpha_\mu(x)} = -\bar{\chi} T_\alpha \gamma^\mu \chi, \tag{8.55}$$

on s'intéresse à la contribution à une boucle de la fonction à 3 points,

$$\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) = \langle j^\mu_\alpha(x) j^\nu_\beta(y) j^\rho_\gamma(z)\rangle, \tag{8.56}$$

et à sa divergence $\partial^x_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}$. Le résultat naïf pour cette divergence est

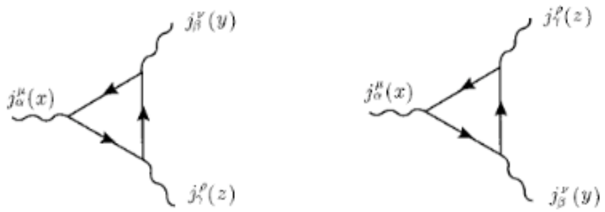
$$\partial^x_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) = i\delta^4(x-y) f^\delta_{\alpha\beta}\langle j^\nu_\delta(y) j^\rho_\gamma(z)\rangle + i\delta^4(x-z) f^\delta_{\alpha\gamma}\langle j^\nu_\beta(y) j^\rho_\delta(z)\rangle. \tag{8.57}$$

En effet,

$$\partial_\mu j^\mu_\alpha = -\frac{\delta^R \mathcal{L}}{\delta\chi} T_\alpha \chi + \bar{\chi} T_\alpha \frac{\delta^L \mathcal{L}}{\delta\bar{\chi}} - A^\beta_\nu f^\gamma_{\alpha\beta} j^\nu_\gamma \iff D_\mu j^\mu_\alpha = -\frac{\delta^R \mathcal{L}}{\delta\chi} T_\alpha \chi + \bar{\chi} T_\alpha \frac{\delta^L \mathcal{L}}{\delta\bar{\chi}}.$$

Avec $\delta_\alpha = -\frac{\partial^R}{\partial\chi} T_\alpha \chi + \bar{\chi} T_\alpha \frac{\partial^L}{\partial\bar{\chi}}$ on a $\delta_\alpha j^\mu_\beta = f^\gamma_{\alpha\beta} j_\gamma$ ce qui donne le résultat en adaptant la dérivation de (3.70) et en annulant les champs de jauge externes à la fin du calcul.

En particulier, deux diagrammes en triangles contribuent,



(figure extraite de [25]).
Ces diagrammes donnent

$$\begin{aligned}\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}|_{tri} = \mathrm{tr}_I\big[&(-i)S_F(x-y)T_\beta\gamma^\nu P_+(-i)S_F(y-z)T_\gamma\gamma^\rho P_+(-i)S_F(z-x)T_\alpha\gamma^\mu P_+\big] \\ +\mathrm{tr}_I\big[&(-i)S_F(x-z)T_\gamma\gamma^\rho P_+(-i)S_F(z-y)T_\beta\gamma^\nu P_+(-i)S_F(y-x)T_\alpha\gamma^\mu P_+\big].\end{aligned} \tag{8.58}$$

En vertu de (2.230), (2.232), le propagateur fermionique est donné par $S_F(x) = \frac{1}{(2\pi)^4}\int d^4p \frac{-i\gamma^\mu p_\mu}{p^2 - i\epsilon} e^{ip\cdot x}$.

En choisissant les moments comme k_1, p, k_2 avec $e^{i(p-k_1+a)\cdot(x-y)}$, $e^{i(p+a)\cdot(y-z)}$, $e^{i(p+k_2+a)\cdot(z-x)}$ pour la première expression et $e^{i(p-k_2+b)\cdot(x-z)}$, $e^{i(p+b)\cdot(z-y)}$, $e^{i(p+k_1+b)\cdot(y-x)}$ pour la deuxième,

$$\begin{aligned}
\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}|_{tri} &= -\frac{1}{(2\pi)^{12}}\int d^4k_1 d^4k_2\, e^{-i(k_1+k_2)\cdot x} e^{ik_1\cdot y} e^{ik_2\cdot z} \int d^4p \\
&\Big[\mathrm{tr}_S\big(\frac{\not p - \not k_1 + \not a}{(p-k_1+a)^2 - i\epsilon}\gamma^\nu \frac{\not p + \not a}{(p+a)^2 - i\epsilon}\gamma^\rho \frac{\not p + \not k_2 + \not a}{(p+k_2+a)^2 - i\epsilon}\gamma^\mu \frac{1+\gamma_5}{2}\big)\mathrm{tr}_G[T_\beta T_\gamma T_\alpha] \\
&+\mathrm{tr}_S\big(\frac{\not p - \not k_2 + \not b}{(p-k_2+b)^2 - i\epsilon}\gamma^\rho \frac{\not p + \not b}{(p+b)^2 - i\epsilon}\gamma^\nu \frac{\not p + \not k_1 + \not b}{(p+k_1+b)^2 - i\epsilon}\gamma^\mu \frac{1+\gamma_5}{2}\big)\mathrm{tr}_G[T_\gamma T_\beta T_\alpha]\Big],
\end{aligned} \tag{8.59}$$

où les constantes a^μ, b^μ sont arbitraires et n'affectent pas l'expression à ce stade. En effet, l'intégrale en p semble diverger de manière linéaire. Le facteur de l'intégrand qui diverge linéairement est

$$p_\sigma p_\tau p_\lambda \mathrm{tr}_S(\gamma^\sigma\gamma^\nu\gamma^\tau\gamma^\rho\gamma^\lambda\gamma^\mu\frac{1+\gamma_5}{2}) = 0.$$

Il en est de même du facteur qui diverge de manière logarithmique car il fait intervenir un nombre impair de matrices γ.

On veut maintenant calculer $\partial^x_\mu\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}$. Pour ce calcul, on utilise l'identité

$$\not k_1 + \not k_2 = (\not p + \not k_2 + \not a) - (\not p - \not k_1 + \not a) = (\not p + \not k_1 + \not b) - (\not p - \not k_2 + \not b), \tag{8.60}$$

ce qui donne

$$\begin{aligned}
\partial^x_\mu\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}|_{tri} &= i\frac{1}{(2\pi)^{12}}\int d^4k_1 d^4k_2\, e^{-i(k_1+k_2)\cdot x} e^{ik_1\cdot y} e^{ik_2\cdot z} \int d^4p \\
&\Big[\mathrm{tr}_G[T_\beta T_\gamma T_\alpha]\mathrm{tr}_S\big(\frac{\not p - \not k_1 + \not a}{(p-k_1+a)^2 - i\epsilon}\gamma^\nu \frac{\not p + \not a}{(p+a)^2 - i\epsilon}\gamma^\rho \frac{1+\gamma_5}{2}\big) \\
&-\mathrm{tr}_G[T_\beta T_\gamma T_\alpha]\mathrm{tr}_S\big(\frac{\not p + \not a}{(p+a)^2 - i\epsilon}\gamma^\rho \frac{\not p + \not k_2 + \not a}{(p+k_2+a)^2 - i\epsilon}\gamma^\nu \frac{1+\gamma_5}{2}\big) \\
&+\mathrm{tr}_G[T_\gamma T_\beta T_\alpha]\mathrm{tr}_S\big(\frac{\not p - \not k_2 + \not b}{(p-k_2+b)^2 - i\epsilon}\gamma^\rho \frac{\not p + \not b}{(p+b)^2 - i\epsilon}\gamma^\nu \frac{1+\gamma_5}{2}\big) \\
&-\mathrm{tr}_G[T_\gamma T_\beta T_\alpha]\mathrm{tr}_S\big(\frac{\not p + \not b}{(p+b)^2 - i\epsilon}\gamma^\nu \frac{\not p + \not k_1 + \not b}{(p+k_1+b)^2 - i\epsilon}\gamma^\rho \frac{1+\gamma_5}{2}\big)\Big].
\end{aligned} \tag{8.61}$$

Pour le facteur de group,

$$\mathrm{tr}_G(T_\beta T_\gamma T_\alpha) = \frac{1}{2}\mathrm{tr}_G(\{T_\beta, T_\gamma\}T_\alpha + [T_\beta, T_\gamma]T_\alpha) \equiv -iD_{\alpha\beta\gamma} - \frac{1}{2}Nf_{\alpha\beta\gamma}, \tag{8.62}$$

avec $-iD_{\alpha\beta\gamma} = \frac{1}{2}\mathrm{tr}_G(\{T_\beta, T_\gamma\}T_\alpha)$ complètement symétrique et $\mathrm{tr}_G(T_\delta T_\alpha) = -N\delta_{\alpha\beta}$. En regroupant le premier avec le quatrième et le deuxième avec le troisième, les termes avec les constantes de structures donnent le membre de droite de (8.57). Les autres donnent

$$\partial^x_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}|_{tri,anom} = \frac{1}{(2\pi)^{12}} D_{\alpha\beta\gamma} \int d^4k_1 d^4k_2\, e^{-i(k_1+k_2)\cdot x} e^{ik_1\cdot y} e^{ik_2\cdot z}$$
$$\Big[\mathrm{tr}_S\big(\gamma^\kappa\gamma^\nu\gamma^\lambda\gamma^\rho\frac{1+\gamma_5}{2}\big) I_{\kappa\lambda}(a-b-k_1, b, b+k_1) + \mathrm{tr}_S\big(\gamma^\kappa\gamma^\rho\gamma^\lambda\gamma^\nu\frac{1+\gamma_5}{2}\big) I_{\kappa\lambda}(b-a-k_2, a, a+k_2)\Big] \tag{8.63}$$

avec

$$I_{\kappa\lambda}(k,c,d) = \int d^4p\big(f_{\kappa\lambda}(p+k,c,d) - f_{\kappa\lambda}(p,c,d), \tag{8.64}$$
$$f_{\kappa\lambda}(p,c,d) = \frac{(p+c)_\kappa (p+d)_\lambda}{[(p+c)^2 - i\epsilon][(p+d)^2 - i\epsilon]}. \tag{8.65}$$

Pour calculer les intégrales, on considère l'expansion en série de Taylor autour de k,

$$f_{\kappa\lambda}(p+k,c,d) = \sum_{n=0}^{\infty} \frac{1}{n!} k^{\mu_1} \dots k^{\mu_n} \frac{\partial^n f_{\kappa\lambda}(p,c,d)}{\partial p^{\mu_1}\dots\partial p^{\mu_n}}. \tag{8.66}$$

Puisque le terme d'ordre zéro ne contribue pas à $I_{\kappa\lambda}(k,c,d)$, tous les termes sont des dérivées totales. Après une rotation de Wick dont l'effet est de donner un facteur i, ils peuvent donc être écrits comme une intégrale de surface sur une 3-sphère de rayon P en vertu du théorème de Stokes. L'aire de la 3-sphere est $2\pi^2 P^3$, tandis que la dérivée n-ième de $f_{\kappa\lambda}(p,c,d)$ donne l'intégrale de surface d'une fonction qui se comporte comme comme $P^{-2-(n-1)}$ car on perd une dérivée en utilisant le théorème de Stokes. À la limite $P \to \infty$, seul les termes avec $n = 1, 2$ contribuent et

$$I_{\kappa\lambda}(k,c,d) = k^\mu \int d^4p \frac{\partial f_{\kappa\lambda}(p,c,d)}{\partial p^\mu} + \frac{1}{2} k^\mu k^\nu \int d^4p \frac{\partial^2 f_{\kappa\lambda}(p,c,d)}{\partial p^\mu \partial p^\nu}. \tag{8.67}$$

On trouve par calcul direct[1]

$$I_{\kappa\lambda}(k,c,d) = \frac{i}{6}\pi^2\big[k_\kappa k_\lambda + 2k_\lambda c_\kappa + 2k_\kappa d_\lambda - k_\lambda d_\kappa - k_\kappa c_\lambda - \eta_{\kappa\lambda} k\cdot(k+c+d)\big]. \tag{8.68}$$

[1]Il y a une coquille dans [25] eq. (22.3.19) où le premier terme manque.

En effet, regardant d'abord le théorème de Stokes en coordonnées sphériques. $\int dk = \int d^3x \partial_i k^i = \int r^2 \sin\theta d\theta d\phi [\frac{\partial r}{\partial x^i}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x^i}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x^i}\frac{\partial}{\partial \phi}]k^i$ Posons $x_i = re_i$ et considérons la surface $r = cte$. On trouve $\int d^3x \partial_i k^i = \int r^2 \sin\theta d\theta d\phi [e_i k^i]$.
En utilisant la variance des intégrales sous les transformations de Lorentz, on établit d'abord les identités bien connues

$$\int d^4p\, p_{\lambda_1} \dots p_{\lambda_k} f(p^2) = 0, \quad \text{pour } k \text{ impair} \tag{8.69}$$

$$\int d^4p\, p_\lambda p_\mu f(p^2) = \frac{\eta_{\lambda\mu}}{4} \int d^4p\, p^2 f(p^2), \tag{8.70}$$

$$\int d^4p\, p_\mu p_\nu p_\rho p_\lambda f(p^2) = \frac{1}{24}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\rho}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho}) \int d^4p\, (p^2)^2 f(p^2). \tag{8.71}$$

Puis on utilise $d^4p = idPP^3d^3\Omega$ en Euclidien et, $\int d^3\Omega = 2\pi^2$ pour conclure que si $p_\mu = pe_\mu$,

$$\int d^3\Omega\, e_{\lambda_1}\dots e_{\lambda_k} = 0, \quad \text{pour } k \text{ impair} \tag{8.72}$$

$$\int d^3\Omega\, e_\lambda e_\mu = 2i\pi^2 \frac{\eta_{\lambda\mu}}{4}, \tag{8.73}$$

$$\int d^3\Omega\, e_\mu e_\nu e_\rho e_\lambda = 2i\pi^2 \frac{1}{24}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\rho}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho}). \tag{8.74}$$

Comme $\int d^4p \dfrac{\partial(k^\mu f)}{\partial p^\mu} = \int d^3\Omega\, e_\mu k^\mu f$ en vertu du théorème de la divergence, on a

$$\begin{aligned} k^\mu \int d^3\Omega \frac{P^5}{P^4} e_\mu \Big[\frac{(e+\frac{c}{P})_\kappa (e+\frac{d}{P})_\lambda}{(e+\frac{c}{P})^2(e+\frac{d}{P})^2} = \\ = (2i\pi^2)\Big[\frac{1}{4}(k_\lambda c_\kappa + k_\kappa d_\lambda) - \frac{2}{24}(k_\kappa(c+d)_\lambda + k_\lambda(c+d)_\kappa + \eta_{\kappa\lambda} k\cdot(c+d))\Big], \end{aligned} \tag{8.75}$$

et aussi

$$\begin{aligned} \frac{1}{2} k^\mu k^\nu \int d^3\Omega e_\mu \Big[\frac{P^4}{P^4} \frac{\eta_{\kappa\nu}(e+\frac{d}{P})_\lambda + \eta_{\lambda\nu}(e+\frac{c}{P})_\kappa}{(e+\frac{c}{P})^2(e+\frac{d}{P})^2} \\ - 2\frac{P^6}{P^6}\Big(\frac{(e+\frac{c}{P})_\kappa (e+\frac{d}{P})_\lambda (e+\frac{c}{P})_\nu}{(e+\frac{c}{P})^4(e+\frac{d}{P})^2} + \frac{(e+\frac{c}{P})_\kappa (e+\frac{d}{P})_\lambda (e+\frac{d}{P})_\nu}{(e+\frac{c}{P})^2(e+\frac{d}{P})^4}\Big)\Big] \\ = \frac{1}{2}(2i\pi^2)\Big[\frac{2}{4}k_\kappa k_\lambda - \frac{4}{24}(2k_\kappa k_\lambda + k^2\eta_{\lambda\kappa})\Big] \quad \square \end{aligned} \tag{8.76}$$

Établissons encore

$$I_{\kappa\lambda}(k,c,d) + I_{\lambda\kappa}(k,c,d) = \frac{i}{6}\pi^2\big[2k_\kappa k_\lambda + k_\lambda(c+d)_\kappa + k_\kappa(c+d)_\lambda - 2\eta_{\kappa\lambda} k\cdot(k+c+d)\big], \tag{8.77}$$

$$\epsilon^{\kappa\nu\lambda\rho} I_{\kappa\lambda}(k,c,d) = \frac{i}{2}\pi^2 \epsilon^{\kappa\nu\lambda\rho}(k_\lambda c_\kappa + k_\kappa d_\lambda). \tag{8.78}$$

Pour la trace spinorielle, on effectue d'abord celle avec 1. Puisque

$$\mathrm{tr}_S\big(\gamma^\kappa\gamma^\nu\gamma^\lambda\gamma^\rho\big) = 4(\eta^{\kappa\nu}\eta^{\lambda\rho} - \eta^{\kappa\lambda}\eta^{\nu\rho} + \eta^{\kappa\rho}\eta^{\nu\lambda}) \tag{8.79}$$

est symétrique en κ, λ et en ν, ρ, les intégrales apparaissent comme

$$\begin{aligned} I_{\kappa\lambda}(a-b-k_1, b, b+k_1) + I_{\lambda\kappa}(a-b-k_1, b, b+k_1) + I_{\kappa\lambda}(b-a-k_2, a, a+k_2) + I_{\lambda\kappa}(b-a-k_2, a, a+k_2) \\ = \frac{i}{6}\pi^2\big[(a-b-k_1)_\kappa(a-b-k_1)_\lambda + (a-b-k_1)_\lambda(2b+k_1)_\kappa + (a-b-k_1)_\kappa(2b+k_1)_\lambda \\ - 2\eta_{\kappa\lambda}(a-b-k_1)\cdot(a+b) + (b-a-k_2)_\kappa(b-a-k_2)_\lambda + (b-a-k_2)_\lambda(2a+k_2)_\kappa \\ + (b-a-k_2)_\kappa(2a+k_2)_\lambda - 2\eta_{\kappa\lambda}(b-a-k_2)\cdot(a+b)\big]. \end{aligned} \tag{8.80}$$

Cette expression s'annule ssi $b = -a$.

$$
\begin{aligned}
&I_{\kappa\lambda}(2a-k_1,-a,-a+k_1)+I_{\lambda\kappa}(2a-k_1,-a,-a+k_1)+I_{\kappa\lambda}(-2a-k_2,a,a+k_2)+I_{\lambda\kappa}(-2a-k_2,a,a+k_2) \\
&\quad = \frac{i}{6}\pi^2\big[(2a-k_1)_\kappa(2a-k_1)_\lambda + (2a-k_1)_\lambda(-2a+k_1)_\kappa + (2a-k_1)_\kappa(-2a+k_1)_\lambda + \\
&\quad (-2a-k_2)_\kappa(-2a-k_2)_\lambda + (-2a-k_2)_\lambda(2a+k_2)_\kappa + (-2a-k_2)_\kappa(2a+k_2)_\lambda\big] = 0. \quad (8.81)
\end{aligned}
$$

Il faut maintenant considérer la trace spinorielle avec γ_5 utilisant

$$
\mathrm{tr}_S\big(\gamma^\kappa\gamma^\nu\gamma^\lambda\gamma^\rho\gamma_5\big) = -4i\epsilon^{\kappa\nu\lambda\rho}, \quad \epsilon^{0123} = 1. \quad (8.82)
$$

Ceci donne

$$
\begin{aligned}
\partial^x_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}|_{tri,anom} &= \frac{1}{(2\pi)^{12}} D_{\alpha\beta\gamma} \int d^4k_1 d^4k_2\, e^{-i(k_1+k_2)\cdot x} e^{ik_1\cdot y} e^{ik_2\cdot z} \\
(-2i)\epsilon^{\kappa\nu\lambda\rho}&\Big[I_{\kappa\lambda}(2a-k_1,-a,-a+k_1) - I_{\kappa\lambda}(-2a-k_2,a,a+k_2)\Big].
\end{aligned} \quad (8.83)
$$

Utilisant (8.78) on trouve

$$
\begin{aligned}
\partial^x_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}|_{tri,anom} &= \epsilon^{\kappa\nu\lambda\rho}\frac{\pi^2}{(2\pi)^{12}} D_{\alpha\beta\gamma} \int d^4k_1 d^4k_2\, e^{-i(k_1+k_2)\cdot x} e^{ik_1\cdot y} e^{ik_2\cdot z} \\
\Big[(2a-k_1)_\lambda(-a)_\kappa &+ (2a-k_1)_\kappa(-a+k_1)_\lambda - (-2a-k_2)_\lambda a_\kappa - (-2a-k_2)_\kappa(a+k_2)_\lambda\Big] \\
&= \frac{2\pi^2}{(2\pi)^{12}} D_{\alpha\beta\gamma} \int d^4k_1 d^4k_2\, e^{-i(k_1+k_2)\cdot x} e^{ik_1\cdot y} e^{ik_2\cdot z} \big[\epsilon^{\kappa\nu\lambda\rho} a_\kappa (k_1+k_2)_\lambda\big].
\end{aligned} \quad (8.84)
$$

On pourrait choisir $a \parallel k_1 + k_2$ pour enlever l'anomalie ou même $a = 0$. Mais alors elle apparaîtrait dans la divergence ∂^y_ν ou ∂^z_ρ: pour éviter l'anomalie dans ∂^y_ν, il faut $a + k_2 \parallel k_1$ et dans ∂^z_ρ, il faut $a - k_1 \parallel k_2$. Les 3 conditions ne peuvent être satisfaites pour $k_1 \nparallel k_2$.

Pour évaluer la divergence ∂^y_ν, on utilise l'identité

$$\not{k}_1 = (\not{p} + \not{a}) - (\not{p} - \not{k}_1 + \not{a}) = (\not{p} + \not{k}_1 + \not{b}) - (\not{p} + \not{b}), \tag{8.85}$$

ce qui donne au lieu de (8.61) un expression du type

$$\begin{aligned}(\not{p} - \not{k}_1 + \not{a})\gamma^\rho(\not{p} + \not{k}_2 + \not{a})\gamma^\mu P_+ - (\not{p} + \not{k}_2 + \not{a})\gamma^\mu(\not{p} + \not{a})\gamma^\rho P_+ \\ + (\not{p} + \not{b})\gamma^\mu(\not{p} - \not{k}_2 + \not{b})\gamma^\rho P_+ - (\not{p} - \not{k}_2 + \not{b})\gamma^\rho(\not{p} + \not{k}_1 + \not{b})\gamma^\mu P_+\end{aligned} \tag{8.86}$$

On obtient alors les intégrales $I_{\kappa\lambda}(a - b - k_1 + k_2, b - k_2, b + k_1)$ et $I_{\kappa\lambda}(b - a - k_2, a + k_2, a)$ et le calcul se ramène au calcul précédent en posant

$$b - k_2 = b', \qquad a + k_2 = a', \tag{8.87}$$

$$k_1 + k_2 = k_1', \qquad k_2 = -k_2'. \tag{8.88}$$

L'anomalie dans la partie non chirale de la divergence ∂^y_ν s'annule donc ssi $a' = -b'$ ce qui est équivalent à $a = -b$, comme précédemment. L'absence de la partie chirale de l'anomalie dans la divergence ∂^y_ν implique $a' \parallel k_1' + k_2'$ et donc $a + k_2 \parallel k_1$. Le choix $a = 0 = b$ n'enlève donc pas non plus l'anomalie de la divergence ∂^y_ν car $k_2 \nparallel k_1$.
Pour la divergence ∂^z_ρ, on utilise l'identité

$$\not{k}_2 = (\not{p} + \not{k}_2 + \not{a}) - (\not{p} + \not{a}) = (\not{p} + \not{b}) - (\not{p} - \not{k}_2 + \not{b}), \tag{8.89}$$

ce qui donne au lieu de (8.61) une expression du type

$$\begin{aligned}(\not{p} + \not{a})\gamma^\mu(\not{p} - \not{k}_1 + \not{a})\gamma^\nu P_+ - (\not{p} - \not{k}_1 + \not{a})\gamma^\nu(\not{p} + \not{k}_2 + \not{a})\gamma^\mu P_+ \\ + (\not{p} - \not{k}_2 + \not{b})\gamma^\nu(\not{p} + \not{k}_1 + \not{b})\gamma^\mu P_+ - (\not{p} + \not{k}_1 + \not{b})\gamma^\mu(\not{p} + \not{b})\gamma^\nu P_+\end{aligned} \tag{8.90}$$

On obtient alors les intégrales $I_{\kappa\lambda}(a - b - k_1, b + k_1, b)$ et $I_{\kappa\lambda}(b - a - k_2 + k_1, a - k_1, a + k_2)$ et le calcul se ramène au calcul précédent en posant

$$b + k_1 = b'', \qquad a - k_1 = a'', \tag{8.91}$$

$$k_1 = -k_1'', \qquad k_1 + k_2 = k_2''. \tag{8.92}$$

L'anomalie dans la partie non chirale de la divergence ∂^z_ρ s'annule donc ssi $a'' = -b''$ ce qui est équivalent à $a = -b$, comme précédemment. L'absence de la partie chirale de l'anomalie dans la divergence ∂^z_ρ implique $a'' \parallel k_1'' + k_2''$ et donc $a - k_1 \parallel k_2$.

On peut donc donc shifter l'anomalie d'un courant à un autre, mais

Il n'y a pas de choix qui l'enlève partout si $D_{\alpha\beta\gamma} \neq 0$.

Remarque: Intégrales divergeantes et shift de variables cf. [47]. Considérons

$$\Delta(a) = \int_{-\infty}^{+\infty} dx\, [f(x+a) - f(x)] \tag{8.93}$$
$$= \int_{-\infty}^{+\infty} dx\, [af'(x) + \frac{a^2}{2} f''(x) + \dots] \tag{8.94}$$
$$= a[f(+\infty) - f(-\infty)] + \frac{a^2}{2}[f'(+\infty) - f'(-\infty)] + \dots. \tag{8.95}$$

Si l'intégrale converge ou diverge au plus de manière logarithmique, alors $f(+\infty) = f(-\infty) = f'(+\infty) = f'(-\infty) = \cdots = 0$ et $\Delta(a) = 0$ et on peut shifter $x \to x - a$ dans l'intégrale. C'est le cas de l'expression (8.59). Si l'intégrale diverge de manière linéaire, $f(\pm\infty) \neq 0$ et $f'(+\infty) = f'(-\infty) = \cdots = 0$, alors $\Delta(a)$ peut créer un terme de surface non nul qui dépend du shift a,

$$\Delta(a) = a[f(+\infty) - f(-\infty)]. \tag{8.96}$$

C'est ce que l'on a trouvé pour l'intégrale dans (8.61).

8.7 Anomalie "singlet"

Supposons maintenant que j^μ_α est une symétrie de Noether de la théorie, $\partial_\mu j^\mu_\alpha \approx 0$, c-à-d que T_α commute avec les autres générateur, $f^\gamma_{\alpha\beta} = 0$ pour α fixé et quelque soit β, γ. Les courants $j^\nu_\beta(x)$, $j^\rho_\gamma(x)$ sont couplés à des champs de jauge.

Dans ce cas on ne veut pas d'anomalies dans la divergence (covariante) de $\partial^y_\nu, \partial^z_\rho$ pour garantir l'invariance de jauge du résultat, ce qui impose

$$a = k_1 - k_2, \tag{8.97}$$

et on obtient

$$\partial^x_\mu \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}|_{tri,anom} = \frac{(2\pi)^2}{(2\pi)^{12}} D_{\alpha\beta\gamma} \int d^4k_1 d^4k_2\, e^{-i(k_1+k_2)\cdot x} e^{ik_1\cdot y} e^{ik_2\cdot z} \epsilon^{\kappa\nu\lambda\rho} k^1_\kappa k^2_\lambda \tag{8.98}$$
$$= -\frac{1}{(2\pi)^2} D_{\alpha\beta\gamma} \epsilon^{\kappa\nu\lambda\rho} \frac{\partial\delta^4(y-x)}{\partial y^\kappa} \frac{\partial\delta^4(z-y)}{\partial z^\lambda}. \tag{8.99}$$

Si on considère $\langle j^\mu_\alpha(x)\rangle^A$ calculé avec l'action (8.51), c-à-d en présence des autres courants couplés à des champs de jauge externes, le résultat pour les diagrammes de triangles peut s'exprimer comme

$$\langle j^\mu_\alpha(x)\rangle^A_\Delta = \frac{1}{2} \int d^4y\, d^4z\, \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) A^\beta_\nu(y) A^\gamma_\rho(z), \tag{8.100}$$
$$\langle \partial^x_\mu j^\mu_\alpha(x)\rangle^A_{\Delta,anom} = -\frac{1}{8\pi^2} D_{\alpha\beta\gamma} \epsilon^{\kappa\nu\lambda\rho} \partial_\kappa A^\beta_\nu(x) \partial_\lambda A^\gamma_\rho(x). \tag{8.101}$$

Il y a d'autres diagrammes à une boucle qui contribuent

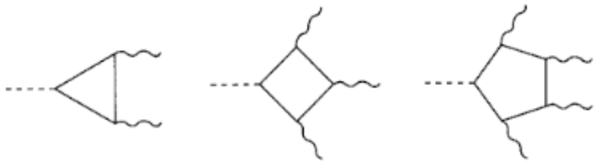


Figure 22.2 One-loop diagrams for the anomaly in a current indicated by the dashed line. Solid lines are fermions; wavy lines are gauge fields with which they interact.

(figure extraite de [25]).

L'invariance de jauge du résultat final implique que

$$\langle \partial^x_\mu j^\mu_\alpha(x) \rangle^A_{anom} = -\frac{1}{32\pi^2} D_{\alpha\beta\gamma} \epsilon^{\kappa\nu\lambda\rho} F^\beta_{\kappa\nu}(x) F^\gamma_{\lambda\rho}(x). \tag{8.102}$$

Pour des fermions de Dirac, on a en vertu de (8.53)

$$-iD_{\alpha\beta\gamma} = \frac{1}{2}\mathrm{tr}_G(\{t^L_\alpha, t^L_\beta\} t^L_\gamma) - \frac{1}{2}\mathrm{tr}_G(\{t^R_\alpha, t^R_\beta\} t^R_\gamma). \tag{8.103}$$

Dans la section précédente, on a calculé la divergence d'un courant axial j^μ_5 avec $t^L = -t^R \equiv t$ ce qui est l'effet de

$$\gamma_5 = \begin{pmatrix} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & -1_{2\times 2} \end{pmatrix} \tag{8.104}$$

dans (8.55)) avec des courants $j^\nu_\beta, j^\rho_\gamma$ pour lesquels $t^L_\beta = t^R_\beta = t_\beta$ et $t^L_\gamma = t^R_\gamma = t_\gamma$. On a donc que $-iD_{\alpha\beta\gamma} = \mathrm{tr}_G(\{t, t_\beta\} t_\gamma) = \mathrm{tr}_G(\{t_\beta, t_\gamma\} t)$ et

$$\langle \partial^x_\mu j^\mu_\alpha(x) \rangle^A_{anom} = -i\frac{1}{32\pi^2} \mathrm{tr}_G(\{t_\beta, t_\gamma\} t) \epsilon^{\kappa\nu\lambda\rho} F^\beta_{\kappa\nu}(x) F^\gamma_{\lambda\rho}(x). \tag{8.105}$$

en accord avec (8.30).

8.8 Anomalie non-abélienne

Supposons maintenant qu'aucun champ de jauge ne couple aux courants $j^\mu_\alpha(x), j^\nu_\beta(y) j^\rho_\gamma(z)$ associés à des symétries globales, mais que les générateurs de ces symétries sont de la forme (8.53) et soit vectoriels $t^R = t^L$ ou axiaux, $t^R = -t^L$. L'équation (8.103) implique alors que les graphes triangulaires anomaux sont ceux avec un courant axial et deux courants vectoriels ou avec trois courants axiaux.

Dans le premier cas, si $j^\mu_\alpha(x)$ est le courant axial et $j^\nu_\beta(y) j^\rho_\gamma(z)$ les courants vectoriels, on choisit a tel que la divergence des courants vectoriels soit non-anomale et donc comme précédemment $a = k_1 - k_2$ avec comme résultat (8.99).

Pour trois courants axiaux, il n'y a pas de raisons d'en privilégier l'un d'entre eux et on voudrait respecter la symétrie entre ces courants en choisissant $a = \alpha k_1 + \beta k_2$ de manière appropriée.

Pour ∂^x_μ, l'anomalie est proportionnelle à

$$\epsilon^{\kappa\nu\lambda\rho} a_\kappa (k_1 + k_2)_\lambda = \epsilon^{\kappa\nu\lambda\rho} (\alpha - \beta)(k_1)_\kappa (k_2)_\lambda, \tag{8.106}$$

pour ∂^y_ν à

$$-\epsilon^{\kappa\rho\lambda\mu} a'_\kappa (k'_1 + k'_2)_\lambda = -\epsilon^{\kappa\rho\lambda\mu} (\alpha k_1 + (\beta + 1) k_2)_\kappa (k_1)_\lambda = -\epsilon^{\kappa\rho\lambda\mu} (\beta + 1)(k_2)_\kappa (k_1)_\lambda, \tag{8.107}$$

et pour ∂^z_ρ à

$$-\epsilon^{\kappa\mu\lambda\nu} a''_\kappa (k''_1 + k''_2)_\lambda = -\epsilon^{\kappa\mu\lambda\nu} ((\alpha - 1) k_1 + \beta k_2)_\kappa (k_2)_\lambda = -\epsilon^{\kappa\mu\lambda\nu} (\alpha - 1)(k_1)_\kappa (k_2)_\lambda. \tag{8.108}$$

Ceci implique $\alpha = -\beta = \frac{1}{3}$,

$$a = \frac{1}{3}(k_1 - k_2) \tag{8.109}$$

et donc :

L'anomalie dans le courant axial pour une amplitude avec trois courants axiaux est un tiers de l'anomalie pour une amplitude avec un courant axial et deux courants vectoriels.

De nouveau, les autres diagrammes contribuent à la divergence du courant. On voudrait calculer l'anomalie pour la symétrie chirale $SU(3) \times SU(3)$ des interactions fortes, en choisissant que les courants vectoriels sont conservés et que les graphes avec seulement des courants axiaux sont symétriques.

Il s'agit donc de symétries globales agissant sur les saveurs des quarks et non les couleurs. Malgré cela, il est pratique de coupler les courants à des champs de jauge fictifs V^a_μ, A^a_μ et d'exprimer l'anomalie comme la non-invariance d'une fonctionnelle $\Gamma[V, A]$. Le Lagrangien associé

$$\mathcal{L} = -\bar{\psi}(\partial\!\!\!/ + V\!\!\!/ + A\!\!\!/ \gamma_5)\psi, \tag{8.110}$$

est invariant de jauge sous

$$\delta\psi = (-\alpha(x) - \beta(x)\gamma_5)\psi, \quad \delta\bar{\psi} = \bar{\psi}(\alpha(x) - \beta(x)\gamma_5), \tag{8.111}$$

$$\delta V_\mu = \partial_\mu \alpha + [V_\mu, \alpha] + [A_\mu, \beta], \quad \delta A_\mu = \partial_\mu \beta + [A_\mu, \alpha] + [V_\mu, \beta], \tag{8.112}$$

avec les lois de conservation pour $j^\mu_a = \bar{\psi}\gamma^\mu t_a \psi$ et $j^\mu_{5\alpha} = \bar{\psi}\gamma^\mu \gamma_5 t_a \psi$

$$\partial_\mu j^\mu_a - f^c_{ba} V^b_\mu j^\mu_c - f^c_{ba} A^b_\mu j^\mu_{5c} \approx 0, \tag{8.113}$$

$$\partial_\mu j^\mu_{5a} - f^c_{ba} V^b_\mu j^\mu_{5c} - f^c_{ba} A^b_\mu j^\mu_c \approx 0. \tag{8.114}$$

Remarque: Pour découpler les transformations $SU(3) \times SU(3)$, on peut définir

$$A_\mu^{L,R} = V_\mu \pm A_\mu, \qquad \Lambda^{L,R} = \alpha \pm \beta, \tag{8.115}$$

$$j_a^{\mu L,R} = \frac{1}{2}(j_a^\mu \pm j_{5a}^\mu), \qquad D_\mu^{L,R} j^{\mu L,R} \approx 0. \tag{8.116}$$

Les transformations de jauge des potentiels sont alors générées par

$$\delta V_\mu^b(y) = \int d^4x\, \alpha^a(x) Y_a(x) V_\mu^b(y) + \int d^4x\, \beta^a(x) X_a(x) V_\mu^b(y), \tag{8.117}$$

$$\delta A_\mu^b(y) = \int d^4x\, \alpha^a(x) Y_a(x) A_\mu^b(y) + \int d^4x\, \beta^a(x) X_a(x) A_\mu^b(y), \tag{8.118}$$

avec

$$Y_a(x) = -\frac{\partial}{\partial x^\mu} \frac{\delta}{\delta V_\mu^a(x)} - f_{ab}^c V_\mu^b(x) \frac{\delta}{\delta V_\mu^c(x)} - f_{ab}^c A_\mu^b(x) \frac{\delta}{\delta A_\mu^c(x)}, \tag{8.119}$$

$$X_a(x) = -\frac{\partial}{\partial x^\mu} \frac{\delta}{\delta A_\mu^a(x)} - f_{ab}^c V_\mu^b(x) \frac{\delta}{\delta A_\mu^c(x)} - f_{ab}^c A_\mu^b(x) \frac{\delta}{\delta V_\mu^c(x)}. \tag{8.120}$$

et on exige que

$$Y_a(x)\Gamma[V, A] = 0. \tag{8.121}$$

Un calcul long et difficile donne

$$\begin{aligned} X_a(x)\Gamma[V, A] = -\frac{i}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\Big[t_a \big(F_{\mu\nu}^V F_{\rho\sigma}^V + \frac{1}{3} F_{\mu\nu}^A F_{\rho\sigma}^A \\ - \frac{8}{3}(A_\mu A_\nu F_{\rho\sigma}^V + A_\mu F_{\nu\rho}^V A_\sigma + F_{\mu\nu}^V A_\rho A_\sigma) + \frac{32}{3} A_\mu A_\nu A_\rho A_\sigma \big) \Big], \end{aligned} \tag{8.122}$$

avec

$$F_{\mu\nu}^V = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] + [A_\mu, A_\nu], \tag{8.123}$$

$$F_{\mu\nu}^A = \partial_\mu A_\nu - \partial_\nu A_\mu + [V_\mu, A_\nu] + [A_\mu, V_\nu]. \tag{8.124}$$

Le facteur 1/3 dans le 2ème terme a déjà été expliqué comme conséquence du choix différent de a dans les AVV et les AAA graphes. Dans la section suivante, on va donner des conditions de cohérence permettant de déterminer les termes cubiques et quartiques à partir des termes quadratiques que l'on a calculés explicitement.

Lorsqu'il n'y a que des courants axiaux, on fait le choix symétrique (8.109) et l'inclusion de tous les diagrammes complète (8.100) en

$$\langle D_\mu^x j_\alpha^\mu(x) \rangle_{anom}^A = -\epsilon^{\kappa\nu\lambda\rho} \frac{1}{24\pi^2} \mathrm{Tr}\Big[t_\alpha \partial_\kappa \big(A_\nu \partial_\lambda A_\rho + \frac{1}{2} A_\nu A_\lambda A_\rho \big) \Big]. \tag{8.125}$$

De nouveau, on a déjà calculé le terme quadratique et les autres termes seront déterminés par la condition de cohérence.

Dans ce cas, les transformations de jauge (fictives) $\delta A_\mu^\alpha = \partial_\mu \epsilon^\alpha + f^\alpha_{\beta\gamma} A_\mu^\beta \epsilon^\gamma$ sont engendrées par $\delta A_\mu^\beta(y) = \int d^4x \, \epsilon^\alpha(x) \mathcal{T}_\alpha(x) A_\mu^\beta(y)$ avec

$$\mathcal{T}_\alpha(x) = -\frac{\partial}{\partial x^\mu} \frac{\delta}{\delta A_\mu^\alpha(x)} - f^\gamma_{\alpha\beta} A_\mu^\beta(x) \frac{\delta}{\delta A_\mu^\gamma(x)} \tag{8.126}$$

Si $Z[A] = e^{\frac{i}{\hbar}\Gamma[A]}$, avec $Z[A]$ basé sur le Lagrangien (8.51), $\langle j_\alpha^\mu(x)\rangle^A = \dfrac{\delta\Gamma[A]}{\delta A_\mu^\alpha(x)}$ et l'anomalie se manifeste alors de la manière suivante,

$$\mathcal{T}_\alpha(x)\Gamma[A] = D_\mu^x \langle j_\alpha^\mu(x)\rangle^A \equiv G_\alpha[x; A]. \tag{8.127}$$

8.9 Condition de cohérence de Wess-Zumino

8.9.1 Commutateurs

Par calcul direct, on peut vérifier que les générateurs $\mathcal{T}_\alpha(x)$ forment un représentation de l'algèbre de Lie dans le sens où

$$[\mathcal{T}_\alpha(x), \mathcal{T}_\beta(y)] = f^\gamma_{\alpha\beta} \delta^4(x,y) \mathcal{T}_\gamma(x). \tag{8.128}$$

En agissant avec cette relation sur $\Gamma[A]$ et en utilisant la définition de l'anomalie (8.127), on déduit la relation de cohérence de Wess-Zumino

$$\mathcal{T}_\alpha(x) G_\beta[y; A] - \mathcal{T}_\beta(y) G_\alpha[x; A] = f^\gamma_{\alpha\beta} \delta^4(x,y) G_\gamma[x; A]. \tag{8.129}$$

Dans le contexte original de la symétrie $SU(3) \times SU(3)$ des interactions fortes, on trouve les relations de commutation suivantes:

$$[Y_a(x), Y_b(y)] = f^c_{ab} \delta^4(x,y) Y_c(x), \tag{8.130}$$

$$[Y_a(x), X_b(y)] = f^c_{ab} \delta^4(x,y) X_c(x), \tag{8.131}$$

$$[X_a(x), X_b(y)] = f^c_{ab} \delta^4(x,y) Y_c(x). \tag{8.132}$$

En appliquant sur $\Gamma[A, V]$ et en utilisant (8.121) et $X_a(x)\Gamma[A, V] = G_a(x)$ avec $G_a(x)$ défini par le membre de droite de (8.122), on trouve les conditions de cohérence

$$Y_a(x) G_b(y) = f^c_{ab} \delta^4(x,y) G_c(x), \quad X_a(x) G_b(y) - X_b(y) G_a(x) = 0. \tag{8.133}$$

La première équation exprime le fait que l'anomalie se transforme dans la représentation adjointe sous une transformation $SU(3)$ ordinaire. On peut vérifier que la deuxième est aussi satisfaite pour le membre de droite de (8.122).

Nous allons nous concentrer sur le cas symétrique de l'équation (8.125). On a explicitement calculé le terme quadratique en les champs de jauge. On va montrer que les autres termes sont déterminés par la condition de cohérence.

8.9.2 Fantômes et cohomologie

Comme pour la fixation de jauge, on remplace $\epsilon^\alpha(x) \to C^\alpha(x)$, avec $C^\alpha(x)$ un champ fermionique externe. La transformation BRST est définie par

$$sA^\alpha_\mu = D_\mu C^\alpha, \quad sC^\alpha = -\frac{1}{2} f^\alpha_{\beta\gamma} C^\beta C^\gamma, \tag{8.134}$$

et est nilpotente, $s^2 = 0$. Définissant $G[C, A] = \int d^4x\, C^\alpha G_\alpha[x; A]$, la définition de l'anomalie (8.127) et la condition de cohérence (8.129) se réécrivent, en réinstaurant $\hbar$, comme

$$s\Gamma[A] = \hbar G[C, A], \qquad sG[A, C] = 0. \tag{8.135}$$

Ceci se voit en appliquant $\frac{\delta}{\delta C^\alpha(x)}$ respectivement $\frac{\delta^2}{\delta C^\alpha(x)\delta C^\beta(y)}$.

Si l'anomalie peut s'écrire comme $G[A, C] = sF[A]$ avec $F[A]$ une fonctionnelle locale en les A^α_μ, la condition de cohérence est automatiquement satisfaite car $s^2 = 0$. Dans ce cas, on peut l'absorber en redéfinissant l'action de départ S par un contreterme fini d'ordre $\hbar$. Pour des champs de jauge externes, ceci revient à redéfinir l'action effective par le même contreterme, $\Gamma'[A] = \Gamma[A] - \hbar F[A]$, de manière à ce que $s\Gamma'[A] = 0$. De la même manière, ce genre de contretermes permet de modifier l'expression de l'anomalie.

à faire: des choix différents de a correspondent à de tels contretermes

Sur les exemples que l'on a vu, les anomalies $G[A, C]$ sont des fonctionnelles locales en A, C. On peut montrer que c'est vrai en général. Dans le contexte présent, les anomalies non triviales sont donc des classes d'équivalences de fonctionnelles en A, C, linéaire en C^α et ses dérivées, qui sont BRST fermées et deux fonctionnelles sont équivalentes si elles diffèrent par un terme BRST exact,

$$[G[A, C]] \in H^1(s) : \quad \begin{cases} sG[A, C] = 0 \\ G[A, C] \sim G'[A, C] \iff G[A, C] = G'[A, C] + sF[A]. \end{cases} \tag{8.136}$$

$$\boxed{\text{Les anomalies non triviales sont des représentants de } H^1(s).}$$

En termes des intégrands, il faut admettre des dérivées totales comme pour les symétries, $G[A, C] = \int d^4x\, \mathcal{G}$, $s\mathcal{G} = \partial_\mu k^\mu$, $\mathcal{G} = \mathcal{G}' + s\mathcal{F} + \partial_\mu l^\mu$.

On va maintenant utiliser les conditions de cohérence pour calculer les termes cubiques et quartiques dans (8.125). En choisissant la dimension canonique de C^α nulle, la transformation BRST ne modifie pas la dimension canonique et celle de $G[A, C]$ doit être zéro, tout comme celle de $\Gamma[A]$. En mettant tous le termes possibles avec des coefficients arbitraires, on a donc

$$G[A, C] = -\frac{1}{24\pi^2} \epsilon^{\kappa\nu\lambda\rho} \int d^4x\, \mathrm{Tr}\Big(C\big[\partial_\kappa A_\nu \partial_\lambda A_\rho + c_1 \partial_\kappa A_\nu A_\lambda A_\rho + c_2 A_\kappa \partial_\nu A_\lambda A_\rho + c_3 A_\kappa A_\nu \partial_\lambda A_\rho + c_4 A_\kappa A_\nu A_\lambda A_\rho\big]\Big). \tag{8.137}$$

On simplifie le calcul en travaillant avec des formes, $A = A^\alpha_\mu T_\alpha dx^\mu$ et $C = C^\alpha T_\alpha$, $d = dx^\mu \partial_\mu$ et on considérant que les dx^μ anticommutent entre eux et avec les C^α. Dans la suite, on omet le produit extérieur des formes. On a alors

$$sA = -dC - \{A, C\}, \qquad sC = -C^2, \qquad d^2 = 0, \qquad \{s, d\} = 0. \tag{8.138}$$

Le calcul direct de $sG[A, C] = 0$ donne alors

$$G[A, C] = -\frac{1}{24\pi^2}\mathrm{Tr}\Big(Cd\big[AdA + \tfrac{1}{2}A^3\big]\Big), \tag{8.139}$$

comme annoncé dans (8.125). On ne fait pas explicitement le calcul ici car on va le voir comme conséquence d'une méthode plus générale pour dériver des solutions de l'équation de cohérence.

8.9.3 Équations de descente

Considérons $F = dA + \frac{1}{2}A^2 = \frac{1}{2}F^\alpha_{\mu\nu} t_\alpha dx^\mu dx^\nu$ dans l'algèbre "universelle" des polynômes en A, C, dA, dC. Un autre ensemble de générateurs est A, sA, C, F, avec $sF = [F, C], dF = [F, A]$. Sous quelles conditions on peut travailler librement avec une telle algèbre est discuté par exemple dans la section 10.2 de [50].

Considérons les "classes caractéristiques" $\mathrm{Tr}\, F^{n+1}$. On a $d\mathrm{Tr}\, F^{n+1} = 0$. Dans la base A, dA de l'algèbre des polynômes en A, dA, l'homotopie contractante pour d est $\sigma = A\frac{\partial}{\partial dA}$ et on a

$$\{d, \sigma\} = A\frac{\partial}{\partial A} + dA\frac{\partial}{\partial dA}, \tag{8.140}$$

Un polynôme $\omega(A, dA)$ peut s'écrire

$$\omega(A, dA) - \omega(0, 0) = \int_0^1 dt\frac{d}{dt}\omega(tA, tdA) = \int_0^1 \frac{dt}{t}[\{d, \sigma\}\omega](tA, tdA). \tag{8.141}$$

Si $d\omega = 0$ et $\omega(0, 0) = 0$, alors on trouve $\omega = d\eta$ avec

$$\eta = \int_0^1 \frac{dt}{t}[\sigma\omega](tA, tdA). \tag{8.142}$$

En particulier,

$$\mathrm{Tr}F^{n+1} = d\Omega_{2n+1}(A, F), \qquad \Omega_{2n+1}(A, F) = (n + 1)\int_0^1 dt\, \mathrm{Tr}\, A(tF + (t^2 - t)A^2)^n. \tag{8.143}$$

Pour $n = 2$, on trouve

$$\Omega_5(A, F) = \mathrm{Tr}\,\big[AF^2 - \frac{1}{2}A^3F + \frac{1}{10}A^5\big]. \tag{8.144}$$

À partir de la condition de “horizontalité”

$$(d+s)(A+C)+(A+C)^2=F, \tag{8.145}$$

on déduit ensuite que

$$\mathrm{Tr}\, F^{n+1}=(d+s)\Omega_{2n+1}(A+C,F) \tag{8.146}$$

et donc aussi, si $\Omega_{2n+1}(A+C,F)=\Omega_{2n+1}(A,F)+\Omega^1_{2n}(A,C,F)+\cdots+\Omega^{2n+1}_0(C)$,

$$s\Omega^1_{2n}(A,C,F)+d\Omega^2_{2n-1}(A,C,F)=0. \tag{8.147}$$

Pour $n=2$, on trouve

$$\Omega^1_4(A,C,F)=\mathrm{Tr}\Big(C\big[F^2-\frac{1}{2}(A^2F+AFA+FA^2)+\frac{1}{2}A^4\big]\Big)=\mathrm{Tr}\Big[Cd\big(AdA+\frac{1}{2}A^3\big)\Big], \tag{8.148}$$

comme il faut.

8.9.4 Résultat à tous les ordres et antichamps

Les résultats discutés jusqu’à présent sont des résultats valables à une boucle. Dans une théorie de Yang-Mills avec champs de jauge quantifiés, une anomalie de jauge se manifeste par le fait que l’équation de Zinn-Justin naïve

$$\frac{1}{2}(\Gamma,\Gamma)_{\phi,\widetilde{\phi}^*}=0, \tag{8.149}$$

est violée par des anomalies. À une boucle, en utilisant que $\Gamma=S_{gf}+\hbar\Gamma^1+O(\hbar^2)$, et

$$\frac{1}{2}(S_{gf},S_{gf})_{\phi,\widetilde{\phi}^*}=0, \tag{8.150}$$

on trouve

$$(S_{gf},\Gamma^1)_{\phi,\widetilde{\phi}^*}=G_1. \tag{8.151}$$

On peut montrer que G_1 est une fonctionnelle locale en les champs ϕ et les antichamps $\widetilde{\phi}^*$. En appliquant $(S_{gf},\cdot)_{\phi,\widetilde{\phi}^*}$ et en utilisant l’identité de Jacobi gradué pour l’anticrochet ainsi que (8.150), on obtient la condition de cohérence pour l’anomalie

$$(S_{gf},G_1)_{\phi,\widetilde{\phi}^*}=0. \tag{8.152}$$

De nouveau, si l’anomalie est de la forme triviale, $G_1=(S_{gf},F_1)_{\phi,\widetilde{\phi}^*}$ avec F_1 une fonctionnelle locale des champs et antichamps de nombre de fantôme 0, l’anomalie peut être absorbée en rajoutant un contreterme fini à l’action fixée de jauge.

On trouve donc encore une fois que les anomalies non triviales sont des représentants de $H^1(s)$, mais cette fois-ci la différentielle est la différentielle BRST fixée de jauge $s=(S_{gf},\cdot)_{\phi,\widetilde{\phi}^*}$ dans l’espace des fonctionnelles locales en les champs et antichamps $\phi,\widetilde{\phi}^*$. Comme le changement vers les ϕ,ϕ^* de la base invariante de jauge est une transformation anticanonique, la cohomolgie est isomorphe à celle de la différentielle associée à la solution non fixée de jauge de l'équation maîtresse. En utilisant une homotopie contractante analogue à celle pour d, on montre facilement que la cohomologie est indépendante des champs $B^\alpha, \bar{C}^\alpha$ et leurs antichamps. Puis on peut montrer [50] que pour un groupe semi-simple, les seuls candidats pour les anomalies sont de la forme

$$\sum_m c_m \mathrm{Tr}_{G_m}\Big(Cd\big[AdA + \frac{1}{2}A^3\big]\Big), \tag{8.153}$$

avec des coefficients c_m à déterminer pour chaque facteur simple G_m.

On va voir qu'il y a des groupes de jauge pour lesquels $D_{\alpha\beta\gamma} = 0$ pour tout contenu fermionique. Dans ce cas, $H^1(s)$ est vide, ce que l'on supposera par la suite. Il ne peut donc pas y avoir d'anomalie non triviale à une boucle. Il pourrait y avoir une anomalie à 2 boucles,

$$(S_{gf}, \Gamma^2)_{\phi,\widetilde{\phi}^*} = 0, \tag{8.154}$$

$$(S_{gf}, \Gamma^2)_{\phi,\widetilde{\phi}^*} + \frac{1}{2}(\Gamma^1, \Gamma^1)_{\phi,\widetilde{\phi}^*} = G_2. \tag{8.155}$$

et de nouveau, on peut montrer que G_2 est une fonctionnelle locale de nombre de fantômes 1. On trouve alors, en appliquant $(S_{gf}, \cdot)_{\phi,\widetilde{\phi}^*}$ à la deuxième équation et en utilisant la première que

$$(S_{gf}, G_2)_{\phi,\widetilde{\phi}^*} = 0. \tag{8.156}$$

Puisque la cohomologie est vide, $G_2 = (S_{gf}, F_2)_{\phi,\widetilde{\phi}^*}$ peut être absorbée par un contreterme. Par récurrence on peut montrer que c'est le cas à tous les ordres et on peut donc établir l'équation de Zinn-Justin (8.149) à tous les ordres par un choix approprié de contretermes locaux.

8.10 Théories sans anomalies

L'anomalie chirale est proportionnelle au tenseur symétrique invariant

$$-iD_{\alpha\beta\gamma} = \frac{1}{2}\mathrm{Tr}\,(\{T_\alpha, T_\beta\}T_\gamma). \tag{8.157}$$

Si les courants sont couplés à des vrais champs de jauge, une condition suffisante pour ne pas avoir d'anomalie dans une symétrie de jauge est que ce tenseur s'annule,

$$D_{\alpha\beta\gamma} = 0. \tag{8.158}$$

Cette dernière propriété est désirable car si l'invariance de jauge ne peut être préservée dans la théorie quantique, on rencontre des problèmes avec l'unitarité: la théorie quantifiée n'est pas équivalente à une théorie pour laquelle on ne quantifie que des degrés de libertés physiques. **à faire: montrer que l'équation de Zinn-Justin implique l'unitarité**.

T_α est la représentation de l'algèbre de jauge sur tous les fermions et anti-fermions gauches.

Pour certains groupes de jauge, (8.158) est valable pour toute représentation des fermions. C'est le cas si la représentation est équivalente à sa complexe conjuguée,

$$T_\alpha^* = ST_\alpha S^{-1}. \tag{8.159}$$

Puisqu'on considère des représentations antihermitiennes, $T_\alpha^\dagger = -T_\alpha \iff T_\alpha^* = -T_\alpha^T$, ceci est équivalent à

$$T_\alpha^T = -ST_\alpha S^{-1}. \tag{8.160}$$

En insérant cette dernière relation dans la définition (8.157), on trouve alors $D_{\alpha\beta\gamma} = -D_{\alpha\beta\gamma}$.

Une telle représentation est soit réelle (s'il existe une matrice R non dégénérée R tel que $T'_\alpha = RT_\alpha R^{-1}$ avec T'_α réelle et antisymétrique) ou pseudo-réelle (si une telle matrice R n'existe pas). Par exemple, la représentation irréductible de $SU(2)$ à 3 dimensions est réelle (matrices ϵ_{ik}^j), mais celle à 2 dimensions est pseudo-réelle (matrices $\tau_i = -\frac{i}{2}\sigma_i$, $S = i\sigma_2$).

Il n'y a donc pas d'anomalie pour les algèbres de jauge n'ayant que des représentations réelles ou pseudo-réelles. C'est le cas pour les algèbres de Lie de $SO(2n+1)$ y compris $SU(2) \equiv SO(3)$, $SO(4n)$ pour $n \geqslant 2$, $USp(2n)$ pour $n \geqslant 3$, G_2, F_4, E_7, E_8 et leurs sommes directes. D'autres algèbres n'ont que des représentations pour lesquelles (8.158) est valable même si certaines de leurs représentations ne sont ni réelles ni pseudo-réelles. C'est le cas des algèbres de $SO(4n+2)$ (mais pas $SO(2) \equiv U(1)$).

De même, s'il n'y a que des fermions de Dirac dans la théorie se transformant comme (8.53) avec $t_\alpha^L = t_\alpha = t_\alpha^R$,

$$T_\alpha = \begin{pmatrix} t_\alpha & 0 \\ 0 & -(t_\alpha)^T \end{pmatrix}. \tag{8.161}$$

la représentation est (pseudo-)réelle avec

$$S = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{8.162}$$

et (8.158) est valable.

Les anomalies ne sont donc que possibles pour des algèbres de jauge contenant des facteurs $SU(n)$ avec $n \geqslant 3$ ou $U(1)$ et des couplages à des fermions chiraux. C'est justement celle-là qui sont importantes en physique des particules où le groupe de jauge est $SU(3) \times SU(2) \times U(1)$ et le couplage aux leptons est chiral. Il faudra donc utiliser des cancellations entre les fermions de la théorie pour garantir l'absence d'anomalies.

Avant de passer à la jauge unitaire suite à la brisure spontanée de symétrie, on a comme groupe de jauge $SU(2)_L \times U(1)_Y$ pour le Lagrangien leptonique du secteur électrofaible du modèle standard:

$$\psi_l = \begin{pmatrix} \nu_l \\ l \end{pmatrix}, \tag{8.163}$$

où l dénote le type (e pour électronique, μ pour muonique et τ pour tauonique) et

$$L_{lept} = -\sum_{l=e,\mu,\tau} \bar{\psi}_l\big(\gamma^\mu D_\mu^{B,C}\big)\psi_l \tag{8.164}$$

avec

$$D_\mu^{B,C}\psi_l = (\partial_\mu + gB_\mu^i t_i^L + g'C_\mu y)\psi_l, \tag{8.165}$$

$$t_i^L = P_+\frac{-i}{2}\sigma_i, \qquad y = -i\Big[P_+\tfrac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + P_-\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\Big]. \tag{8.166}$$

Notons que la charge électrique est associée au générateur

$$q = t_3^L - y = -i\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}. \tag{8.167}$$

Pour les quarks, on a $t_\alpha = \frac{-i}{2}\lambda_\alpha$, $q = u, c, t, d, s, b$, qui sont regroupés comme $q^+ = (u, c, t)$, $q^- = (d, s, b)$ et

$$\psi_{q^+}^L = P_+\begin{pmatrix} q^+ \\ \sum_{q^-} V_{q^+q^-}q^-\end{pmatrix} \tag{8.168}$$

avec $V_{q^+q^-}$ une matrice unitaire 3×3, la matrice de Kobayashi-Maskawa, et

$$L_{quarks} = -\sum_q \bar{q}(\gamma^\mu D_\mu^G)q - \sum_{q^+}\bar{\psi}_{q^+}^L\gamma^\mu\big(gB_\mu^i t_i + g'C_\mu y^{q_L^+}\big)\psi_{q^+}^L - \sum_q \bar{q}_R\gamma^\mu g'C_\mu y^{q_R}q_R, \tag{8.169}$$

avec

$$D_\mu^G q = (\partial_\mu + g_s G_\mu^\alpha t_\alpha)q, \tag{8.170}$$

$$y^{q_L^+} = -i\begin{pmatrix}-\frac{1}{6} & 0\\ 0 & -\frac{1}{6}\end{pmatrix} \qquad y^{q_R^+} = -i(-\frac{2}{3}), \qquad y^{q_R^-} = -i(\frac{1}{3}). \tag{8.171}$$

Ceci explique la table suivante extraite de [25] pour une génération:

Table 22.1. First-generation left-handed fermion and antifermion fields of the standard model.

Fermions	$SU(3)$	$SU(2)$	$U(1)$ $[y/g']$
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	3	2	−1/6
u_R^*	$\bar{3}$	1	+2/3
d_R^*	$\bar{3}$	1	−1/3
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	1	2	1/2
e_R^*	1	1	−1

Comme les anomalies avec 1 facteur $SU(3)$ ou 1 facteur $SU(2)$ sont proportionnelles à $\mathrm{tr}\, t_\alpha = 0 = \mathrm{tr}\, t_i$. On doit vérifier les cas suivants pour chaque génération:

- $SU(3) - SU(3) - SU(3)$: pas d'anomalie car couplage non-chiral
- $SU(3) - SU(3) - U(1)$:

$$\mathrm{tr}\,[\{t_\alpha, t_\beta\} y^q] = -\frac{1}{2}\delta_{\alpha\beta}\mathrm{tr}\, y^q = -\frac{1}{2}\delta_{\alpha\beta}(-\frac{1}{6} - \frac{1}{6} + \frac{2}{3} - \frac{1}{3}) = 0. \tag{8.172}$$

- $SU(2) - SU(2) - SU(2)$: pas d'anomalie car représentation réelle ou pseudo-réelle
- $SU(2) - SU(2) - U(1)$:

$$\mathrm{tr}\,[\{t_i, t_j\}(y^{q_L^+} + y)] = -\frac{1}{2}\delta_{ij}\mathrm{tr}\,(y^{q_L^+} + y) = -\frac{1}{2}\delta_{ij}[3(-\frac{1}{6}) + \frac{1}{2}] = 0. \tag{8.173}$$

- $U(1) - U(1) - U(1)$:

$$\mathrm{tr}\,[(y^{q_L^+} + y^{q_R} + y)^3] = 6(\frac{-1}{6})^3 + 3(\frac{2}{3})^3 + 3(\frac{-1}{3})^3 + 2(\frac{1}{2})^3 + (-1)^3 = 0. \tag{8.174}$$

On trouve donc que le modèle standard est exempt d'anomalies chirales de jauge et la cancellation dépend de l'organisation des quarks et des leptons à l'intérieur de chaque génération.

Des anomalies supplémentaires sont dues au couplage à la gravitation. On peut montrer que l'anomalie en présence d'un champ de gravitation externe est proportionnelle à

$$\mathrm{tr}(T_\alpha)\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\kappa\lambda} R_{\rho\sigma}{}^{\kappa\lambda}. \tag{8.175}$$

à faire par le heat kernel

Le seul cas non trivial à vérifier est $U(1)$. Or

$$\mathrm{tr}(y^{q_L^+} + y^{q_R} + y) = 6(-\frac{1}{6}) + 3(\frac{2}{3}) + 3(-\frac{1}{3}) + 2(\frac{1}{2}) + (-1) = 0, \tag{8.176}$$

et il n'y a donc pas non plus d'anomalies gravitationnelles dans le modèle standard.

8.11 Exercices

8.11.1 Théorème d'Atiyah-Singer à toute dimension

En commençant en $d = 2$, montrer que le théorème de l'indice pour $d = 4$ donné par (8.50), se généralise en dimensions paires $d = 2n$ de la manière suivante: si

$$\boxed{\mathrm{ch}\,(F) = \mathrm{tr}_I\ \exp\frac{F}{2\pi}}, \tag{8.177}$$

alors

$$\boxed{\mathrm{index}\, i\not{D}_+ = \int_{M_{2n}} \mathrm{ch}\,(F)} = \frac{1}{n!}\int_{M_{2n}} \mathrm{tr}_I(\frac{F}{2\pi})^n. \tag{8.178}$$

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