4e

9 COUNTER-TERMS AND RENORMALIZATION CONDITION

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→ We'll now adress renormalized perturbation theory. We'll reformulate the computation such that renormalized quantities appear in the results, while counterterms will arise, splitting the bare Lagrangian from the physical one. This splitting is implemented by renormalization conditions. We'll also see a different regularization method.

9.1 Renormalized perturbation theory

→ We exemplify the procedure with our favorite theory: $\lambda\phi^4$

DEF We declare the usual Lagrangian to be the bare Lagrangian $\mathcal{L}_b$:

$$\mathcal{L}_b = \frac{1}{2}(\partial_\mu \phi_b)^2 - \frac{1}{2} m_b^2 \phi_b^2 - \frac{1}{4}\lambda_b \phi_b^4$$

We write ϕ_b, m_b, λ_b to emphasize that these are the bare values.

→ The field need to be renormalized in order to be related to the propagator of the exact quantum particle associated to it:

↳ First, we eliminate the shift in the field strength. We had

(p39) $$\langle\Omega|\phi_b(p)\,\phi_b(-p)|\Omega\rangle = \frac{iZ}{p^2 - m_r^2} + (\text{terms regular at } p^2 = m^2)$$

where m_r is the renormalized mass (physical). We eliminate the residue Z by rescaling the field: $\phi_b = \sqrt{Z}\,\phi_r$ so that

$$\langle\Omega|\phi_r\,\phi_r|\Omega\rangle = \frac{i}{p^2 - m_r^2} + (\dots)$$

↳ The Lagrangian transform as:

$$\mathcal{L}_b = \frac{1}{2} Z (\partial_\mu \phi_r)^2 - \frac{1}{2} m_b^2 Z \phi_r^2 - \frac{1}{4}\lambda_b Z^2 \phi_r^4$$

To get rid of the bare mass and coupling constant, we define:

DEF $\delta_Z \equiv Z - 1$; $\delta_{m^2} \equiv m_b^2 Z - m_r^2$ and $\delta_\lambda \equiv \lambda_b Z^2 - \lambda_r$

$\Leftrightarrow Z = 1 + \delta_Z$; $\Leftrightarrow m_b^2 Z = m_r^2 + \delta_{m^2}$ and $\Leftrightarrow \lambda_b Z^2 = \lambda_r + \delta_\lambda$

↳ The Lagrangian transform as:

$$\mathcal{L}_b = \frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{1}{2} m_r^2 \phi_r^2 - \frac{\lambda_r}{4}\phi_r^4$$
$$+\frac{1}{2}\delta_Z (\partial_\mu \phi_r)^2 - \frac{1}{2}\delta_{m^2} \phi_r^2 - \frac{1}{4}\delta_\lambda \phi_r^4$$

DEF We define the renormalized Lagrangian $\mathcal{L}_r$ as

$$\mathcal{L}_r \equiv \frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{1}{2} m_r^2 \phi_r^2 - \frac{1}{4}\lambda_r \phi_r^4$$

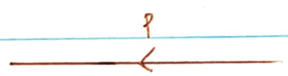
and the counterterms $\mathcal{L}_{ct}$ as

$$\mathcal{L}_{ct} \equiv \frac{1}{2}\delta_Z (\partial_\mu \phi_r)^2 - \frac{1}{2}\delta_{m^2} \phi_r^2 - \frac{1}{4}\delta_\lambda \phi_r^4$$

→ We've showed that $\mathcal{L}_b = \mathcal{L}_r + \mathcal{L}_{ct}$

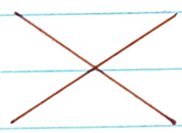
→ The renormalized perturbation theory is performed using the counterterms as perturbations in the computations. Note that there are now "interactions" with only 2 legs.

⊙ Feynman rules for $\lambda\phi^4$ theory in renormalized perturbation th.

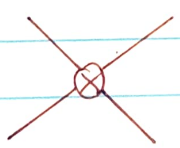


$= \dfrac{i}{p^2 - m^2 + i\epsilon}$

$= i(p^2 \delta_Z - \delta_{m^2})$ from the term $\frac{1}{2}\delta_Z (\partial_\mu \phi_r)^2 - \frac{1}{2}\delta_{m^2} \phi_r$



$= -i6\lambda$



$= -i6\delta_\lambda$ from the term $-\frac{1}{4}\delta_\lambda \phi_r^4$

Not We denote counterterm in RPT (renormalized perturbation theory) by a ⊗ vertex.

→ The coefficients of the counterterms δ_Z, δ_{m^2} and δ_λ are parameters that need to be adjusted order by order, to cancel $\mathcal{L}_b - \mathcal{L}_r$. Such δ's are usually divergents, wich is ok since $\mathcal{L}_r$ is unobservable.

9.2 Renormalization conditions

→ There is an arbitrariness in the split $\mathcal{L}_b \mapsto \mathcal{L}_r + \mathcal{L}_{ct}$. The way to fix it is to set some conditions: enforce that m_r^2, λ_r and ϕ_r are indeed the physical quantities. However, the radiative corrections typically depend on the external momenta.

⊙ <u>On shell renormalisation condition:</u>

→ The condition that will determine δ_Z and δ_{m^2} is to require that, on shell (at $p^2 = m^2$), the exact propagator is exactly $\frac{i}{p^2 - m^2}$.

→ In general, we have $\frac{i}{p^2 - m^2 - M(p^2)}$ = —●— = —— + —(1PI)— + —(1PI)—(1PI)—

Reminder: we defined the sum of all one-particle-irreducible^{+} insertions into the propagator —(1PI)— $= -iM(p^2)$. Then the full 2-pt function is given by the geometric series

—●— = —— + —①—①— + ... $= \frac{i}{p^2 - m^2 - M(p^2)}$

↳ Expand $M(p^2)$ around $p^2 = m^2$:

$$M(p^2) = M(m^2) + \frac{d}{dp^2} M(p^2)\Big|_{\text{shell}} (p^2 - m^2) + \mathcal{O}\left[(p^2 - m^2)^2\right]$$

? We see that $M(m^2)$ would correct m^2 while $\frac{d}{dp^2} M(p^2)\Big|_{p^2 = m^2}$ would correct Z.

DEF The renormalization conditions are

① $M(m^2) \overset{!}{=} 0$ ② $\frac{d}{dp^2} M(p^2)\Big|_{p^2 = m^2} \overset{!}{=} 0$

Indeed, we have

$\frac{i}{p^2 - m^2 - M^2} \mapsto \frac{i}{p^2 - m^2 - M(m^2) - M'^2(m^2)(p^2 - m^2)}$ After setting $M(m^2) \overset{!}{=} 0$, we're left with $\frac{i(1 - M'(m^2))^{-1}}{p^2 - m^2}$ which corrects $\frac{iZ}{p^2 - m^2}$.

→ To determine δ_λ, we need to impose a condition on the exact 4-pt function  at some value for the external momenta.

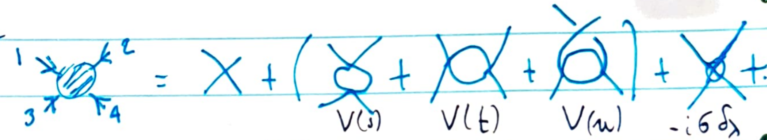
↳ We consider $1,2 \to 3,4$ scattering, on shell and collinear. Then, $s \equiv (p_1+p_2)^2 = 4m^2$ and $t \equiv (p_1+p_3)^2 = 0 = (p_1+p_4)^2 \equiv u$.

↳ We ask:  $\stackrel{!}{=} -6i\lambda$ ⊛

→ Let's now go through a series of examples, actually using the radiative corrections that we had already computed.

9.3 Fix δ_λ through NLO of $\langle\phi^4\rangle$

→ We'll fix δ_λ by computing the one-loop radiative corrections to the 1PI 4-pt function. We consider the basic 2-part. scattering amplitude

$i\mathcal{M}(p_1 p_2 \to p_3 p_4) =$ 

Again, defining $p = p_1+p_2 = \sqrt{s}$, the  is

$$\frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{i}{(k-p)^2-m^2} \equiv (-i\lambda)^2 \cdot iV(p^2) = (-i\lambda)^2 \cdot iV(s)$$

↳ The entire amplitude is therefore

$$\mathcal{M} = -i6\lambda + V(s=4m^2) + V(t=0) + V(u=0) - i6\delta_\lambda \stackrel{!⊛}{=} -i6\lambda$$

$$\Leftrightarrow 6i\delta_\lambda \stackrel{⊛}{=} V(4m) + 2V(0)$$

→ From section 6.2 (p32), we have:

$$V(s) = 18i\lambda^2\int_0^1 dx\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+m^2-sx(1-x)]^2}$$

$$= \frac{18i\lambda^2}{(4\pi)^2}\int_0^1 dx\int_0^{\Lambda^2} dk_E^2\,\frac{k_E^2}{\{k_E^2+m^2-sx(1-x)\}^2}$$

$$= -\frac{18i\lambda^2}{(4\pi)^2}\int_0^1 dx\,\log\left[\frac{m^2-sx(1-x)}{\Lambda^2}\right]$$

$$\Rightarrow 6i\delta_\lambda \stackrel{⊛}{=} -\frac{18i\lambda^2}{(4\pi)^2}\int_0^1 dx\,\log\left[\frac{m^2}{\Lambda^2}(1-2x)^2\right] - \frac{2\cdot 18i\lambda^2}{(4\pi)^2}\log\left[\frac{m^2}{\Lambda^2}\right]$$

$$\simeq \frac{54i\lambda^2}{(4\pi)^2}\log\left[\frac{m^2}{\Lambda^2}\right]$$

We find $\delta_\lambda = \frac{9\lambda^2}{(4\pi)^2} \log\left[\frac{\Lambda^2}{m^2}\right]$ neglecting $\int_0^1 dx \log(1-2x)^2 = -2$

↳ We hid the divergence in δ_λ.

→ Actually, if we want to be precise, we shouldn't neglect any finite term. But any specific prescription of substracting the infinite part is actually called a renormalisation scheme. Observable quantities should be scheme independent.

9.4 Dimensional regularization

→ There is an alternating regularization for the same integral. Properties of QFT's change when the dimension of spacetime changes. We're going to consider our theory in d dimensions, and analytically continue in d.

→ Recall (p32), we had:

$$V(s) = 18i\lambda^2 \int_0^1 dx \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{\{k_E^2 - px(1-x) + m^2\}^2}$$

$$= 18i\lambda^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k_E^2 + \Delta)^2} \qquad \text{with } \Delta \equiv m^2 - sx(1-x)$$

$$\sim \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2 + \Delta)^2}$$

→ Let's be more general and compute

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{(k_E^2)^n}{(k_E^2 + \Delta)^m}$$ We need some math first.

⊙ Volume of a $(d-1)$ dim. sphere $V(S^{d-1})$:

$$\rightarrow \int d^d x_i \, e^{-x_i^2} = \left(\int dx \, e^{-x^2}\right)^d = \pi^{d/2} = V(S^{d-1}) \int_0^\infty d\rho \, \rho^{d-1} e^{-\rho^2}$$

$$= V(S^{d-1}) \int_0^\infty \tfrac{1}{2} dt \, t^{(d-2)/2} e^{-t} = \tfrac{1}{2} V(S^{d-1}) \Gamma(d/2)$$

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t} dt$$

$$\Leftrightarrow V(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

Properties of Γ-functions:

DEF The gamma function $\Gamma(z)$ is defined as:
$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t} dt$$

→ Product of Γ-functions are given by:
$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty dt\, t^{\alpha-1} e^{-t} \int_0^\infty du\, u^{\beta-1} e^{-u} = \int_0^\infty dt \int_0^\infty du\, t^{\alpha-1} u^{\beta-1} e^{-(t+u)}$$

Setting $t+u = x$, $t-u = y$: $t = \frac{1}{2}(x+y)$; $u = \frac{1}{2}(x-y)$ and $dt\,du = \frac{1}{2} dx dy$,
$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty dx \int_{-x}^{x} dy \frac{1}{2^{\alpha+\beta-1}} (x+y)^{\alpha-1} (x-y)^{\beta-1} e^{-x}$$

Setting $z \equiv y/x$, we get:
$$\frac{1}{2^{\alpha+\beta-1}} \int_0^1 dx\, x^{\alpha+\beta-1} e^{-x} \int_{-1}^{1} dz\, (1+z)^{\alpha-1} (1-z)^{\beta-1}.$$ Setting $1+z \equiv v$
$$= \Gamma(\alpha+\beta) \int_0^1 dv\, v^{\alpha-1} (1-v)^{\beta-1}$$

DEF The Euler beta function $B(z_1, z_2)$ is defined as
$$B(z_1, z_2) = \int_0^1 t^{z_1-1} (1-t)^{z_2-1} dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(\alpha+\beta)}$$

→ Finally, another usefull definition is linked to the derivation of the Γ-function.

DEF The Euler-Mascheroni constant γ is defined (among others) as
$$\gamma \equiv -\Gamma'(1) \simeq 0{,}58$$

Back to the diverging integral:

→ We had: $$\int \frac{d^d k_E}{(2\pi)^d} \frac{(k_E^2)^m}{(k_E^2+\Delta)^n} = \frac{V(S^{d-1})}{(2\pi)^d} \int_0^\infty d|k_E| \frac{|k_E|^{d-1} (k_E^2)^m}{(k_E^2+\Delta)^n}$$
$$= \frac{V(S^{d-1})}{2(2\pi)^d} \int_0^\infty dk_E^2 \frac{(k_E^2)^{\frac{d}{2}-1+m}}{(k_E^2+\Delta)^n}$$

We set $k_E^2 \equiv \Delta u$ (suppose $\Delta > 0$). Then:
$$= \frac{V(S^{d-1})}{2(2\pi)^d} \Delta^{\frac{d}{2}+m-n} \int_0^\infty du \frac{u^{\frac{d}{2}-1+m}}{(1+u)^n}$$

↳ The power of Δ $\left(\frac{d}{2}+m-n\right)$, the only dimensionful quantity, could have been guessed just by dimensional analysis.

→ From $\int_0^\infty du \frac{u^{\frac{d}{2}-1+m}}{(1+u)^n}$ we set $1+u \equiv 1/t \Leftrightarrow u = 1/t - 1 = (1-t)/t$. Then

$$= \int_0^1 dt\, t^{-2} (1-t)^{\frac{d}{2}-1+m}\, t^{-\frac{d}{2}+1-m}\, t^n$$

$$= \int_0^1 dt\, t^{n-m-\frac{d}{2}-1} (1-t)^{m+\frac{d}{2}-1} = B\left(n-m-\frac{d}{2}, m+\frac{d}{2}\right)$$

$$= \frac{\Gamma(n-m-d/2)\,\Gamma(m+d/2)}{\Gamma(n)}$$

→ Gathering those long computations, we get:

$$\int \frac{d^dk_E}{(2\pi)^d} \frac{(k_E^2)^m}{(k_E^2+\Delta)^n} = \frac{\Delta^{\frac{d}{2}+m-n}}{(4\pi)^{d/2}} \frac{\Gamma(n-m-d/2)\Gamma(m+d/2)}{\Gamma(d/2)\,\Gamma(n)}$$

↳ The only potential divergencies hide in $\Gamma(n-m-d/2)$. We expect it to be divergent when $\frac{d}{2}+m-n \geqslant 0$

⇒ We regularize the integral taking d to be continuous and close to 4: $d \equiv 4-2\epsilon$

→ Going back to our integral, we had $m=0$ and $n=2$:

$$\int \frac{d^dk_E}{(4\pi)^d} \frac{1}{(k_E^2+\Delta)^2} = \frac{\Delta^{\frac{d}{2}-2}}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(d/2)\,\Gamma(2)} = \frac{\Delta^{(d-4)/2}}{(4\pi)^{d/2}} \cdot \Gamma(2-d/2)$$

→ We make the integral dimensionless by multiplying by the appropriate power of an arbitrary function scale μ. We have:

$$\mu^{4-d} \int \frac{d^dk_E}{(2\pi)^d} \frac{1}{(k_E^2+\Delta)^2} = \frac{1}{(4\pi)^2} \left(\frac{\Delta}{4\pi\mu^2}\right)^{\frac{d}{2}-2} \Gamma\left(2-\frac{d}{2}\right)$$

$d = 4-2\epsilon$

$$\hookrightarrow = \frac{1}{(4\pi)^2}\left(\frac{\Delta}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(\epsilon) = \frac{1}{(4\pi)^2}\left(1-\epsilon\log\left[\frac{\Delta}{4\pi\mu^2}\right]\right)\frac{1}{\epsilon}\Gamma(1+\epsilon)$$

$$\hookrightarrow = \frac{1}{\epsilon}\Gamma(1+\epsilon) \simeq \frac{1}{\epsilon}\left(\Gamma(1)+\epsilon\Gamma'(1)\right) = \frac{1}{\epsilon}\left(\Gamma(1)-\epsilon\gamma\right)$$

$$= \frac{1}{(4\pi)^2}\left(1-\epsilon\log\left[\frac{\Delta}{4\pi\mu^2}\right]\right)\frac{1}{\epsilon}(1-\epsilon\gamma)$$

↳ Taking the limit:

$$\lim_{d\to 4} \mu^{4-d} \int \frac{d^dk_E}{(2\pi)^d} \frac{1}{(k_E^2+\Delta)^2} = \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + \log 4\pi - \gamma - \log\left[\frac{\Delta}{\mu^2}\right]\right)$$

→ Eventually, we have:

$$V(s) = \frac{18i\lambda}{(4\pi)^2}\int_0^1 dx \left\{ \frac{1}{\epsilon} + \log 4\pi - \gamma - \log\left[\frac{m^2 - s\,x(1-x)}{\mu^2}\right]\right\}$$

The renormalization proceeds exactly as before:

$$6i\delta_\lambda = V(4m^2) + 2V(0)$$

so that the correction to the 4-pt vertex reads

$$\{V(s) - V(4m^2)\} + \{V(t) - V(0)\} + \{V(u) - V(0)\}$$

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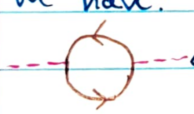
$$= -\frac{18i\lambda}{(4\pi)^2}\int_0^1 dx \log\left[\frac{m^2 - s x(1-x)}{m^2(1-2x)^2}\right] - \frac{18i\lambda^2}{(4\pi)^2}\int_0^1 dx\, \frac{m^2 - t\,x(1-x)}{m^2}$$

$$- \frac{18i\lambda^2}{(4\pi)^2}\int_0^1 dx\, \frac{m^2 - u(1-x)}{m^2}$$

→ Same as before, ie: independant on regularization and its parameters (Λ on one side, ϵ and μ on the other).

9.5 Field strength and mass renormalization

→ To determine δ_Z and δ_{m^2}, let's consider the correction to the scalar propagator in Yukawa theory. From section 6.4 (p34), we have:



$$\Leftrightarrow -i\mathcal{M}(p^2) = 4ig^2 \int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4}\frac{k_E^2 - \Delta}{(k_E^2+\Delta)^2}$$

where $\Delta = m_\phi^2 - p^2 x(1-x)$

→ We use dimensional regularization:

$$\int \frac{d^dk_E}{(2\pi)^d}\frac{k_E^2}{(k_E^2+\Delta)^2} = \frac{\Delta^{(d-2)/2}}{(4\pi)^{d/2}}\frac{\Gamma(1-d/2)\Gamma(1+d/2)}{\Gamma(d/2)\,\Gamma(2)} \quad \text{and } \Gamma(1+\alpha) = \alpha\Gamma(\alpha)$$

$$= \frac{\Delta^{\frac{d}{2}-1}}{(4\pi)^{d/2}}\cdot\frac{d}{2}\cdot\Gamma(1-d/2)$$

And $$\int \frac{d^dk_E}{(2\pi)^d}\frac{\Delta}{(k_E^2+\Delta)^2} = \frac{\Delta^{\frac{d}{2}-1}}{(4\pi)^{d/2}}\Gamma(2-d/2)$$

$\to$ We get: $$\int \frac{d^d k_E}{(2\pi)^d} \frac{k_E^2-\Delta}{(\Delta+k_E^2)^2} = \frac{\Delta^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\left(\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)-\Gamma\left(2-\frac{d}{2}\right)\right)$$

$$= \frac{\Delta^{1-\epsilon}}{(4\pi)^{2-\epsilon}}\left\{(2-\epsilon)\Gamma(-1+\epsilon)-\Gamma(\epsilon)\right\} \qquad \Gamma(-1+\alpha) = \frac{\Gamma(\alpha)}{-1+\alpha}$$

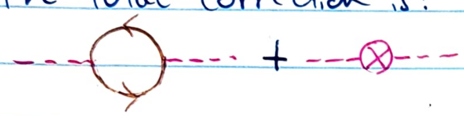
$$= \frac{\Delta}{(4\pi)^2}\left(\frac{\Delta}{4\pi\mu^2}\right)^{-\epsilon}\left[-\frac{2-\epsilon}{1-\epsilon}-1\right]\Gamma(\epsilon)$$

$$= \frac{\Delta}{(4\pi)^2}\left(1-\epsilon\log\left[\frac{\Delta}{4\pi\mu^2}\right]\right)(-3-\epsilon)\left(\frac{1}{\epsilon}-\gamma\right)$$

$$= -\frac{3\Delta}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma+\frac{1}{3}-\log\left[\frac{\Delta}{4\pi\mu^2}\right]\right)$$

$\hookrightarrow$ The quadratic and the log. divergence show up as a simple $1/\epsilon$ pole $\to$ peculiarity of dimensional regularization

$\to$ The total correction is:



$$-i\mathcal{M}(p^2) = \frac{12ig^2}{(4\pi)^2}\int_0^1 dx\left\{m_\chi^2-p^2x(1-x)\right\}\left\{\frac{1}{\epsilon}-\gamma+\frac{1}{3}-\log\left[\frac{m_\chi^2-p^2x(1-x)}{4\pi\mu^2}\right]\right\}$$
$$+\, i(p^2\delta_Z-\delta_{m^2})$$

$\hookrightarrow$ The renormalization conditions being $\mathcal{M}(m^2)=0$ and $\mathcal{M}'(m^2)=0$, we get:

$$-\frac{12ig^2}{(4\pi)^2}\int_0^1 dx\left\{m_\chi^2-m^2x(1-x)\right\}\left\{\frac{1}{\epsilon}-\gamma+\frac{1}{3}-\log\left[\frac{m_\chi^2-m^2x(1-x)}{4\pi\mu^2}\right]\right\}+i(m^2\delta_Z-\delta_{m^2})=0$$

and $$\frac{12ig^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\left(\frac{1}{\epsilon}-\gamma-\frac{2}{3}-\log\left[\frac{m_\chi^2-m^2x(1-x)}{4\pi\mu^2}\right]\right\}+i\delta_Z=0$$

$\hookrightarrow$ By doing long computation, we eventually get:

$$-i\mathcal{M}(p^2) = \frac{12ig^2}{(4\pi)^2}\int_0^1 dx\left\{(m_\chi^2-p^2x(1-x))\log\left[\frac{m_\chi^2-p^2x(1-x)}{m_\chi^2-m^2x(1-x)}\right]+(p^2-m^2)x(1-x)\right\}$$

$\to$ We see that δ_Z and δ_{m^2} are divergent, but $\mathcal{M}(p^2)$ is eventually finite, even though p-dependent.