$\boxed{8}$ POWER COUNTING, DIVERGENCIES AND RENORMALIZABILITY

→ On one hand, we need to renormalize the fields. On the other one, we saw that some quantum corrections are divergent. To adress this issue, we make the following statement.

DEF | The physical, renormalized quantities are always finite. The bare quantities might diverge.

↳ The bare quantities where already diverging, in a compensating way. In any case, we only measure the physical quantities.

→ We will be concerned here only with UV divergences, ie those arising for large momenta $|k| \to \infty$. In the presence of massless particles, there can be also IR divergences at $|k| \to 0$.

→ Since we need to reabsorb divergencies of a QFT in its bare parameters of the Lagrangian, there is only a finite number of divergencies that can be treated in this way ⇒ some QFT will be renormalizable, and some not.

8.1 Example : $\lambda\phi^4$ theory

→ In the $\lambda\phi^4$ theory, we encountered different types of divergences.

1) 2-pt function:  $\sim \lambda \int d^4k \; 1/k^2 \sim \lambda\Lambda^2$: quadratic

↳ 1 loop, 1 internal propagator (with limit $|k| \gg |p|, m$, i.e. UV)

2) 4-pt function:  $\sim \lambda^2 \int d^4k \; 1/(k^2)^2 \sim \lambda^2 \log \Lambda^2$

log-divergence

3) 6-pt function:  $\sim \lambda^3 \int d^4k \; 1/(k^2)^3 \sim \lambda^3 \frac{1}{m^2}$: finite !

→ This theory, at 1-loop, has only divergencies for $\langle\phi^2\rangle$ and $\langle\phi^4\rangle$!

8.2 Powercounting

→ In a more systematic way, we consider a $\lambda\phi^n$ interaction in d dimension:

$$S = \int d^dx \left\{ \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{n}\lambda\phi^n \right\}$$

→ The K-G equation in d dimensions is the same:

$(-\partial^2 - m^2)\phi = 0$ free theory

so that the propagator satisfies:

$(-\partial^2 - m^2) D_0(x-y) = \delta^d(x-y)$

$\Leftrightarrow D(x-y) = \dfrac{1}{-\partial^2 - m^2}\delta^d(x-y)$

$\Leftrightarrow D(p) = \dfrac{1}{p^2 - m^2}$. In any d, the propagator of a scalar goes like $1/p^2$ at large momenta

p315 Peskin ⊙ Characterizing a generic diagram:

DEF A generic diagram will be characterized by a set of discrete numbers:

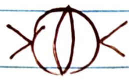
L: # of loops (= # of $\int d^dk$)
N: # of external lines (points)
V: # of vertices ($\lambda\phi^n$)
P: # of internal propagators

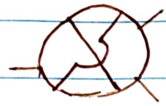
→ They're 2 types of relation among these number:

PROP A topological relation: $nV = 2P + N$
Indeed, since each vertex has n lines attached to it, and each internal line attaches to 2 vertices, we have $nV = 2P + N$

PROP A dynamical relation: $L = P - V + 1$
Indeed, L is equal to the # of d-dim. momentum integrals left after implementing momentum conservation at every vertex on internal propagators, plus the overall momentum conservation.

↳ Examples:  $L = 1 - V + 1 = 1$ loops

 $L = 6 - 4 + 1 = 3$ loops

 $L = 9 - 7 + 1 = 3$ loops

DEF The superficial degree of divergence D is defined as
$$D = dL - 2P \qquad (\sim \Lambda^D)$$

↳ Indeed, each loop counts d dimensions because of $\int d^dk$, and each internal propagator counts -2 because of $1/k^2$

↳ The degree of divergence of a diagram is found, essentially, by its mass dimension: this is power counting

→ Let's massage the definition a bit:
$$D = dL - 2P = d(P - V + 1) - 2P = dP - dV + d - 2P$$
$$= d + (d-2)P - dV$$
$$= d + (d-2) \cdot \frac{nV - N}{2} - dV = d - V\left(d - n\frac{d-2}{2}\right) - N\left(\frac{d-2}{2}\right)$$

↳ Notice that in d-dim, we have $[S] = 0 \Rightarrow [(\partial\phi)^2] = d$
$\Rightarrow 2[\phi] + 2 = d \Leftrightarrow [\phi] = \frac{d-2}{2}$ and the interaction term has:

$[\lambda\phi^n] = d \Leftrightarrow [\lambda] = d - n\left(\frac{d-2}{2}\right)$. Replacing, we get:

PROP
$$D = d - V[\lambda] - [\phi]N$$
$$= d - V\left(d - \left(\frac{d-2}{2}\right) \cdot n\right) - N \cdot (d-2)/2$$

⊙ Example: $d = 4$ and $\lambda\phi^4$:

→ We have $[\phi] = (4-2)/2 = 1$ and $[\lambda] = 4 - 4 \cdot 1 = 0$ so that $D = 4 - N$

↳ We reobtain the result that:
- for $N = 2$; $D = 2 \Rightarrow$ quadratic divergence
- for $N = 4$; $D = 0 \Rightarrow$ logarithmic divergence
- for $N \geq 6$; $D < 0 \Rightarrow$ no UV divergency

Example: $d=4$ and $\lambda\phi^6$:

→ We have $[\phi]=1$ still, and $[\lambda]=4-6\cdot1=-2$ so $D=4+2V-N$
↳ For any N, at high enough order (V large), we'll have $D>0$.
Hence, any N-pt function at some order will diverge.

8.3 Characterizing renormalizability:

→ Supposing $[\phi]>0$ ($d>2$), they're 3 ≠ cases: the effect of increasing V uniquely depends on $[\lambda]$:

DEF

① A theory is super-renormalizable if $[\lambda]>0$. There is only a finite number of diagrams that diverges (low V and N)

② A theory is renormalizable if $[\lambda]=0$. There is a finite number of N-pt functions wich have diagrams that diverge, though at any order in V.

③ A theory is non-renormalizable if $[\lambda]<0$. For any N, there are diagrams with V large enough that diverge.

→ Remember D is a superficial degree of divergence, since a diagram can have sub-divergences wich make it divergent even if $D<0$, or some cancellations due to symmetries can make a diagram less divergent or finite, even if $D\geq0$

→ Example in $d=4$: ϕ^4 → renormalizable
ϕ^3 → superrenormalizable (with $V<0$)
ϕ^6 → non-renormalizable

→ Other renormalizable theories are $\lambda\phi^n$ in $n=\dfrac{2n}{n-2}$