$\boxed{7}$ PHYSICS OF RENORMALIZATION

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We've computed quantum corrections to propagators and vertices of QFT's. These affects the effective action, wich means both the kinetic term and the interaction term. We want to know what is the physical mass of a particle, or what is the physical field that creates it.

p 211 7.1 Field-Strength Renormalization

→ For simplicity, we consider the propagator of a scalar field theory

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle_{int} = \langle \Omega|T\phi_x\phi_y|\Omega\rangle$$

What is the general structure of the propagator in the exact theory (with quantum corrections and interactions)? To dissect the 2-pt function we'll insert the identity operator in the form of a sum of complete set of states

→ Recall that $\mathbb{1} = \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_\lambda(\bar{p})}|\lambda_p\rangle\langle\lambda_p|$ where $|\lambda_p\rangle$ is a state of momentum p: $\hat{P}|\lambda_p\rangle = p$, and $\omega_\lambda = \sqrt{\bar{p}^2+m_\lambda^2}$, $\frac{d^3p}{\omega_\lambda}$ are Lorentz invariant.

↳ The states are normalized as $\langle\lambda_p|\lambda'_q\rangle = \delta_{\lambda,\lambda'}(2\pi)^3\, 2\omega_p\, \delta^3(\bar{p}-\bar{q})$

→ Assuming $x^0 > y^0$. We can write:

$$\langle\Omega|\phi_x\phi_y|\Omega\rangle = \sum_\lambda\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_\lambda}\langle\Omega|\phi_x|\lambda_p\rangle\langle\lambda_p|\phi_y|\Omega\rangle$$

Let's write the matrix element as:

$$\langle\Omega|\phi_x|\lambda_p\rangle = \langle\Omega|e^{iP.x}\phi(0)e^{-iP.x}|\lambda_p\rangle$$
$$= \langle\Omega|\phi(0)|\lambda_p\rangle e^{-ip.x}|_{p^0=\omega_p}$$
? $$= \langle\Omega|\phi(0)|\lambda_0\rangle e^{-ip.x}|_{p^0=\omega_p}$$

↳ We get:

$$\langle\Omega|\phi_x\phi_y|\Omega\rangle = \sum_\lambda\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_\lambda}e^{-ip(x-y)}|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

We now want to integrate over p_0 (contour with $i\epsilon$ prescription):

$$\langle\Omega|\phi_x\phi_y|\Omega\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_\lambda^2+i\epsilon}\, e^{-ip(x-y)}\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

DEF We introduce the spectral density function $\rho(m^2)$ defined as

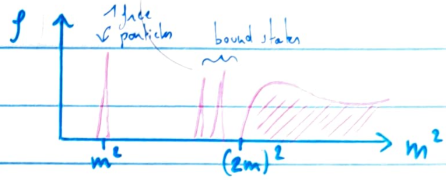
$$\rho(m^2) \equiv \sum_\lambda \delta(m^2-m_\lambda^2)\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$; real and positive,

and the Källén-Lehmann spectral representation

$$\langle\Omega|T\phi_x\phi_y|\Omega\rangle = \int_0^\infty dm^2\,\rho(m^2)\,D_F(x-y\,;\,m^2)$$

$$= \int_0^\infty dm^2\,\rho(m^2)\int\frac{d^4p}{(2\pi)^4}\,\frac{ie^{-ip(x-y)}}{p^2-m^2+i\epsilon}$$

⊙ Typical functional form of $\rho(m^2)$:



→ If $|\lambda\rangle$ is a single-particle state, we expect $\rho(m^2)\propto\delta(m^2-m_\lambda^2)$

→ If $|\lambda\rangle$ is a multi-particle state, we expect a continuous function of m^2.

Example: Yukawa theory, propagator of ϕ at NLO

For $p^2 \geq (2m_\psi)^2$, the virtual fermion pair can be on-shell. Above this threshold, we expect a continuous profile of $\rho(m^2)$.

→ If $\langle\phi\phi\rangle$ is related to an elementary field, we expect a 1st δ-function related to 1-particle (free), then additional δ-functions related to bound states. Those are usually related to strong coupling effects.

→ We then split ρ into the 1st δ-function and the rest.

DEF We write the spectral density function as:

$$\rho(s) = Z\,.\,\delta(s-m^2)+\tilde\rho(s) \quad \text{with } \tilde\rho(s)\begin{cases} =0 & \text{for } s\leq(2m)^2\\ \neq 0 & \text{for } s>(2m)^2\end{cases}$$

We refer to Z as the field-strength renormalization:

$$Z \equiv |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

→ In $\rho(s)$, m^2 is the exact mass of a single particle (the exact energy eigenvalue at rest)

7.2 Physical and bare quantities

→ In the Fourier space, the Källén-Lehmann spectral rep becomes:

$$\int d^4x\, e^{ip\cdot x} \langle\Omega|T\phi_x\phi_0|\Omega\rangle = \langle\Omega|T\phi(p)\,\phi(-p)|\Omega\rangle$$
$$= \int_0^\infty ds\, \rho(s) \frac{i}{p^2-m_\lambda^2+i\epsilon} = \underbrace{\frac{iZ}{p^2-m^2}}_{\otimes} + \int_{\sim 4m^2}^{\infty} ds\, \frac{i\hat{\rho}(s)}{p^2-s}$$

→ We had $\langle 0|\phi(p)\,\phi(-p)|0\rangle = \frac{i}{p^2-m_0^2}$ where m_0 is the mass appearing in the Lagrangian. On the other hand, the ($\otimes$) term contains the exact mass of the particle, as it could be eventually measured.

→ A "physical" field need to be renormalized such that
$$Z=1 \Leftrightarrow |\langle\Omega|\phi_{ren}|\lambda\rangle|^2=1 \Leftrightarrow \phi = \sqrt{Z}\,\phi_{ren}.$$
↳ The transition between the original Lagrangian (the bare Lagrangian) and the physical particle propagating requires:

① renormalizing the field

② renormalizing the parameters.

7.3 LSZ reduction formula

→ Skiping technicalities, we generalize to n-pt correlators. Stripping the correlators of external legs, for wich we have to take into account the pole at the exact mass, and the normalization of the bare field w.r.t the physical one, we get a general relation between correlation functions and S-matrix elements.

Thm (LSZ reduction formula simple version)

Given k incoming and n-k outgoing physical particles, we have
$$\langle\Omega|T\phi_1\cdots\phi_n|\Omega\rangle = \prod_{i=1}^{n} \frac{i\sqrt{Z}}{p_i^2-m^2}\, \underbrace{\langle p_1\cdots p_n|\Omega\rangle}_{S\text{-matrix}}$$

→ Each field is associated with a $\sqrt{Z}$ factor.