6 RADIATIVE CORRECTIONS: LOOPS AND DIVERGENCIES

→ Now that we know the formalism to compute an observable in any QFT (ie through correlators), we're gonna perform some computations.
Giving an n-pt function, connected and 1PI, there will be leading order and subleading order (next to leading order NLO, NNLO, ..., N^pLO).
Those NPO's includes loops.

→ Ex: the $\lambda\phi^4$ theory → 2-pt function: LO: $\mathcal{O}(\lambda^0)$

NLO:  $\mathcal{O}(\lambda^1)$

→ 4-pt function: LO:  $\mathcal{O}(\lambda^0)$

NLO:  $\mathcal{O}(\lambda^2)$

→ Ex: Yukawa + $\lambda\phi^4$: 

→ Ex: 4Ψ interaction in Yukawa:  $\mathcal{O}(y^2)$

DEF | Terms arising at L.O. and with no loops are called tree-level, and are refered to as representing the semi-classical limit of the QFT amplitudes.

DEF | The radiative corrections are quantum effects that correct the above semi-classical quantities, or generate new terms.

6.1 A first computation: 2-pt function in $\lambda\phi^4$

→ Consider the 1st radiative correction that arise in $\lambda\phi^4$ theory:

At the connected 2-pt function level, we have:

$$-\frac{i}{4}\lambda\int d^4x\,\langle\phi_1\phi_2\phi_x^4\rangle = -3i\lambda\int d^4x\,\langle\phi_1\phi_x\rangle\langle\phi_2\phi_x\rangle\langle\phi_x\phi_x\rangle\langle\phi_x\phi_x\rangle$$

In Fourier space, $-3i\lambda\,\underbrace{\frac{i}{p^2-m^2}}_{\langle\phi_1\phi_x\rangle}\underbrace{\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}}_{\langle\phi_x\phi_x\rangle}\underbrace{\frac{i}{p^2-m^2}}_{\langle\phi_x\phi_2\rangle}$

→ The 1PI part is $-i\mathcal{M} = -3i\lambda \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2}$

↳ It doesn't depend on p^μ ⇒ it's a constant! ⇝ a shift in m^2

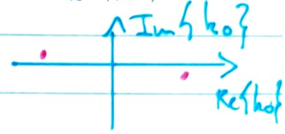
↳ Note that $\left[\lambda \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2}\right] = [m^2]$ and $[\lambda] = L^0$, then m is the only dimensionful parameter in the theory.

→ Evaluation of $\int d^4k\, (k^2-m^2)^{-1} = \int d^4k\, (k_0^2 - \vec{k}^2 - m^2)$

→ It has poles in $k_0^2 = \vec{k}^2 + m^2 - i\epsilon$ (free)

$$\Leftrightarrow k_0 = \pm\sqrt{\vec{k}^2+m^2-i\epsilon} = \pm\sqrt{\vec{k}^2+m^2}\left(1 - \frac{i\epsilon}{2}\frac{1}{\vec{k}^2+m^2}\right)$$

$$= \pm\sqrt{\vec{k}^2+m^2} \mp i\frac{\epsilon}{2\sqrt{\vec{k}^2+m^2}}$$



→ We rotate the contour to the imaginary axis

DEF The Wick rotation is the rotation of the contour (as long as it doesn't cross any pole) defining a Euclidean 4-momentum

$$k_0 \equiv i k_{0E} \text{ and } \vec{k}_E \equiv \vec{k} \Leftrightarrow t \mapsto -it_E \Leftrightarrow t_E \mapsto it$$

↳ We have $k^2 \mapsto -k_E^2$; $dk^0 \mapsto i\,dk_E^0$; $dk^4 \mapsto i\,dk_E^4$ and

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2} = -i\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2} = -i\int \frac{dk_E\, d^3\Omega}{(2\pi)^4}\frac{k_E^3}{k_E^2+m^2}$$ and $\int d^3\Omega = 2\pi^2 = \mathrm{vol}(S^3)$

$$= -i\frac{2\pi^2}{(2\pi)^4}\int_0^\infty dk_E \frac{k_E^3}{k_E^2+m^2} = \frac{-i}{(4\pi)^2}\int_0^\infty dk_E^2 \frac{k_E^2}{k_E^2+m^2}$$ since $2k\,dk = dk^2$

$$= \frac{-i}{(4\pi)^2}\int_0^\infty dp\, \frac{p}{p+m^2} = \infty$$ it diverges!

→ We regularize the integral by introducing a cut-off Λ on $|k|$:

$$\int_0^{\Lambda^2} du\, \frac{u}{u+m^2} = \int_0^{\Lambda^2} du\left(1 - \frac{m^2}{u+m^2}\right) = \Lambda^2 - m^2 \log\frac{\Lambda^2+m^2}{m^2} \sim \Lambda^2$$

We find $-i\mathcal{M} = \frac{3\lambda i}{(4\pi)^2}\int_0^{\Lambda^2} dk_E^2\, \frac{k_E^2}{k_E^2+m^2} = \frac{-3i\lambda}{(4\pi)^2}\Lambda^2$

⇒ The correction to m^2 is proportional to: (recall: $D_{tot} = \frac{1}{k^2-m^2-\mathcal{M}}$)

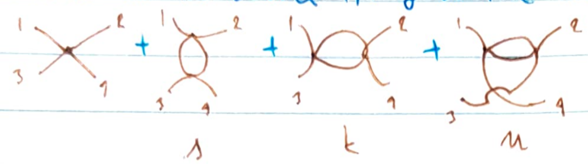
1) the loop factor $\lambda/(4\pi)^2$

2) the cut-off scale Λ^2

↳ not Lorentz invariant!

6.2 2^{nd} computation: vertex in $\lambda\phi^4$

→ Corrections to the vertex itself in the $\lambda\phi^4$ theory are:

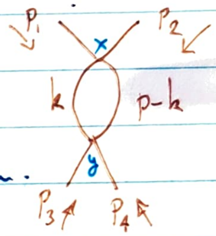


DEF The Mandelstam variables s, t, u are given by:

$s \equiv (p_1+p_2)^2 = (p_3+p_4)^2$ $\qquad t \equiv (p_1+p_3)^2 = (p_2+p_4)^2$ $\qquad u \equiv (p_1+p_4)^2 = (p_2+p_3)^2$

→ The amputated tree-level diagram was just $-6i\lambda$. The 1 loop diagrams differ only by the momentum flowing through the loop. Let's focus on the s-channel:

→ All the p_j are incoming: $p_1+p_2+p_3+p_4=0$ and we let $p \equiv p_1+p_2$, the momentum flowing through the loop, and k be the internal momentum.



$$\to \frac{1}{2}\left(\frac{-i\lambda}{4}\right)^2 \int_x\int_y \langle \phi_1 \phi_2 \phi_x^4 \phi_y^4 \phi_3 \phi_4 \rangle$$

$$= (-i\lambda)^2 \frac{1}{2\cdot 4^2} \int_x\int_y 2\cdot 4 \langle \phi_1\phi_x\rangle \cdot 3\langle\phi_2\phi_x\rangle \cdot 4\langle\phi_3\phi_y\rangle . 3\langle\phi_4\phi_y\rangle \cdot 2\langle\phi_x\phi_y\rangle^2$$

$$= -18\lambda^2 \iint_{x\,y} \langle\phi_1\phi_x\rangle\langle\phi_2\phi_x\rangle \underbrace{\langle\phi_x\phi_y\rangle^2}_{1PI} \langle\phi_3\phi_y\rangle\langle\phi_4\phi_y\rangle$$

$$\hookrightarrow -18\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(p-k)^2-m^2}$$

⊙ Feynman parametrization:

Prop $$\frac{1}{ab} = \int_0^1 dx \frac{1}{(xa+(1-x)b)^2} = \int_0^\infty dx \frac{1}{(xa+b)^2}$$

DEMO Indeed, $\int_0^1 dx\, (ax+b(1-x))^{-2} = \int_0^1 dx\, ((a-b)x+b)^{-2}$

$$= \frac{1}{a-b}\left[-\frac{1}{(a-b)x+b}\right]_0^1 = \frac{1}{a-b}\left[\frac{-1}{a}+\frac{1}{b}\right] = \frac{1}{a-b}\,\frac{a-b}{ab} = \frac{1}{a.b}$$ ▨

→ Using this Feynman trick, the 1PI part of the matrix element of the s-channel becomes:

$$18\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)([p-k]^2-m^2)}$$

$$= 18\lambda^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\{(p-k)^2x - mx^2 + k^2(1-x) - m^2(1-x)\}^2}$$

$$= 18\lambda^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\{k^2 - 2k\cdot px + p^2x - m^2\}^2}$$

The $2k\cdot px$ term prevents us from integrating over k_E^2.

→ Let's shift k so this linear term in k disappears:

$k^\mu \mapsto k^\mu + xp^\mu$. Then, $d^4k \mapsto d^4k$

$$k^2 - 2k\cdot px + p^2x \mapsto k^2 + 2k\cdot px + x^2p^2 - 2k\cdot px - 2x^2p^2 + p^2x$$
$$= k^2 + p^2x(1-x)$$

The integral becomes:

$$18\lambda^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\{k^2 + p^2x(1-x) - m^2\}^2}$$

→ Going to Euclidean coord. ($k^0 \mapsto ik_E^0$) and integrating over the angular part, we get

$$⊛\ \frac{i18\lambda^2}{(4\pi)^2} \int_0^1 dx \int_0^\infty dk_E^2 \frac{k_E^2}{\{k_E^2 + m^2 - p^2x(1-x)\}^2}$$

↳ We now let $\Delta \equiv m^2 - p^2x(1-x)$ and $k_E^2 \equiv u$, we have for the diverging part:

$$\int_0^{\Lambda^2} du \frac{u}{(u+\Delta)^2} = \int_0^{\Lambda^2} du \left(\frac{1}{u+\Delta} - \frac{\Delta}{(u+\Delta)^2}\right) = \left[\log(u+\Delta) + \frac{\Delta}{u+\Delta}\right]_0^{\Lambda^2}$$

$$\doteq \log\frac{\Lambda^2}{\Delta} - 1 \simeq \log\left[\frac{\Lambda^2}{\Delta}\right]$$

$$\Rightarrow ⊛ = \frac{-18i\lambda^2}{(4\pi)^2} \int_0^1 dx \log\left(\frac{m^2 - p^2x(1-x)}{\Lambda^2}\right)$$

→ The 1st 2 orders of the vertex in the $\lambda\phi^4$ theory is:

$$-6i\lambda - \frac{18i\lambda^2}{(4\pi)^2} \int_0^1 dx \left\{ \log\left[\frac{m^2 - sx(1-x)}{\Lambda^2}\right] + \log\left[\frac{m^2 - tx(1-x)}{\Lambda^2}\right] + \log\left[\frac{m^2 - ux(1-x)}{\Lambda^2}\right]\right\}$$

⊙ Remarks:

1) It corrects λ by $\lambda \cdot \frac{\lambda}{(4\pi)^2}$ depends only on λ itself

2) It depends on the external momenta (s, t, u) but also on the cut-off Λ

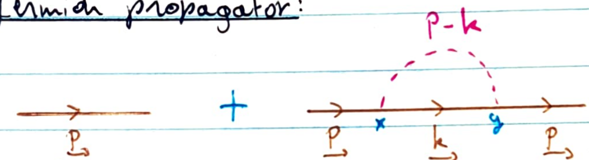
3) It diverges logarithmically, essentially because $[\lambda] = L^0$

6.3 3rd computation: $\mathcal{L}_I = g\, \phi \bar{\psi} \psi$

→ Consider a Yukawa theory: $\mathcal{L}_I = g\, \phi \bar{\psi} \psi$. We can compute the correction both from the scalar propagator $\langle \phi^2 \rangle$ and from the fermion propagator $\langle \bar{\psi} \psi \rangle$.

⊙ Corrections to the fermion propagator:

→ Let's compute:



↳ The $\mathcal{O}(g^2)$ (NLO) term is:

$$(ig)^2 \frac{1}{2} \langle \psi_1 \bar{\psi}_2 \int_x \phi_x \bar{\psi}_x \psi_x \int_y \phi_y \bar{\psi}_y \psi_y \rangle$$

$$= (ig)^2 \cdot \frac{1}{2} \iint (-1) \cdot 2 \langle \psi_1 \bar{\psi}_x \rangle \langle \phi_x \phi_y \rangle \langle \psi_x \bar{\psi}_y \rangle (-1)^3 \langle \psi_y \bar{\psi}_2 \rangle \quad \text{since } \{\psi_i, \psi_j\} = 0$$

$$= (ig)^2 \iint \langle \psi_1 \bar{\psi}_x \rangle \langle \phi_x \phi_y \rangle \langle \psi_x \bar{\psi}_y \rangle \langle \psi_y \bar{\psi}_2 \rangle$$

→ The 1PI part is:

$$(ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p-k)^2 - m_\phi^2} \cdot \frac{i}{\not{k} - m_\psi} \cdot \frac{\not{k} + m_\psi}{\not{k} + m_\psi} \quad \text{and } \not{k} \cdot \not{k} = k^2$$

$$= g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2 - m_\phi^2} \cdot \frac{\not{k} + m_\psi}{k^2 - m_\psi^2}$$

→ Using Feynman parametrization and shifting $k \mapsto k + px$, we get:

$$g^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\cancel{\not{k}} + \not{p}x + m_\psi}{\{k^2 + p^2 x(1-x) - m^2(x)\}^2} \quad \text{where } m^2(x) \equiv m_\phi^2 x + m_\psi^2 (1-x)$$

→ The integral over $\not{k}$ drops because it's odd in k! We get

$$i\mathcal{M} = \frac{-ig^2}{(4\pi)^2} \int_0^1 dx\, (\not{p}x + m_\psi) \log\left[\frac{m^2(x) - p^2 x(1-x)}{\Lambda^2}\right]$$

with $\frac{i}{\not{p} - m} \mapsto \frac{i}{\not{p} - m - \mathcal{M}}$

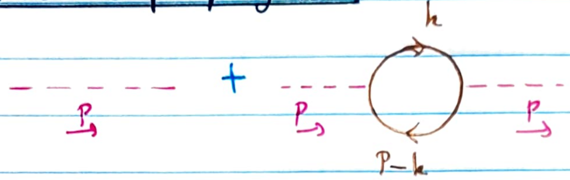
→ $i\mathcal{M} \propto \log \Lambda^2$ is log-divergent

→ It is proportional to $\not{p}$ and m_ψ ⇒ it's not linearly divergent in Λ^2

Prop Symmetries protect physical quantities from quantum corrections.

⊙ Correction to the scalar propagator:

→ Let's compute



↳ The NLO term is:

$$(ig)^2 \frac{1}{2} \langle \phi_1 \phi_2 \int_x \phi_x \bar{\psi}_x \psi_x \int_y \phi_y \bar{\psi}_y \psi_y \rangle$$

$$= (ig)^2 \iint_{xy} \langle \phi_1 \phi_x \rangle \langle \psi_x \bar{\psi}_y \rangle (-1)^3 \langle \psi_y \bar{\psi}_x \rangle \langle \phi_y \phi_2 \rangle$$

→ The 1PI part is:

$$-(ig)^2 \int \frac{d^4k}{(2\pi)^4} \left(\frac{i}{\not{p} - \not{k} - m} \right)_{\alpha\beta} \left(\frac{i}{-\not{k} - m} \right)_{\beta\alpha}$$ with α, β = clifford indices

$$= -g^2 \int \frac{d^4k}{(2\pi)^4} \operatorname{tr} \left\{ \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2} \cdot \frac{-\not{k} + m}{k^2 - m^2} \right\}$$

$$= -g^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \operatorname{tr} \left\{ \frac{(\not{p}(1-x) - \not{k} + m)(-\not{k} - \not{p}x + m)}{[k^2 + p^2 x(1-x) - m^2]^2} \right\}$$

$$= -g^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \operatorname{tr} \left\{ \frac{k^2 - p^2 x(1-x) + m\not{p}(1-2x) + m^2}{[k^2 + p^2 x(1-x) - m^2]^2} \right\}$$

$$= \frac{4ig^2}{(4\pi)^2} \int_0^1 dx \int_0^{\Lambda^2} dk_E^2 \, \frac{k_E^2 + p^2 x(1-x) - m^2}{\{k_E^2 - p^2 x(1-x) m^2\}^2}$$ since $\operatorname{tr}(\mathbb{1}) = 4$

$$= \frac{4ig^2}{(4\pi)^2} \int_0^1 dx \int_0^{\Lambda^2} du \, \frac{u - \Delta}{(u + \Delta)^2} \simeq \frac{4ig^2 \Lambda^2}{(4\pi)^4}$$

→ The leading divergence is $\frac{4ig^2}{(4\pi)^2} \Lambda^2$ wich is the opposite sign from

the $-\frac{3i\lambda}{(4\pi)^2} \Lambda^2$ correction from  !