5 SYMMETRIES, WARD ID AND THE PATH INTEGRAL

→ We'll use the path integral to derive the quantum consequences of the presence of global symmetries in a QFT, for instance the quantum equiv. of the Noether theorem and of the classical equations of motion

5.1 Noether theorem

Thm (Noether) An action invariant under a (global) continuous symmetry is associated to a corresponding conserved current.

→ We've seen 2 examples: a Dirac fermion and a complex scalar, both invariant under a $U(1)$.

⊙ Example: the fermion:

→ We have: $S_F = \int d^4x \{\bar{\psi}(i\not{\partial}-m)\psi\}$ and we transform $\psi \mapsto (1+i\alpha\psi)$
so $\delta\psi = i\alpha\psi$ and $\delta\bar{\psi} = -i\alpha\bar{\psi}$
$\hookrightarrow \delta S_F = \int d^4x \{ i\delta\bar{\psi}.\not{\partial}\psi + i\bar{\psi}\not{\partial}\delta\psi - m\,\delta\bar{\psi}.\psi - m\bar{\psi}\delta\psi\}$
$= \int d^4x \{\alpha\bar{\psi}\not{\partial}\psi + (-\alpha)\bar{\psi}\not{\partial}\psi + i\alpha m\bar{\psi}\psi - i\alpha m\bar{\psi}\psi\} = 0$

→ Gauging the symmetry, we have $\delta S_F \propto \partial_\mu\alpha$ because if $\partial_\mu\alpha = 0$, we have $\delta S_F = 0$. By definition, $\delta S_F = -\int d^4x\, \partial_\mu\alpha \cdot J^\mu_\psi$
$\Leftrightarrow \delta S_F = +\int d^4x\, \alpha(x).\,\partial_\mu J^\mu_\psi$. Since $\delta\psi = i\alpha(x)\psi$ is a particular case of variation, and that the EOM are stationary under any variation, we have $\delta S_F = 0\big|_{EOM} \Leftrightarrow \partial_\mu J^\mu_\psi \approx 0$

true on shell at least.

⊙ General theorem of classical field theory:

→ Take a field φ such that under $\delta\varphi = \alpha\Delta\varphi$ the action is invariant. We have for α constant:
$\delta\mathcal{L} = \mathcal{L}[\varphi+\delta\varphi] - \mathcal{L}[\varphi] = \alpha\,\partial_\mu \mathcal{B}^\mu$, some total derivative

→ When $\alpha = \alpha(x)$, we have:

$$\delta S = \int d^4x \left\{ \frac{\delta \mathcal{L}}{\delta \varphi} \delta\varphi + \frac{\delta \mathcal{L}}{\delta \varphi_\mu} \partial_\mu \delta\varphi \right\}$$

$$= \int d^4x \left\{ \alpha \partial_\mu \mathcal{V}^\mu + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \Delta\varphi \cdot \partial_\mu \alpha \right\}$$

$$= \int d^4x \, \alpha \, \partial_\mu \left\{ \mathcal{V}^\mu - \frac{\delta \mathcal{L}}{\delta \varphi_\mu} \Delta\varphi \right\} \equiv \int d^4x \, \alpha \, \partial_\mu J^\mu \approx 0$$

⇒ We have a conserved current $J^\mu \equiv -\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \Delta\varphi + \mathcal{V}^\mu$ such that $\partial_\mu J^\mu \approx 0$

5.2 Quantum conservation equation

→ At a quantum level, the conservation equation $\partial_\mu J^\mu = 0$ is an operator inserted in correlation functions:

$$\langle \partial_\mu J^\mu \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$$

Intuitively, we could expect that any insertion of $\partial_\mu J^\mu$ should render the correlation functions vanishing. But we have a time ordering inside that might interfere with ∂_μ.

→ Consider $\langle \hat{\mathcal{O}}_1 \cdots \hat{\mathcal{O}}_n \rangle \equiv \int \mathcal{D}\varphi \, \mathcal{O}_1 \cdots \mathcal{O}_n e^{iS} / \int \mathcal{D}\varphi \, e^{iS}$ under $\delta\varphi = \alpha \Delta\varphi$.
Then, $\delta \mathcal{O}_j = \alpha \Delta \mathcal{O}_j$, but any x-dependent transformation of the fields can be considered a field redefinition and absorbed by $\mathcal{D}\varphi \mapsto \mathcal{D}\varphi'$ (assuming invariant measure ⇔ no anomalies).
We have:

$$0 = \int \mathcal{D}\varphi \, \delta [\mathcal{O}_1 \cdots \mathcal{O}_n e^{iS}]$$

$$= \int \mathcal{D}\varphi \{ \delta\mathcal{O}_1 \cdots \mathcal{O}_n + \cdots + \mathcal{O}_1 \cdots \delta\mathcal{O}_n + \mathcal{O}_1 \cdots \mathcal{O}_n \cdot i \delta S \} e^{iS}$$

$$= \int \mathcal{D}\varphi \{ \alpha_1 \Delta \mathcal{O}_1 \cdots \mathcal{O}_n + \cdots + \mathcal{O}_1 \cdots \alpha(x_n) \Delta \mathcal{O}_n + i \int \mathcal{O}_1 \cdots \mathcal{O}_n \, \alpha(x) \, \partial_\mu J^\mu \} e^{iS}$$

$$= \alpha(x_1) \langle \Delta \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle + \cdots + \alpha(x_n) \langle \mathcal{O}_1(x_1) \cdots \Delta \mathcal{O}_n(x_n) \rangle$$
$$+ i \int d^4x \, . \, \alpha(x) \, \langle \partial_\mu J^\mu(x) . \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle$$

$$= \int d^4x \, . \, \alpha(x) \{ \delta(x - x_1) \langle \Delta \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle + \cdots + \delta(x - x_n) \langle \mathcal{O}_1(x_1) \cdots \Delta \mathcal{O}_n(x_n) \rangle$$
$$+ i \langle \partial_\mu J^\mu(x) \, \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle \}$$

We see that

$$\langle \partial_\mu J^\mu(x) \, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = i \delta(x - x_1) \langle \Delta \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle + \cdots$$
$$+ i \delta(x - x_n) \langle \mathcal{O}_1(x_1) \cdots \Delta \mathcal{O}_n(x_n) \rangle$$

↳ The expectation that $\partial_\mu J^\mu(x)$ makes a correlator zero is valid as long as x doesn't hit one of the locations of the other operators: there are contact terms.

DEF The Ward-Takahashi Identities are given by

$$\langle \partial_\mu J^\mu(x) \prod_{j=1}^{n} O_j(x_j) \rangle = \sum_{j=1}^{n} i\,\delta(x-x_j) \langle O_1(x_1) \cdots \Delta O_j(x_j) \cdots O_n(x_n) \rangle$$

→ The Ward identities are a generalization of those that appear in QED, in

$$\langle \partial_\mu J^\mu(x) \psi(y) \bar\psi(z) \rangle = -e\,\delta(x-y) \langle \psi(y) \bar\psi(z) \rangle + e\,\delta(x-z) \langle \psi(y) \bar\psi(z) \rangle$$

when $\Delta\psi = ie\psi$ and $\Delta\bar\psi = -ie\bar\psi$

→ For $n=0$: $\langle \partial_\mu J^\mu(x) \rangle = 0$ is a 1-pt function

→ For $n=1$: $\langle \partial_\mu J^\mu(x)\, O(y) \rangle = i\,\delta(x-y) \langle \Delta O(y) \rangle$ not trivial only if $\langle O \rangle \neq 0$, i.e. if the symmetry is spontaneously broken (see Goldstone theorem).

5.3 Quantum equations of motion

→ Consider the variation $\varphi(x) \mapsto \varphi(x) + \epsilon(x) \Leftrightarrow \delta\varphi(x) = \epsilon(x)$, computed in the n-pt correlator: $\langle \varphi_1 \cdots \varphi_n \rangle \equiv \int \mathcal{D}\varphi\, \varphi_1 \cdots \varphi_n\, e^{iS} / \int \mathcal{D}\varphi\, e^{iS}$

By the same argument, we should have:

$$0 = \int \mathcal{D}\varphi\, \delta[\varphi_1 \cdots \varphi_n] e^{iS}$$

$$= \int \mathcal{D}\varphi \{ \delta\varphi_1 \cdots \varphi_n + \dots + \varphi_1 \cdots \delta\varphi_n + \varphi_1 \cdots \varphi_n\, i\,\delta S \} e^{iS}$$

$$= \int \mathcal{D}\varphi \{ \epsilon(x_1) \varphi_2 \cdots \varphi_n + \dots + \varphi_1 \cdots \varphi_{n-1} \epsilon(x_n) + i \int d^4x\, \varphi_1 \cdots \varphi_n \frac{\delta S}{\delta \varphi(x)} \epsilon(x) \} e^{iS}$$

$$= \int \mathcal{D}\varphi \int d^4x\, \epsilon(x) \{ \varphi_2 \cdots \varphi_n + \dots + \varphi_1 \cdots \varphi_{n-1} + i\, \varphi_1 \cdots \varphi_n \frac{\delta S}{\delta \varphi_x} \} e^{iS}$$

DEF The Schwinger-Dyson equations are given by

$$\left\langle \frac{\delta S}{\delta \varphi(x)} \varphi(x_1) \cdots \varphi(x_n) \right\rangle = i\,\delta(x-x_1) \langle \varphi_2 \cdots \varphi_n \rangle + \dots + i\,\delta(x-x_n) \langle \varphi_1 \cdots \varphi_{n-1} \rangle$$

→ ex: real scalar field. We have $\delta S/\delta\varphi = (-\partial^2 - m^2)\phi$. For $n=2$, we have $\langle (-\partial^2 - m^2) \varphi_x \varphi_y \rangle = (-\partial^2 - m^2) \langle 0 | T[\varphi_x \varphi_y] | 0 \rangle = i\,\delta(x-y)$. It defines the propagator of the free theory.