4 PATH INTEGRAL FOR VECTOR FIELDS

→ We only consider the massless case, relevant for QED. The action is the following:

$$S_A \equiv \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

with $F_{\mu\nu} \hat{=} \partial_\mu A_\nu - \partial_\nu A_\mu$

4.1 Gauge freedom

→ The action $S[A]$ has a redundancy in the off-shell degrees of freedom: we seem to have 4 dof: (A_0, A_1, A_2, A_3), but $S[A]$ has a gauge freedom; the action doesn't change under a gauge transformation:

$$S[A] \mapsto S[\overbrace{A_\mu(x) + \partial_\mu \alpha(x)}] = S[A_\mu(x)]$$

↳ This gauge freedom can be used to set to 0 one of the A_μ.

→ Furthermore, there is no momentum conjugate to A_0: $\frac{\delta S_A}{\delta \dot{A}_0} = 0$

Indeed, $F^2 = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$= 2(\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha)$$

$$\sim \partial_0 A_0 \partial^0 A^0 - \partial_0 A_0 \partial^0 A^0 = 0$$ On the contrary,

$$\Pi_i = \frac{\delta S_A}{\delta \dot{A}_i} = \partial_0 A_i - \partial_i A_0 = E_i$$

→ Alternatively, notice that the EOM $\partial^\mu F_{\mu\nu} = 0 \Leftrightarrow \partial^\mu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu = 0$
$\Leftrightarrow \{ \partial^2 \delta^\mu_\nu - \partial_\nu \partial^\mu \} A_\mu = 0$

This kinetic operator $\partial^2 \delta^\mu_\nu - \partial_\nu \partial^\mu$ has eigenfunction $\partial_\mu \alpha$ with eigenvalue 0

Indeed, $\{ \partial^2 \delta^\mu_\nu - \partial_\nu \partial^\mu \} \partial_\mu \alpha = \partial^2 \partial_\nu \alpha - \partial_\nu \partial^2 \alpha = 0$

Since $S_A = \int d^4x \{ -\frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) \partial^\mu A^\nu \} = \int d^4x \{ \frac{1}{2} A_\mu (\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu \}$
it implies that $\int \mathcal{D}A \, e^{iS_A} \longrightarrow \infty$ since $\partial^2 \delta^\nu_\mu - \partial_\mu \partial^\nu$ has a vanishing eigenvalue.

⊙ Fixing the gauge:

→ Fixing the gauge requires to add a term to our action S_A to make the kinetic operator invertible. Indeed, we want to find $D_{\alpha\beta}(x-y)$ such that $\{\partial^2\eta^{\alpha\beta}+\partial^\alpha\partial^\beta\} D_{\beta\gamma}(x-y) = \delta^\alpha_\gamma \delta^4(x-y)$

→ We introduce a parameter ξ in S_A:

$$S_A \mapsto \int d^4x \left\{ \frac{1}{2} A_\mu (\partial^2\eta^{\mu\nu} - \partial^\mu\partial^\nu) A_\nu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \right\}$$

$$= \int d^4x \left\{ \frac{1}{2} A_\mu \left(\partial^2\eta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right) A_\nu \right\}$$

→ Inverting the kinetic operator,

$\left[-\partial^2\eta^{\mu\nu} + \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] D_{\nu\rho} = \delta^\mu_\rho \delta^4(x-y)$. Go to Fourier space:

$\left[-k^2\eta^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right] D_{\nu\rho} = \delta^\mu_\rho$. Let $D_{\nu\rho} = A(k^2)\eta_{\nu\rho} + B(k^2) k_\nu k_\rho$

Then:

$+k^2 A \delta^\mu_\rho - k^2 B k^\mu k_\rho - \left(1-\frac{1}{\xi}\right)\left(A k^\mu k_\rho + B k^2 k^\mu k_\rho\right) = \delta^\mu_\rho$

$\Leftrightarrow A = 1/k^2$ and $B = (\xi-1).A/k^2 = (\xi-1)/k^4$

Thus, $D_{\mu\nu} = \frac{1}{k^2}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$

? Still ill-defined when $\xi \to \infty$. We can choose convenient values of ξ.

DEF We can $\xi=1$: the Feynmann gauge. Then,

$$D_{\mu\nu} = \frac{\eta_{\mu\nu}}{k^2}$$

For $\xi=0$, it's the Landau (or transverse) gauge. Then,

$$D_{\mu\nu} = \frac{1}{k^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

Prop In the Landau gauge, we have $D_{\alpha\beta}k^\beta = 0$

Indeed, $\frac{1}{k^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)k^\nu = \frac{1}{k^2}\left(k_\mu - \frac{k_\mu k^2}{k^2}\right) = 0$ ∎

4.2 Faddeev - Popov procedure

→ We need to make sure this extra term $(\partial\cdot A)^2/2\xi$ can be addet in S_A in the path integral Z_A without changing the theory.
We'll insert a δ-functional to impose a gauge condition $G(A)=0$, such $G(A)=\partial_\mu A^\mu=0$

→ We want to write $Z_A=\int \mathcal{D}A\, e^{iS[A]} \overset{?}{=} \int \mathcal{D}\alpha \int \mathcal{D}A\, e^{i[A+\partial\alpha]}$

→ Let's recall that:
$1=\int dx\, \delta(x)=\int df(x)\, \delta[f(x)]=\int dx\, |f'(x)|\, \delta[f(x)]$
At a functional level:
$1=\int \mathcal{D}\alpha\, \delta[\alpha]=\int \mathcal{D}\alpha\, \delta[G({}^\alpha A)]\left|\det \dfrac{\delta G({}^\alpha A)}{\delta\alpha}\right|$

→ We let ${}^\alpha A_\mu \equiv A_\mu+\partial_\mu\alpha$ and $G(A)=\partial_\mu A^\mu$.
Then, $G({}^\alpha A)=\partial_\mu A^\mu+\partial^2\alpha$
So that $\dfrac{\delta G({}^\alpha A)}{\delta\alpha}=\partial^2$ and $\det\{\partial^2\}$ doesn't depend on α; we can factor it out in N.

↳ We can write $1=\left|\det\dfrac{\delta G({}^\alpha A)}{\delta\alpha}\right|\int\mathcal{D}\alpha\, \delta[G({}^\alpha A)]=N\int\mathcal{D}\alpha\,\delta[G({}^\alpha A)]$
and thus,

$$\begin{aligned}Z_A^{fix}&=N\int\mathcal{D}\alpha\int\mathcal{D}A\,\delta[G({}^\alpha A)]\,e^{iS_A}\\&=N\int\mathcal{D}\alpha\,\mathcal{D}{}^\alpha A\,\delta[G({}^\alpha A)]\,e^{iS_{{}^\alpha A}}\\&=N'\int\mathcal{D}{}^\alpha A\,\delta[G({}^\alpha A)]\,e^{iS_{{}^\alpha A}}\\&=N'\int\mathcal{D}A\,\delta[G(A)]\,e^{iS_A}\end{aligned}$$
as desired.

→ Finally, we consider a family of gauge fixing condition $G(A)=\partial_\mu A^\mu-\omega(x)$ and integrate with a gaussian weight over them:
$$Z_A=N'\int\mathcal{D}\omega\,\exp\left\{-i\int d^4x\,\frac{1}{2\xi}\omega^2\right\}\int\mathcal{D}A\,\delta[\partial_\mu A^\mu-\omega]\,e^{iS_A}$$
$$=N'\int\mathcal{D}A\,\exp\left\{iS_A-i\int d^4x\,\frac{1}{2\xi}(\partial_\mu A^\mu)^2\right\}$$
Indeed, $\delta[G(A)]=\int\mathcal{D}\lambda\,\exp\{i\int d^4x\,\lambda(x)\,G(A)\}$ and
$\int\mathcal{D}\lambda\int\mathcal{D}A\,\exp\{iS[A]+i\int d^4x\,\lambda\,\partial_\mu A^\mu+i\int d^4x\,\frac{1}{2}\xi\lambda^2\}=\int\mathcal{D}A\,\exp\{i(S_A-\int d^4x\,\frac{(\partial\cdot A)^2}{2\xi})\}$
since $\frac{1}{2}\xi\lambda^2+\partial_\mu A^\mu=\frac{1}{2}\xi(\lambda^2+\frac{1}{\xi}\partial_\mu A^\mu)^2-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$

→ We should have all observables independants from the gauge fixing, hence from ξ.
↳ Let's notice that the ξ-dependent part of the photon propagator $\xi \frac{k_\mu k_\nu}{k^2} \propto k_\mu k_\nu$ and is completely longitudinal

4.3 Adding sources

→ In order to compute $\langle 0|T A_\mu(x) A_\nu(y)|0\rangle$, let introduce sources for A_μ:

$$Z[J_\mu] = \int \mathcal{D}A \exp\left\{ i S_A + i \int d^4x \, J^\mu(x) A_\mu(x) \right\}$$

→ Because of the gauge invariance, J_μ must satisfy

$$i\int d^4x \, J^\mu A_\mu \mapsto i\int d^4x \, J^\mu (A_\mu + \partial_\mu \alpha) \overset{\text{IPP}}{=} i\int d^4x \, (J^\mu A_\mu - \alpha \, \partial_\mu J^\mu)$$
$$\overset{!}{=} i\int d^4x \, J^\mu A_\mu \qquad \forall \alpha$$

$$\Leftrightarrow \partial_\mu J^\mu = 0$$

PROP The source for A_μ is a conserved current

→ This is consistent with the fact that A_μ, when it couples to other fields, must gauge a symmetry: it generally couples to conserved currents. Hence, the longitudinal part of the propagator is not physical and drops in any practical computation.

⊙ Photon 2-pt function:

→ With sources, the path integral becomes:

$$Z[J_\mu] = \int \mathcal{D}A \exp\left\{ i \int \tfrac{1}{2} A_\mu [\partial^2 \eta^{\mu\nu} - (1-\tfrac{1}{\xi})\partial^\mu \partial^\nu] A_\nu + J^\mu A_\mu \right\}$$

$$= \int \mathcal{D}A \exp\left\{ i \int \tfrac{1}{2}(A_\mu - J^\rho D_{\rho\mu}) \left[\partial^2 \eta^{\mu\nu} - (1-\tfrac{1}{\xi})\partial^\mu \partial^\nu\right](A_\nu - D_{\nu\alpha} J^\alpha) + i \int \tfrac{1}{2} J^\mu D_{\mu\nu} J^\nu \right\}$$

$$= \mathcal{N} \exp\left\{ i \int d^4x \, d^4y \, \tfrac{1}{2} J^\mu(x) D_{\mu\nu}(x-y) J^\nu(y) \right\}$$

→ $$\langle A_\mu(x) A_\nu(y) \rangle = \frac{1}{Z} \frac{\delta}{\delta i J^\mu(x)} \frac{\delta}{\delta i J^\nu(y)} Z \bigg|_{J=0} = -i D_{\mu\nu}(x-y)$$

↳ The $-i$ factor is consistent with the fact that the spatial components of A_μ have a propagator similar to the one of a real massless scalar: in the Feynman gauge ($\xi=1$),

$$\langle A_1(x) A_1(y)\rangle = \frac{i}{-\partial^2}\delta^4(x-y) \quad \text{since} \quad \langle A_1(k) A_1(-k)\rangle = \frac{-i\eta_{11}}{k^2} = \frac{i}{k^2}$$

4.4 Example : QED

→ Take the Dirac action $S_F \equiv \int d^4x \left(i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi\right)$
It's invariant under a global phase rotation ($U(1)_g$) under wich

$$\Psi \mapsto e^{i\alpha}\Psi \quad \text{and} \quad \bar{\Psi} \mapsto \bar{\Psi}e^{-i\alpha}, \quad \alpha \text{ a constant.}$$

→ If we try to make α local: $\alpha = \alpha(x)$, we get

$$\partial_\mu\Psi \mapsto \partial_\mu(e^{i\alpha}\Psi) = e^{i\alpha}(\partial_\mu\Psi + i\partial\alpha\cdot\Psi)$$

We need a covariant derivative D_μ such that $D_\mu\Psi \mapsto e^{i\alpha(x)}D_\mu\Psi$

↳ We let $D_\mu = \partial_\mu - iA_\mu$ with $A_\mu \mapsto A_\mu + \partial_\mu\alpha$ (connection). Then,

$$D_\mu\Psi = \partial_\mu\Psi - iA_\mu\Psi \mapsto e^{i\alpha}(\partial_\mu\Psi + i\partial_\mu\alpha\cdot\Psi - iA_\mu\Psi - i\partial_\mu\alpha\cdot\Psi)$$
$$= e^{i\alpha}D_\mu\Psi$$

DEF | The QED action S_{QED} is

$$S_{QED} = \int d^4x \left\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\not{D}\Psi - m\bar{\Psi}\Psi \right\}$$

$$= S_A + S_F + e\int d^4x\, A_\mu\bar{\Psi}\gamma^\mu\Psi$$

where e is the electric charge, the EM coupling.

→ Check that A_μ couples to a conserved current:

$$\partial_\mu J^\mu_\Psi = \partial_\mu(\bar{\Psi}\gamma^\mu\Psi e) = e\bar{\Psi}\gamma^\mu\partial_\mu\Psi + e\,\partial_\mu\bar{\Psi}\cdot\gamma^\mu\Psi$$
$$= e\bar{\Psi}\overrightarrow{\not{\partial}}\Psi + e\bar{\Psi}\overleftarrow{\not{\partial}}\Psi = 0 \quad \text{on shell}$$

Indeed, the equations of motion are:

$$i\overrightarrow{\not{D}}\Psi = m\Psi \quad \text{and} \quad -i\bar{\Psi}\overleftarrow{\not{D}} = m\Psi$$

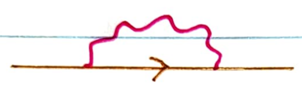
⊙ Feynman rules and diagrams:

→ We denote the propagator of the photon ∿∿ wavy line and the vertex is

with the Feynman rule $-ie\gamma^\mu$

→ The one-loop diagram for the photon self-energy function is

The one for the fermion self-energy function is:



4.5 Scalar QED

→ We consider a complex scalar with a $U(1)_g$ symmetry:

$$\varphi \mapsto e^{i\alpha}\varphi \quad \text{and} \quad \varphi^+ \mapsto e^{-i\alpha}\varphi^+$$

↳ $S_\varphi \equiv \int d^4x \{ \partial_\mu \phi (\partial^\mu \phi)^+ - m^2 \phi^+ \phi \}$

→ Gauging the symmetry, we get the same result as before:

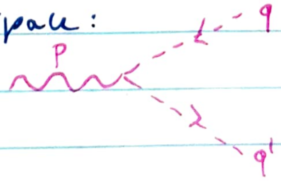
$$S_{\text{scalar QED}} \equiv \int d^4x \{ -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + (D_\mu \phi)^+ D^\mu \phi - m^2 \phi^+ \phi \}$$

$$= S_A + S_\phi + \int d^4x \{ ie A^\mu (\phi^+ \partial_\mu \phi - \phi \partial_\mu \phi^*) + e^2 A_\mu A^\mu \phi^+ \phi \}$$

↳ We have a linear coupling to a conserved current $J^\mu_\phi = ie(\phi^+ \partial^\mu \phi - \phi \partial^\mu \phi^+)$ and a $\mathcal{O}(e^2)$ term ensuring the gauge invariance of the $A_\mu J^\mu_\phi$ term.

→ In QFT, the 3-pt vertex involves derivatives to make it Lorentz covariant, in momentum space:

$\langle A_\mu(p) \phi(q) \phi^+(q') \rangle \equiv$



$\propto ie(q^\mu - q'^\mu)$

with $p \equiv -q - q'$