3 PATH INTEGRAL FOR FERMIONIC FIELDS

→ We want to generalize the path integral approach to QFT's with other fields. We consider fields with spin 1/2 : fermions.

→ We consider the Dirac field:

$$S_{Dirac} \equiv \int d^4x \, (i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi)$$

with the Clifford 4x4 matrices:

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, and $\gamma^{\mu\nu} \equiv [\gamma^\mu, \gamma^\nu]$ is the generator of Lorentz transformations in the spinor representation

→ In order to canonically quantize a field, with a Hamiltonian bounded from below, we need an anticommutation relation between ψ and its conjugate momentum ψ^+:

$$\{\psi(\bar{x}), \psi^+(\bar{y})\} = \delta^3(\bar{x}-\bar{y})$$

3.1 Anticommuting Numbers

→ In the path integral approach, this anticommutation relation, related to the fermionic statistics, translate to considering an anticommuting field $\psi(x) = \sum_i \psi_i \, \phi_i(x)$, with $\{\psi_i, \psi_j\} = 0$

p299 Peskin

DEF We define a Grassmann number by giving algebraic rules for manipulating them. Let θ, η be 2 G-odd numbers. Then,

$$\theta\eta = -\eta\theta$$

Prop 1 This implies $\eta^2 = \eta\eta = -\eta\eta = 0$

→ Any function of a G-odd number has the following taylor expansion: $f(\eta) = f_0 + \eta f_1$

→ Derivatives with respect to G-variable have to be taken specifiying from wich side one takes it:

↳ From the left: $\frac{\partial^L}{\partial \eta}(\theta\eta) = -\theta$ and $\frac{\partial^L}{\partial \eta}(\eta\theta) = +\theta$

→ To define integrals, we want invariance under a shift of η:

$$\int d\eta\, f(\eta+\theta) \overset{!}{=} \int d\eta\, f(\eta) \Leftrightarrow \int d\eta\, f_1 \theta = 0$$

↳ Since we want $\int d\eta\, f(\eta)$ to be linear in $f(\eta)$, we define

DEF 1 $\int d\eta\, 1 = 0, \quad \int d\eta\, \eta = 1 \Rightarrow \int d\eta\,(f_0 + f_1\eta) \equiv f_1$

↳ A Grassmann integral acts much like a derivative

↳ Since $[\int d\eta\, \eta] = [1]$, we have $[d\eta] = [\eta]^{-1}$

DEF Integrating over the complex G-plane is

$$\int d\eta^* d\eta\, \eta\eta^* = 1$$

where we adopted the convention:

$$(\eta\theta)^* = \theta^*\eta^* = -\eta^*\theta^*$$

→ To perform Gaussian integrals over a complex Grassmann variable:

$$\int d\eta^* d\eta\, e^{-\alpha\eta^*\eta} = \int d\eta^* d\eta\,(1+\alpha\eta^*\eta) = \alpha$$

Usually, we had $\int dz^* dz\, e^{-\alpha z^* z} = \pi/\alpha$

↳ Similarly, we have:

$$\int \prod_j d\eta_j^* d\eta_j \exp\{-\eta_j^* k_{ij}\eta_i\} = \int \prod_j d\eta_j'^* d\eta_j' e^{-\eta_j'^* k_j \eta_j'} =$$

$$= \int \prod_j \{ d\eta_j'^* d\eta_j' (1 + k_i \eta_i'^* \eta_i') \} = \prod_j k_j = \det K \quad \left(\text{Usually}, \sim \frac{1}{\det K}\right)$$

↳ With intertion, we get:

$$\int \Big(\prod_j d\eta_j^* d\eta_j\Big) \eta_k \eta_l^* \exp\{-\sum_i \eta_i^* k_i \eta_i\} = \int \prod d\eta\, d\eta^*\, \eta_k \eta_l^* \prod_{j\neq k,l} (1 + k_j \eta_j \eta_j^*)$$

$$= \delta_{kl} \prod_{i\neq k} k_i = (\det K)(K^{-1})_{kl}$$

3.2 Dirac propagator and generating functional

DEF The path integral for fermionic fields is defined by
$$\int \mathcal{D}\bar{\Psi}\, \mathcal{D}\Psi \, \exp\{ i \int d^4x \, (i \bar{\Psi} \not{\partial} \Psi - m \bar{\Psi}\Psi) \}$$

The two-point function is given by:
$$\langle 0 | T\{\Psi_x \bar{\Psi}_y\} | 0 \rangle = \frac{\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \, \Psi(x)\bar{\Psi}(y) \exp\{ i \int d^4x (i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi)\}}{\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\{ i\int d^4x (i\bar{\Psi}\not{\partial}\Psi - m \bar{\Psi}\Psi)\}}$$

→ To derive the Feynman rules for the free Dirac theory, we use a generating functional.

DEF The Dirac generating functional is defined as
$$Z[\eta,\bar{\eta}] \equiv \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\{ i \int d^4x (i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta)\}$$

↳ We need to pay attention to the sign:
$\frac{\delta^L}{\delta i \eta} i \int d^4x \, \bar{\Psi}\eta = -\bar{\Psi}$ and $\frac{\delta^L}{\delta i \bar{\eta}} i \int d^4x \, \bar{\eta}\Psi = +\Psi$ so that
$$\langle \Psi_x \bar{\Psi}_y \rangle = \frac{1}{Z[\eta,\bar{\eta}]} \frac{\delta}{\delta i \bar{\eta}(x)} \left(\frac{-\delta}{\delta i \eta(y)} \right) Z[\eta,\bar{\eta}] \Big|_{\eta = 0 = \bar{\eta}}$$

→ Factorizing Z:
$$\int d^4x \{ \bar{\Psi}(i\not{\partial} - m)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta \}$$
$$= \int d^4x \{ (\bar{\Psi} + \bar{\eta}(i\not{\partial}-m)^{-1})(i\not{\partial}-m)(\Psi + (i\not{\partial}-m)^{-1}\eta) - \bar{\eta}(i\not{\partial}-m)^{-1}\eta \}$$
↳ We write $D_F(x-y) \equiv (i\not{\partial}-m)^{-1}\delta^4(x-y)$

PROP We have $D_F = (i\not{\partial}+m)D \leftarrow$ K-G propagator / scalar Green-fct

Indeed, $(i\not{\partial}+m)(i\not{\partial}-m) = -\gamma^\mu\gamma^\nu \partial_\mu\partial_\nu - m^2 = -\partial^2 - m^2$

and $(-\partial^2 - m^2) D_F = (i\not{\partial}+m)\delta^4(x-y)$

$$\Rightarrow D_F = \frac{i\not{\partial}+m}{-\partial^2 - m^2}\delta^4(x-y) = (i\not{\partial}+m)D$$

↳ We can now write

$$Z[\eta,\bar\eta] = Z[0,0] \cdot \exp\left\{-i\int d^4x\, d^4y\ \bar\eta(x)\, D_F(x-y)\, \eta(y)\right\}$$

→ Let's verify that Z is generating the right quantity:

$$\langle \psi_x \bar\psi_y \rangle = \frac{\delta}{\delta i\bar\eta}\left(\frac{-\delta}{\delta i\eta}\right) \cdot \exp\left\{-i\int d^4x\, d^4y\ \bar\eta D_F \eta\right\}$$

$$= -i\frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta\eta} \exp\left\{\int d^4x\, d^4y\ \bar\eta D_F \eta\right\}$$

$$= +i\frac{\delta}{\delta\bar\eta}\int \bar\eta D_F = i D_F(x-y) \quad \text{as expected}$$

3.3 Interacting fermions

DEF The Yukawa interaction is the interaction between a scalar and a Dirac fermions as follow:

$$S_{tot} = S_\varphi + S_\psi + S_I$$

$$= \int d^4x \left\{ \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 + i\bar\psi\not\partial\psi + y\,\phi\bar\psi\psi \right\}$$

Reminder: $[\varphi] = M \quad [\psi] = M^{3/2} \quad [y] = M^0$

→ The tree-level 3-pt function is:

$$\langle \varphi_1 \psi_2 \bar\psi_3 \rangle_{int} = \int D\varphi\, D\bar\psi\, D\psi\ \varphi_1 \psi_2 \bar\psi_3\, e^{iS_\varphi + iS_\psi} \cdot \underbrace{i\left(1 + \int d^4x\ y\,\phi\bar\psi\psi + \dots\right)}_{\sim e^{iS_I}}$$

$$= \langle \varphi_1 \psi_2 \bar\psi_3 \cdot i\int d^4x\ y\,\varphi_x \bar\psi_x \psi_x \rangle_{free} + \mathcal{O}(y^2)$$

? $$= iy\int d^4x\, (-1)^2 \langle \varphi_1 \varphi_x \rangle \langle \psi_2 \bar\psi_x \rangle \langle \psi_x \bar\psi_3 \rangle$$

⊙ Fermions lines and corrections:

→ We draw

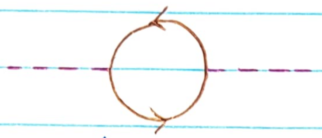
$\langle \psi\bar\psi \rangle \equiv$ ⟶ and $\langle \varphi\varphi \rangle \equiv$ - - - - - -

The vertex being given by:

 $\propto iy$

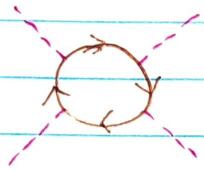
→ Since the scalar interact with a fermion, there is now a lot more radiative corrections.

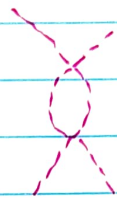
↳ Correction to the propagator:



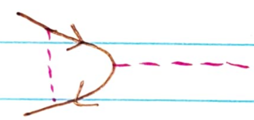
(We had )

↳ Correction to the $\lambda \phi^4$ coupling:



(We had )

↳ Correction to the Yukawa vertex



↳ Correction to the fermion propagator