14 RENORMALIZATION AND SGNG β?

14.1 Renormalization

→ We write the bare Lagrangian as the renormalized one plus the counterterms:

$$\mathcal{L}_{bare} = -\frac{1}{4}\left(\partial_\mu A_{\nu a}^{(b)} - \partial_\nu A_{\mu a}^{(b)}\right)^2 + i\bar{\psi}^{(b)} \not{\partial} \psi^{(b)} + \partial_\mu \bar{c}_a^{(b)} \partial^\mu c_a^{(b)}$$

$$- m^{(b)} \bar{\psi}^{(b)} \psi^{(b)} - g^{(b)} \partial_\mu A_{\nu a}^{(b)} f_{abc} A_b^{\mu (b)} A_c^{\nu (b)}$$

$$-\frac{1}{4} g^{(b)2} f_{abe} f_{cde} A_{\mu a}^{(b)} A_{\nu b}^{(b)} A^{\mu (b)}_c A^{\nu (b)}_d + g(b) A_{\mu a}^{(b)} \bar{\psi}^{(b)} \underbrace{\gamma^\mu t_a}_{\text{fixed by group theory}} \psi^{(b)}$$

$$+ g^{(b)} f_{abc} \partial_\mu \bar{c}_a^{(b)} A^{\mu (b)}_b c_c^{(b)}$$

↳ Remark: the gauge-fixing term being not renormalizable, we didnt write it.

→ The renormalized fields are: $A_{\mu a}^{(b)} = \sqrt{Z_A}\, A_{\mu a}$
$\psi^{(b)} = \sqrt{Z_\psi}\, \psi \qquad c_a^{(b)} = \sqrt{Z_c}\, c_a$ (and $\bar{c}_a^{(b)} = \sqrt{Z_c}\, \bar{c}_a$)

→ Injecting the non-kinetic terms, we can write the bare couplings in terms of the renormalized ones, introducing a Z factor for each vertex:

$$m(b)\, Z_\psi = m\, Z_m \qquad g_{(b)}^2 Z_A^2 = g^2 Z_{A^4} \qquad g(b) Z_A^{1/2} Z_\psi = g\, Z_{A\psi^2}$$

$$g(b)\, Z_A^{3/2} = g\, Z_{A^3} \qquad g_{(b)} Z_A^{1/2} Z_c = g\, Z_{Ac^2}$$

→ $\mathcal{L}_{ren}$ will be the same as $\mathcal{L}_{bare}$ but when we drop all the (b) indices.
$\mathcal{L}_{ct}$ will have a similar structure with $\delta_\bullet = Z_\bullet - 1$:

$$\mathcal{L}_{ct} = -\frac{1}{4}\delta_A \left(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a}\right)^2 + i\,\delta_\psi \bar{\psi} \not{\partial} \psi + \delta_c \partial_\mu \bar{c}_a \partial^\mu c_a - m\,\delta_m \bar{\psi}\psi$$

$$- g\,\delta_{A^3} \partial_\mu A_{\nu a} f_{abc} A_b^\mu A_c^\nu - \frac{1}{4} g^2 \delta_{A^4} f_{abe} f_{cde} A_{\mu a} A_{\nu b} A^\mu_c A^\nu_d$$

$$+ g\,\delta_{A\psi^2} A_{\mu a} \bar{\psi} \gamma^\mu t_a \psi + g\,\delta_{Ac^2} f_{abc} \partial_\mu \bar{c}_a A^\mu_b c_c$$

→ We can write the ratio $g/g^{(b)}$ in several differents ways:

$$\frac{g}{g^{(b)}} = Z_{A^3}^{-1} Z_A^{3/2} = Z_{A^4}^{-1/2} Z_A = Z_{A\psi^2}^{-1} Z_A^{1/2} Z_\psi = Z_{Ac^2}^{-1} Z_A^{1/2} Z_c$$

because only one g must appear in $\mathcal{L}_{ren}$ to preserve gauge invariance. Writing $Z_\bullet = 1 + \delta_\bullet$, we get

? $$\delta_A - \delta_{A^3} = \frac{1}{2}\left(\delta_A - \delta_{A^4}\right) = \delta_\psi - \delta_{A\psi^2} = \delta_c - \delta_{Ac^2}$$

↳ Not as strong as what we got in QED (we had $\delta_\psi = \delta_{A\psi^2}$)

→ In QED, we used the Ward identity to show that $\delta_\psi = \delta_{A\psi^2}$, wich is based on the fact that $\partial_\mu J^\mu = 0$ with $J^\mu = \bar{\psi}\gamma^\mu\psi$

→ In a non abelian theory, one can show that $J_a^\mu = \bar{\psi}\gamma^\mu t_a \psi$ transform in the adjoint rep: $\delta J_a^\mu = i f_{abc}\alpha_b J_c^\mu$

→ Hence, the only covariant conservation equation one can write is: $D_\mu J_a^\mu = 0 \Leftrightarrow \partial_\mu J_a^\mu = ig f_{abc} A_{\mu b} J^\mu{}_c \neq 0$! ~~Ward~~

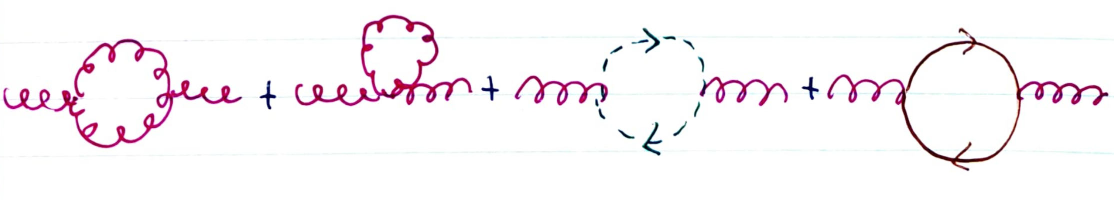
→ As soon as $f_{abc} \neq 0$, $\partial_\mu J^\mu \neq 0$ in the theory coupled to gauge bosons. Hence we cannot conclude that $Z_\psi = Z_{A\psi^2}$ independently of the gauge fixing.

14.2 A long walk to the β-function

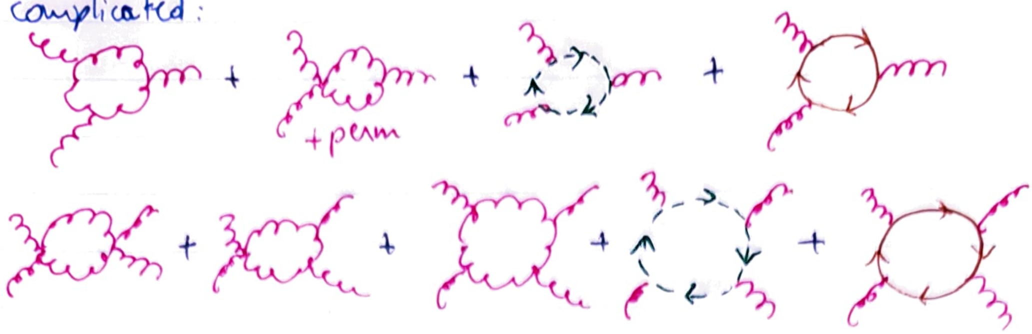
→ In order to compute the β-function for g, we can choose any one of the relations between g and $g(b)$

→ The β-function will be determined by the dependence on the renormalization scale $\tilde{M}$ of the various $\delta_\bullet$. We need in any case to determine δ_A.

→ Z_A is determined by the one-loop corrections to $\langle A A \rangle$:



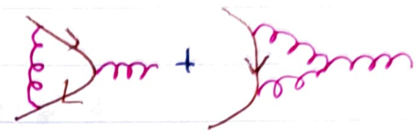
→ The diagrams determining Z_{A^3} or Z_{A^4} are numerous and complicated:



↳ We prefer to compute Z_ψ, $Z_{A\psi^2}$ or Z_c, Z_{Ac^2} instead.

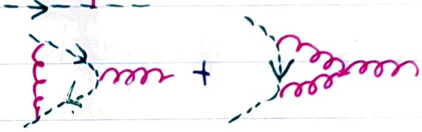
→ Z_ψ: $\langle \psi \bar\psi \rangle$ corrected by

→ $Z_{A\psi^2}$: $A-\psi-\bar\psi$ vertex corrected by



→ Z_c: $\langle c \bar c \rangle$ corrected by

→ Z_{Ac^2}: $A-c-\bar c$ corrected by:

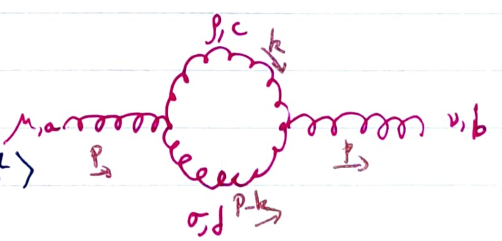


↳ Since ghosts are more intrinsic to the gauge dynamics, they do not involve an arbitrary rep. and don't carry Clifford indices.

→ We will compute the β-function from $g = g(b)\, Z_A^{1/2} Z_c Z_{Ac^2}^{-1}$:

$$\beta(g) = \bar M \frac{\partial}{\partial \bar M} g = g\, \bar M \frac{\partial}{\partial \bar M}\left(\frac{1}{2}\delta_A + \delta_c - \delta_{Ac^2}\right)$$

→ Let's start by computing



$\langle A_1 A_2 \rangle$

$$= \frac{1}{2}\langle A_1 A_2\, i\int g\, \partial A_x A_x^2\, i\int g\, \partial A_y A_y^2 \rangle$$

$$= \frac{1}{2\cdot 6\cdot 6}\langle A_1 A_2 \int g_3 A_x^3 \int g_3 A_y^3 \rangle$$ where g_3 is the 3-gluons vertex

$$= \frac{1}{2\cdot 6\cdot 6}\int g_3^2\; 2\cdot 3\, \langle A_1 A_x \rangle\, 3 \langle A_y A_2 \rangle\, 2 \langle A_x A_y \rangle^2$$

↳ The 1PI part is $\frac{1}{2}\int_k g_3^2 \langle AA \rangle_k \langle AA \rangle_{p-k}$

→ We take the Feynman gauge $\xi = 1$ for all propagators. Quantities might not be gauge invariant, but in the end β is. We get:

$$\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \frac{-i}{(p-k)^2} g^2 f_{acd} \left\{ \eta_{\mu\rho}(-k-p)_\sigma + \eta_{\mu\sigma}(2p-k)_\rho + \eta_{\rho\sigma}(2k-p)_\mu \right\}$$

$$\times f_{bcd} \left\{ \delta_\nu^\rho (k+p)^\sigma + \delta_\nu^\sigma(-2p+k)^\rho + \eta^{\rho\sigma}(p-2k)_\nu \right\}$$

→ The group theory part is evaluated as follows:

DEF given a representation ρ of the group G with generators t_a^ρ, one can define 2 numbers specific to the representation:
the index of ρ $T(\rho)$ and the quadratic Casimir $C(\rho)$ such that:
$\mathrm{tr}\, t_a^{(\rho)} t_b^{(\rho)} = T(\rho)\, \delta_{ab}$ and $t_a^{(\rho)} t_b^{(\rho)} = C(\rho)\, \mathbb{1}_\rho$

PROP Taking the trace on left-over indices, we get:

$$C(\rho)\dim(\rho) = T(\rho)\dim(G)$$

where $\dim(G) \equiv \dim(Ad)$.

→ If $\rho = Ad$, we have $C(Ad) = T(Ad)$ and we can relate it to f_{abc}: $(t_a)_{cd} = i f_{cad}$; $\mathrm{tr}\, t_a^{(Ad)} t_b^{(Ad)} = C(Ad)\,\delta_{ab}$ so
$i f_{cad}\, i f_{dbc} = C(Ad)\,\delta_{ab}$ so $f_{acd} f_{bcd} = C(Ad)\,\delta_{ab}$

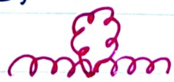
→ The integral becomes, introducing x and $k = \ell + px$:

$$+\frac{1}{2} g^2 C(Ad)\,\delta_{ab} \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta)^2} \left\{ \eta_{\mu\rho}(\ell + p(1+x))_\sigma \right.$$

$$\left. + \eta_{\mu\sigma}(\ell - p(2-x))_\rho + \eta_{\rho\sigma}(p(1-2x) - 2\ell)_\mu \right\} \times \left\{ \delta_\nu^\rho(\ell + p(1+x))^\sigma + \delta_\nu^\sigma(\ell - p(2-x))^\rho \right.$$

$$\left. + \eta^{\rho\sigma}(p(1-2x) - 2\ell)_\nu \right\}$$ where $\Delta = -p^2 x(1-x)$

$$= \dots = \frac{1}{2} g^2 C(Ad)\,\delta_{ab} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^2} \left\{ \eta_{\mu\nu} p^2(1 - 2x + 2x^2) + p_\mu p_\nu(-2 - 10x - 10x^2) \right\}$$

→ Adding the contribution of , we get:

$$-g^2 C(Ad)\,\eta_{\mu\nu} \int \frac{d^dk}{(2\pi)^d} \frac{d-1}{k^2}\,\delta_{ab}$$

→ Adding the contribution of 

we get:

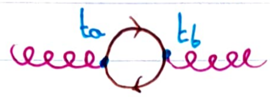
$$\langle A_1 A_2 \rangle = \frac{1}{2} \langle A_1 A_2 \, ig \int \partial \bar{c}_x A_x c_x \, ig \int \partial \bar{c}_y A_y c_y \rangle$$

$$= \int \frac{1}{2} \, 2 \langle A_1 A_x \rangle \langle A_y A_2 \rangle \langle c_x \bar{c}_y \rangle (-1) \langle c_y \bar{c}_x \rangle$$

The 1PI part is $- \int_k g_c \langle c \bar{c} \rangle_k \, g_c \langle c \bar{c} \rangle_{k-p}$

$$= -g^2 f_{adc} f_{bcd} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{(p-k)^2} (k-p)_\mu k_\nu = \dots$$

$$= + g^2 C(Ad) \, \delta_{ab} \, p_\mu p_\nu \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{x(1-x)}{(\ell^2 - \Delta)^2}$$

→ Adding the contribution of 

$\propto \operatorname{tr} t_a t_b = T(\rho) \delta_{ab}$, we have:

$$-\frac{8 i g^2}{(4\pi)^4} T(\rho) \delta_{ab} (p^2 \eta_{\mu\nu} - p_\mu p_\nu) \int_0^1 dx \, \Gamma(2 - d/2) \left(\frac{\Delta_m}{4\pi} \right)^{\frac{d}{2}-2} x(1-x)$$

with $\Delta_m = m^2 - p^2 x(1-x)$

→ The kinetic counterterm for A_μ is 

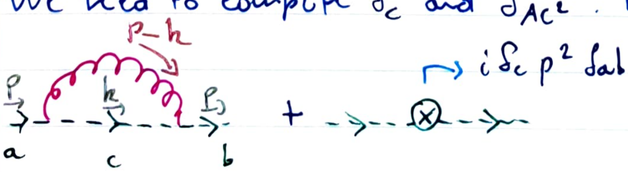
$$-i \delta_A (p^2 \eta_{\mu\nu} - p_\mu p_\nu) \delta_{ab}$$

→ Setting the whole correction to vanish at $p^2 = -M^2$, we find:

$$\delta_A = \frac{g^2}{(4\pi)^4} \int_0^1 dx \left\{ C(Ad)(1 + 4x(1-x)) - T(\rho) \, 8x(1-x) \right\} \left(\frac{1}{\varepsilon} - \gamma - \log\left(\frac{M^2}{4\pi\mu^2} \right) \right)$$

$$= \frac{g^2}{(4\pi)^2} \Big(\underbrace{\tfrac{5}{3} C(Ad)}_{\text{pure gauge}} - \underbrace{\tfrac{4}{3} T(\rho)}_{\text{fermions}} \Big) \left(\frac{1}{\varepsilon} - \gamma - \log \frac{M^2}{4\pi\mu^2} \right)$$

→ We need to compute δ_c and δ_{Acc}. Let's start with δ_c:

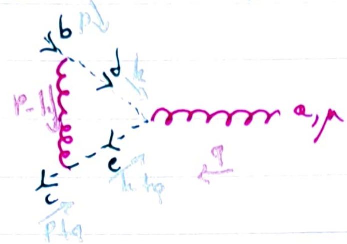


$$\langle c_1 \bar{c}_2 \rangle \Big|_{1PI} = \frac{-i g^2}{(4\pi)^2} C(Ad) \, \delta_{ab} \int_0^1 dx \, x \, \Gamma(2 - d/2) \left(\frac{\Delta}{4\pi} \right)^{\frac{d}{2}-2}$$

so $\delta_c = \frac{g^2}{(4\pi)^2} \frac{1}{2} C(Ad) \left(\frac{1}{\varepsilon} - \gamma - \log(M^2/4\pi\mu^2) \right)$

→ 3 diagrams contribute to δ_{Ac^2}:

→ The 1PI part of



is:

$$g^3 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{(p-k)^2} \frac{i}{k^2} \frac{i}{(k+q)^2} p_\nu (k+q)^\nu k_\mu f_{ade} f_{fbd} f_{fec}$$

↳ Some group theory first:

$$f_{ade} f_{fbd} f_{fec} = -f_{ade} (i f_{dbf})(i f_{fce}) = -f_{ade} (t_b^{Ad})_{df} (t_c^{Ad})_{fe}$$

$$= -\frac{1}{2} f_{ade} \{ (t_b^{Ad})_{df} (t_c^{Ad})_{fe} - (t_b^{Ad})_{ef} (t_c^{Ad})_{fd} \}$$

$$= -\frac{1}{2} f_{ade} \{ (t_b^{Ad})_{df} (t_c^{Ad})_{fe} - (t_c^{Ad})_{df} (t_b^{Ad})_{fe} \}$$

$$= -\frac{1}{2} f_{ade}\, i f_{bcf} (t_f^{Ad})_{de} = \frac{1}{2} f_{ade} f_{bcf} f_{fde} = -\frac{1}{2} f_{ade} f_{fde} f_{fbc}$$

$$= -\frac{1}{2} C(Ad) f_{abc}$$

↳ the diagram is log-div. Hence we take k very large to compute its divergency part, wich is only neded for δ_{Ac^2}:

$$-\frac{1}{2} i g^3 C(Ad) f_{abc} \int \frac{d^4k}{(2\pi)^4} \frac{p^\nu k_\nu k_\mu}{(k^2)^3} = -\frac{1}{8} i g^3 C(Ad) f_{abc} p_\mu \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2}$$

$$= \frac{1}{8} \frac{g^3}{(4\pi)^2} C(Ad) f_{abc} p_\mu \Gamma(2-d/2) \left(\frac{\Delta}{4\pi}\right)^{\frac{d}{2}-2}$$

→ Putting the 2 diagrams together, we find:

$$\delta_{Ac^2} = -\frac{1}{2} \frac{g^2}{(4\pi)^2} C(Ad) \left\{ \frac{1}{\varepsilon} - \gamma - \log \frac{M^2}{4\pi\mu^2} \right\} + \delta_c$$

→ We can finally compute the β-function:

$$\beta = g\bar{M}\frac{\partial}{\partial \bar{M}}\left(\frac{1}{2}\delta_A + \delta_c - \delta_{Ac^2}\right) \text{ and}$$

$$\frac{1}{2}\delta_A + \delta_c - \delta_{Ac^2} = \frac{1}{2}\frac{g^2}{(4\pi)^2}\left\{\frac{5}{3}C(Ad) - \frac{4}{3}T(g) + C(Ad) + C(Ad)\right\}\left(\frac{1}{\varepsilon} + \gamma - \log\frac{\bar{M}^2}{4\pi\mu^2}\right)$$

$$= \frac{1}{2}\frac{g^2}{(4\pi)^2}\left(\frac{11}{3}C(Ad) - \frac{4}{3}T(g)\right)\left(\frac{1}{\varepsilon} - \gamma - \log\frac{\bar{M}^2}{4\pi\mu^2}\right)$$

PROP | The β-function for a non abelian gauge theory is given by

$$\beta = -\frac{g^3}{(4\pi)^2}\underbrace{\left(\frac{11}{3}C(Ad) - \frac{4}{3}T(g)\right)}_{\equiv b_0}$$

14.3 Asymptotic freedom

→ For a pure Yang-Mills theory (no matter), we have

$$\beta = -\frac{g^3}{(4\pi)^2}\frac{11}{3}C(Ad) < 0$$

↳ The coupling decreases towards the UV, i.e. its asymptotically free.

DEF | We denote by $\square$ the fundamental, N-dim. rep of SU(N), and $\overline{\square}$ the complexe conjugate rep.

PROP | From group theory, one can show that for SU(N),

$$C(Ad) = N \quad \text{and} \quad T(\square) = 1/2$$

→ Note that ψ being a Dirac spinor, it's composed of 2 Weyl spinors, with $\chi_1 \in \square$ and $\chi_2 \in \overline{\square}$. If ρ is a real rep., (like Ad) it can be carried by a Weyl spinor.

→ In a QCD-like theory with gauge group SU(N) and N_f flavours of quarks, the total matter rep. is N_f copies of $\square$. ($\square \oplus \overline{\square}$ over Weyl spinors, consistent with the possible presence of a mass term, and the absence of anomalies).

↳ Hence, $T(f) = \sum_{N_f} T(\Box) = \frac{1}{2} N_f$

→ We have then $\beta = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}N_f\right)$

→ For N_f sufficiently small, we always have $\beta < 0$.

PROP In real world QCD, $N=3$ and $N_f = 6$ and $\beta = -7\frac{g^3}{(4\pi)^2} < 0$

PROP Writing $\beta = -\frac{g^3}{(4\pi)^3} b_0$, we need $b_0 > 0$ for asymptotic freedom

→ Since b_0 is an observable, it's automatically gauge invariant and scheme independent. The equation reads:

$$\bar{\mu}\frac{\partial}{\partial\bar{\mu}} g = -\frac{g^3}{(4\pi)^2} b_0 \Leftrightarrow \bar{\mu}\frac{d}{d\bar{\mu}}\frac{1}{g^2} = \frac{b_0}{8\pi^2}$$

The solution is $\frac{1}{g^2(\bar{\mu})} = \frac{1}{g^2(\bar{\mu}_0)} + \frac{b_0}{8\pi^2}\log\left(\frac{\bar{\mu}}{\bar{\mu}_0}\right)$

↳ If $\bar{\mu} > \bar{\mu}_0$, then $\frac{1}{g^2(\bar{\mu})} > \frac{1}{g^2(\bar{\mu}_0)} \Leftrightarrow g^2(\bar{\mu}) < g^2(\bar{\mu}_0)$

→ If we go towards the IR ($\bar{\mu} < \bar{\mu}_0$), we see that there will be an energy scale where the log term cancels the $1/g^2(\bar{\mu}_0)$ term.

DEF We define the IR scale Λ_{IR} such that

$$\frac{1}{g^2(\Lambda_{IR})} = 0 = \frac{1}{g^2(\bar{\mu}_0)} + \frac{b_0}{8\pi^2}\log\left(\frac{\Lambda_{IR}}{\bar{\mu}_0}\right)$$

→ What is intrinsic to QCD is not the coupling (scale dependent) but the IR strong coupling scale Λ_{IR}. This is called dimensional transmutation.

→ Given the coupling g at some scale μ, Λ_{IR} is defined as

$$\Lambda_{IR} \equiv \mu \exp\left\{\frac{-8\pi^2}{b_0 g^2(\mu)}\right\}$$

↳ It's a non perturbative quantity: we cannot expand this quantity as a power series of g^2 ⇒ powers of Λ_{IR} will never appear in QCD perturbation theory.

⇒ In the real-world QCD, from the data at the electroweak scale, one can compute $\Lambda_{QCD} \sim 250$ MeV
↳ We see that $m_p \sim m_n \sim 1$ GeV $\sim \mathcal{O}(\Lambda_{QCD})$

→ We expect the tension of QCD strings (flux tubes between confined quarks being pulled apart) to be $T \sim \Lambda_{QCD}^2$.

→ We expect quark bilinear condensates to form (this breaks spontaneously the chiral flavour symmetry of QCD, SU(3) for light quarks u, d, s) with vacuum expectation value $\langle \bar{\psi} \psi \rangle \sim \Lambda^3_{QCD}$. It is indeed the case, as established from pion physics.