13 QUANTIZATION AND GHOSTS

→ We now proceed to quantize the theory. $\mathcal{L}_I$ will provide us with interaction vertices, while $\mathcal{L}_0$ will give us the propagators.

→ Since the vector part of $\mathcal{L}_0$ is exactly equal to the one for QED, replicated $n = \dim G$ times, we have the same problem of eigenvectors with zero eigenvalue. (p 20)

13.1 Gauge fixing in QED

→ In QED, we added $\mathcal{L}_{g.f} \equiv \frac{-1}{2\xi}(\partial_\mu A^\mu)^2$ to $\mathcal{L}_0$. It's not the only choice but it's the most economical one.

→ We could put more terms in $\mathcal{L}_{g.f.}$, such as $\mathcal{O}(A^2)$: we would have $G(A) = \partial_\mu A^\mu + g A_\mu A^\mu = 0$, but then $\mathcal{L}_{gf} \propto \frac{-1}{2\xi} G^2$ would introduce a ξ-dependent interaction.

13.2 Gauge fixing in QCD

→ We introduce the same gauge-fixing term to $\mathcal{L}_0$:

$$\mathcal{L}_{g.f} \equiv \frac{-1}{2\xi}(\partial_\mu A^\mu_a)^2$$

→ The free propagator will be the same as in QED, with an additional δ spanning the adjoint representation:

$$\langle A_{\mu a}(k) A_{\nu b}(-k)\rangle \equiv -\frac{i\delta_{ab}}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

↳ The th Feynman gauge, $\xi = 1$ and $D_{\mu\nu ab} = \frac{\delta_{ab}\,\eta_{\mu\nu}}{k^2}$

13.3 Faddeev-Popov ghost

→ Let us start the procedure again. We want to insert a δ-functional to impose a gauge condition $G(A)=0$. We have $Z_A = \int \mathcal{D}A\, e^{iS_A}$ and we insert $1 = \int \mathcal{D}\alpha\, \delta[G({}^\alpha A)]. \left|\det \frac{\delta G({}^\alpha A)}{\delta \alpha}\right|$ with $\alpha \equiv \alpha_a t_a$. We get:

$$Z_A = \int \mathcal{D}A \int \mathcal{D}\alpha\, e^{iS_A}\, \delta[G({}^\alpha A)] \left|\det \frac{\delta G({}^\alpha A)}{\delta \alpha}\right|$$

and $G({}^\alpha A) = \partial_\mu (A^\mu + \delta A^\mu) = \partial_\mu (A^\mu + \partial^\mu \alpha + i[\alpha, A_\mu])$
$= \partial_\mu A^\mu + \partial^2 \alpha + i \partial_\mu [\alpha, A_\mu]$

DEF We introduce D_μ the covariant derivative in the adjoint rep. such that: $D_\mu \alpha = \partial_\mu \alpha - i[A_\mu, \alpha]$

↳ Then $G({}^\alpha A) = \partial_\mu (A^\mu + D^\mu \alpha)$. Hence, $\frac{\delta G[{}^\alpha A]}{\delta \alpha} = \partial_\mu D^\mu$
$= \partial_\mu \{ \partial^\mu \cdot - i[A^\mu, \cdot] \}$

⇒ It depends on A_μ! (QED: $\delta G/\delta \alpha = \partial^2$; factor it out in N)
It cannot be factored out as an overall factor.
$Z_A = \int \mathcal{D}\alpha\, \mathcal{D}A\, e^{iS_A}\, \delta[G({}^\alpha A)]\, |\det\{\partial_\mu D^\mu(A)\}|$

→ The usual tricks to modifie $G(A)$ into $\partial_\mu A^\mu - \omega$ and integrate over ω with a Gaussian weight leads to:

(p 22)

$$Z_A = N \int \mathcal{D}A\, e^{iS_A + iS_{gf}}\, |\det \partial_\mu D^\mu|$$

↳ We still have this A_μ dependence in the det. factor, this dependence seems non local (ie it involves an infinite number of derivative. This can be seen by writing $\det = e^{\log \det}$)

?

→ As in the path integral over fermions, $Z_A \propto \det$. We reuse the result that, for η a grassmann number, and $(K)_{ij} = k_{ij}$ an operator,

(p 16)

$\int \prod_j d\eta_j^*\, d\eta_j\, \exp\{-\eta_j^*\, k_{ij}\, \eta_i\} = \det K$

DEF We introduce auxiliary fields c^a, $\bar{c}^a$, the ghosts and antighosts which are Grassman-odd fields such that

$$\int \mathcal{D}\bar{c}\, \mathcal{D}c\, \exp\{-i \int \bar{c}\, \partial_\mu D^\mu c\} = \det\{\partial_\mu D^\mu\}$$

PRop

c and $\bar{c}$ are scalars, and since D^μ is in the adjoint rep., they both carry indices of the adjoint rep: c^a, $\bar{c}^a$.
Given that they're scalars and fermionic, they violate the spin-statistics relation ⇒ they're unphysical, auxiliary fields.

→ We thus have:

$$Z_A = N\int \mathcal{D}A\, e^{iS_A}\, \delta[G({}^\alpha A)]\, |\det \partial_\mu D^\mu|$$

$$= N\int \mathcal{D}A\, \mathcal{D}\bar{c}\, \mathcal{D}c\, \exp\{i(S_A[A] + S_{gf}[A] + S_g[A, c, \bar{c}])\}$$

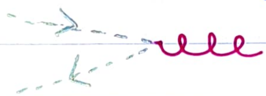
with

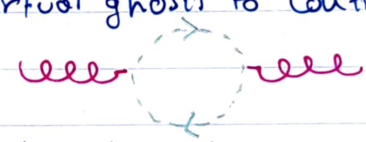
$$S_g \equiv -\int d^4x\, \bar{c}_a\, \partial_\mu (D^\mu c)_a$$

$$= \int d^4x\, \partial_\mu \bar{c}_a \cdot (\partial^\mu c_a - ig A^\mu{}_b (T_b)_{ac}\, c_c)$$

$$= \int d^4x \,\{ \underbrace{\partial_\mu \bar{c}_a\, \partial^\mu c_a}_{\in S_0} + \underbrace{g f_{abc}\, \partial_\mu \bar{c}_a\, A^\mu{}_b\, c_c}_{\in S_I} \}$$

↳ We've introduced a kinetic term for the ghosts, but also an interaction vertex with 2 ghosts and a gauge boson:



↳ This will allow for virtual ghosts to contribute to radiative corrections:  ghost loop

→ Including the matter fields, the full Lagrangian reads as follows:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\mathcal{L}_0 \left\{ = -\frac{1}{4}(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a})^2 - \frac{1}{2\xi}(\partial_\mu A^\mu{}_a)^2 + i\bar{\psi}_A \not{\partial} \psi^A + \partial_\mu \bar{c}_a\, \partial^\mu c_a \right.$$

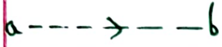
$$\left. \begin{aligned} &+ (-g)\, \partial_\mu A_{\nu a}\, f_{abc}\, A_b^\mu A_c^\nu - \frac{1}{4} g^2 f_{abc} f_{ade}\, A_{\mu b} A_{\nu c} A^\mu{}_d A^\nu{}_e \\ &+ g A_{\mu a}\, \bar{\psi}_A \gamma^\mu (t_a)^A{}_B\, \psi^B + g f_{abc}\, \partial_\mu \bar{c}_a\, A^\mu{}_b\, c_c \end{aligned} \right\} \mathcal{L}_I$$

13.4 Feynman rules

→ In Fourier space, the propagators are:

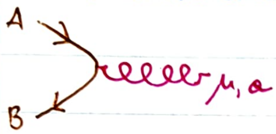
$$\langle A_{\mu a}(k) A_{\nu b}(-k)\rangle = -\frac{i\delta_{ab}}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\, k_\mu k_\nu / k^2\right)$$

$$\langle \psi^A(k)\, \bar\psi_B(-k)\rangle = \frac{i\delta^A_B}{\not k - m} = i\delta^A_B\, \frac{\not k + m}{k^2 - m^2}$$



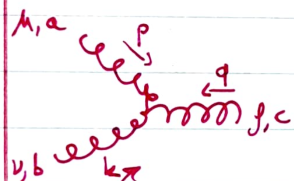
$$\langle c_a(k)\, \bar c_b(-k)\rangle = \frac{i\delta_{ab}}{k^2}$$

→ For the interaction vertices (see notes (p127) for derivations):

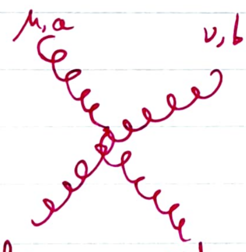


$$ig\gamma^\mu (t_a)^A_B$$

$$g f_{abc}\, p_\mu$$



$$g f_{abc}\left\{\eta_{\mu\nu}(k-p)_\rho + \eta_{\mu\rho}(p-q)_\nu + \eta_{\nu\rho}(q-k)_\mu\right\} \equiv g_3$$



$$-ig^2\left\{\begin{aligned} & f_{abe}f_{cde}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\ & + f_{ace}f_{bde}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\ & + f_{ade}f_{bce}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma}) \end{aligned}\right\} \equiv g_4$$