12 NON-ABELIAN GAUGE THEORIES

ch15 Peskin

→ The Standard Model of particle physics is in great part based on non-abelian gauge theories. They are a generalization of QED, with non trivial interactions among the gauge bosons. This will give us an important example of a class of asymptotically free QFT's.

12.1 Global symmetry

→ In the study of QED, a Dirac fermion is symmetric under constant phase rotations $\begin{cases} \psi \mapsto e^{i\alpha}\psi \\ \bar{\psi} \mapsto e^{-i\alpha}\bar{\psi} \end{cases}$ since $\mathcal{L}_\psi = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$

This is associated to an abelian $U(1)$ group. Let's generalize.

→ Consider the theory of Lie groups. Take a set of Dirac fields $\{\psi^A\}_{A=1}^{r}$ and declare they belong to a representation ρ of a group G (with $\dim\rho = r$): $\psi^A \mapsto U^A{}_B \psi^B$ with

$U(g) \in \rho$ a unitary matrix on the space of ψ

↳ $\bar{\psi}_A \mapsto \bar{\psi}_B (U^\dagger)^B{}_A$

↳ Lagrangian invariant: $\mathcal{L}_\psi \mapsto i\bar{\psi}U^{-1}\not{\partial}U\psi - m\bar{\psi}U^{-1}U\psi = \mathcal{L}_\psi$

→ Going to the algebra, we can write $U = e^{i\alpha_a t_a}$ with t_a the hermitian generators of the algebra $\mathcal{A}$, and $a = 1,..,n \equiv \dim G$. Since $\alpha_a \ll 1$, we can write $\delta\psi^A = i\alpha_a (t_a)^A{}_B \psi^B$

DEF | The group structure implies $[t_a, t_b] = i f_{abc} t_c$ where f_{abc} are the structure constants and can be taken to be completely antisymmetric

12.2 Local symmetry

→ Let us now make U spacetime dependent. As in QED, the kinetic term is no longer invariant. We need to replace $\partial_\mu \psi$ with a covariant derivative D_μ that transforms as

$$D_\mu \psi \mapsto U D_\mu \psi$$

Thus, we introduce a connection A_μ which is in the same representation of U (adjoint rep):

$$(D_\mu \psi)^A = \partial_\mu \psi^A - i (A_\mu)^A{}_B \psi^B$$

↳ We have: $D_\mu \psi \mapsto D'_\mu \psi' = (\partial_\mu - i A'_\mu) \psi'$
$= (\partial_\mu - i A'_\mu) U(x) \psi$
$= U \partial_\mu \psi + \partial_\mu U \cdot \psi - i A'_\mu U \psi$

$D_\mu \psi \mapsto U D_\mu \psi \overset{!}{=} U \partial_\mu \psi - i A_\mu U \psi$

$\Leftrightarrow A'_\mu = -i \partial_\mu U \cdot U^{-1} + U A_\mu U^{-1}$ $\qquad 0 = \partial_\mu (U U^\dagger) = U \partial_\mu U^\dagger + \partial_\mu U \cdot U^\dagger$
$= i U \partial_\mu U^\dagger + U A_\mu U^\dagger$

→ In the algebra $\mathcal{A}$, we have $\alpha \equiv \alpha_a(x) \cdot t_a$. We can write $U = e^{i\alpha}$ and so $\begin{cases} \partial_\mu U^{-1} = \partial_\mu (e^{-i\alpha}) = -i \partial_\mu \alpha \cdot U^{-1} \\ \partial_\mu U = \partial_\mu (e^{i\alpha}) = i \partial_\mu \alpha \cdot U \end{cases}$

Close to the identity, $\qquad$ ↳ $i U \partial_\mu U^\dagger = i e^{i\alpha} \cdot (-i \partial_\mu \alpha) \cdot U^{-1} = \partial_\mu \alpha$

$$A'_\mu = \partial_\mu \alpha + (1 + i\alpha) A_\mu (1 - i\alpha) = \partial_\mu \alpha + i [\alpha, A_\mu]$$

PROP | The infinitesimal gauge transformation of A^μ is

$$\delta A_\mu = \partial_\mu \alpha + i [\alpha, A_\mu]$$

→ A_μ takes values naturally in the algebra $\mathcal{A}$

→ Since $A_\mu \in \mathcal{A}$, we can write it in the $\{t_a\}^n$ basis:

$$\delta A_{\mu a} t_a = \partial_\mu \alpha_a t_a + i [\alpha_a t_a, A_{\mu b} t_b]$$
$$= \partial_\mu \alpha_a t_a + i \alpha_a A_{\mu b} i f_{abc} t_c$$

PROP | We have $\underbrace{\delta A_{\mu a} = \partial_\mu \alpha_a - f_{abc} \alpha_b A_{\mu c}}_{= (D_\mu \alpha)^a} = \partial_\mu \alpha_a + i \alpha_b (T_b)^a{}_c A_{\mu c}$

where $(T_b)^a{}_c \equiv i f_{abc}$ is the adjoint representation generator, i.e. the vector space of the rep. of $\mathcal{A}$ is $\mathcal{A}$ itself.

12.3 Field strength tensor

→ We have generalized the matter part of the QED action to a non abelian symmetry: $\mathcal{L} = i\bar{\Psi} \not{D} \Psi - m\bar{\Psi}\Psi$ but now

$$\mathcal{L}_I = A_{\mu a} \bar{\Psi} \gamma^\mu t_a \Psi \qquad / \ A_\mu \bar{\Psi}\gamma^\mu \Psi$$

→ The transformation of $A_{\mu a}$ acquires a term linear in A_μ:

$$\delta A_{\mu a} = \partial_\mu \alpha_a - f_{abc}\alpha_b A_{\mu c}$$

→ To buid the kinetic term, the usual $\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a}$ is not gauge invariant. Since $[\partial_\mu, \partial_\nu]\Psi = 0$, then by definition of D_μ, we must have $[D_\mu, D_\nu]\Psi \mapsto U[D_\mu, D_\nu]\Psi$.

↳ $$\begin{aligned}[D_\mu, D_\nu]\Psi &= (\partial_\mu - iA_\mu)(\partial_\nu - iA_\nu)\Psi - (\partial_\nu - iA_\nu)(\partial_\mu - iA_\mu)\Psi \\ &= \partial_\mu\partial_\nu\Psi - i\partial_\mu(A_\nu\Psi) - iA_\mu\partial_\nu\Psi - A_\mu A_\nu\Psi \\ &\quad - \partial_\nu\partial_\mu\Psi + i\partial_\nu(A_\mu\Psi) + iA_\nu\partial_\mu\Psi + A_\nu A_\mu\Psi \\ &= -i(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu])\Psi\end{aligned}$$

DEF The field strength tensor in Yang-Mills theories $F_{\mu\nu a}$ is defined as

$$F_{\mu\nu a} \equiv \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + f_{abc} A_{\mu b} A_{\nu c}$$

→ We have $F_{\mu\nu} \to U F_{\mu\nu} U^{-1}$: it transforms covariantly.
Indeed, $F'_{\mu\nu}\Psi' = F'_{\mu\nu} U\Psi = UF_{\mu\nu}\Psi \Leftrightarrow F'_{\mu\nu} = UF_{\mu\nu}U^{-1}$

→ Close to the identity, $F' = (1+i\alpha)F(1-i\alpha) = F + i[\alpha, F] + \mathcal{O}(\alpha^2)$. We have

$$\delta F_{\mu\nu} = i[\alpha, F_{\mu\nu}]$$

In the basis of the algebra $\mathcal{A}$, we have

$$\delta F_{\mu\nu a} = i\alpha_b (T_b)^a{}_c F_{\mu\nu c}$$

↳ $F_{\mu\nu}$ transform linearly in the adjoint representation.

↳ $F_{\mu\nu}$ transform covariantly under U

→ We need to build a gauge invariant bilinear. An easy way is to take the trace over the adjoint rep: $\mathrm{tr}(F_{\mu\nu a} F^{\mu\nu}{}_b) = F_{\mu\nu a} F^{\mu\nu}{}_a$

DEF The Lagrangian of the Yang-Mills pure gauge theory $\mathcal{L}_A$ is:

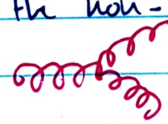
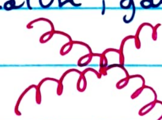
$$\mathcal{L}_A = \frac{-1}{4g^2} F_{\mu\nu a} F^{\mu\nu}{}_a$$

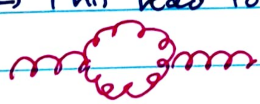
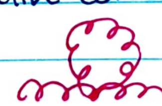
→ Rescaling $A_{\mu a} \mapsto g A_{\mu a}$, we get the usual kinetic term
$-\frac{1}{4} F_{\mu\nu a} F^{\mu\nu}{}_a$ and the covariant derivative become
$$D_\mu \psi \mapsto \partial_\mu \psi - i g A_{\mu a} t_a \psi$$

12.4 Yang-Mills theory

DEF The Yang-Mills Lagrangian is
$$\mathcal{L}[A,\psi] = -\frac{1}{4} F_{\mu\nu a} F^{\mu\nu}{}_a + i \bar{\psi} \not{D} \psi - m \bar{\psi} \psi$$
with:
→ $F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + g f_{abc} A_{\mu b} A_{\nu c}$
→ $D_\mu \psi = \partial_\mu \psi - i g A_{\mu a} t_a \psi$

→ An important feature of $\mathcal{L}$ for quantization is $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ with:
$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a})(\partial^\mu A^\nu{}_a - \partial^\nu A^\mu{}_a) + i \bar{\psi} \not{\partial} \psi - m \bar{\psi} \psi$$
$$= \mathcal{L}_0[A,\psi]$$
and
$$\mathcal{L}_I = -g\, \partial_\mu A_{\nu a} f_{abc} A^\mu{}_b A^\nu{}_c - \frac{1}{4} g^2 f_{abc} f_{ade} A_{\mu b} A_{\nu c} A^{\mu}{}_d A^{\nu}{}_e$$
$$+ g A_{\mu a} \bar{\psi} \gamma^\mu t_a \psi$$
$$= \mathcal{L}_I[A]$$

→ Because of the non-abelian nature, gauge boson interact with themselves:  and 

↳ This lead to radiative corrections of a new kind:
 and  , ...

↳ We expect renormalization and evolution of couplings also in the theory without matter

→ Gauge symmetry imposes that 3 different vertices are associated to one and only one coupling g.