11 ENERGY SCALES AND EVOLUTION OF COUPLINGS

11.1 Renormalization scale $\bar{M}$

→ In RPT (renormalized perturbation theory), the radiative corrections are finite, but typically depend on the momenta of the quantity one is computing. It is tempting to interpret this as an effective coupling. However, in $\lambda\phi^4$ 1PI diagram of the 4-pt function, the cell depends on 3 invariant s, t, u after renormalization. The counter-terms don't depend on the external momenta since they're determined at fixed external momenta through the renormalization conditions.

→ Let's push this logic a bit further by introducing a new scale.

DEF The renormalization scale $\bar{M}$ is the scale at wich we impose the renormalization conditions.

↳ Previously, we had "$\bar{M} = m_{particle}$". Now, we use an arbitrary scale so we can study what happens when we vary it.

→ Practically, we impose the renormalization condition when all external momentum invariants are fixed by M^2. Considering spacelike momenta, we fix the condition for $\lambda\phi^4$ theory:

eq 12.30 Peskin

DEF The renormalization condition is:

$$\xleftarrow{p}\text{(1PI)}\leftarrow \; = \mathcal{M}(p^2=-\bar{M}^2)=0$$

$$\frac{d}{dp^2}\left(\xleftarrow{p}\text{(1PI)}\text{—}\right) = \frac{d}{dp^2}\mathcal{M}(p^2)\Big|_{p^2=-\bar{M}^2} = 0$$

and  $= -6i\lambda$ at $s=t=u=-\bar{M}^2$

→ In any regularization scheme, one introduce a mass scale (Λ in momentum cut-off, μ in dimensional regularization). Through the renormalization condition, one is effectively trading the regularization scale for renormalization scale $\bar{M}$.

11.2 The Callan-Symanzik equation

→ To study how renormalized quantities (ex: coupling constant) evolve when we vary $\bar{M}$, we'll study the n-pt connected Green functions in the bare and renormalized theory:

$$G^{(n)}_{bare} = \langle \Omega | T \phi_b(x_1) \cdots \phi_b(x_n) | \Omega \rangle$$

and $$G^{(n)}_{ren} = \langle \Omega | T \phi_r(x_1) \cdots \phi_r(x_n) | \Omega \rangle$$

→ $G^{(n)}_{bare}$ depends on the bare parameters, and on the cut-off of the theory (or equivalently on μ for reg. dim.): $G^{(n)}_{bare}(x_1, \dots, x_n, \lambda_b, \Lambda)$

→ After renormalization, renormalized quantities shouldn't depend on Λ, but they could depend on $\bar{M}$: $G^{(n)}_{ren}(x_1, \dots, x_n, \lambda_n, \bar{M})$

→ Now, we have $\phi_b = \sqrt{Z}\, \phi_{ren} \Rightarrow G^{(n)}_b = Z^{n/2}\, G_r^{(n)} \Leftrightarrow G_r^{(n)} = Z^{-\frac{n}{2}} G_b^{(n)}$

↳ $G_r^{(n)}$ depends on $\bar{M}$ explicitly and implicitly (in λ_r wich is a function of λ_b and $\bar{M}$). Of course, $Z = Z(\bar{M})$ because of the renormalisation condition of the 2-pt function.

eq 12.37 Peskin → Now, suppose that we shift $\bar{M} \mapsto \bar{M} + \delta\bar{M}$. There is a corresponding shift $\lambda_r \mapsto \lambda_r + \delta\lambda_r$ and $\phi_r \mapsto \phi_r + \delta\eta \cdot \phi_r$ so that the bare $G_b^{(n)}$ remains fixed. Then the shift in any renormalised green's function is simply that induced by the field rescaling:

$$G_r^{(n)} \mapsto (1 + n \cdot \delta\eta)\, G_r^{(n)}$$

→ If we think of $G_r^{(n)}$ as a function of $\bar{M}$ and λ, we have:

$$dG^{(n)} = \frac{\partial G_r^{(n)}}{\partial \bar{M}} \delta\bar{M} + \frac{\partial G_r^{(n)}}{\partial \lambda_r} \delta\lambda_r = n \cdot \delta\eta\, G_r^{(n)}$$

DEF | We define the beta function $\beta(\lambda)$ from the renormalization group as
$$\beta(\lambda) \equiv \frac{\bar{M}}{\delta \bar{M}} \delta\lambda = \frac{\partial \lambda}{\partial \ln \bar{M}}$$
We also define the gamma parameter $\gamma(\phi)$ as
$$\gamma(\phi) \equiv \frac{1}{2} \frac{\bar{M}}{Z_\phi} \frac{\partial Z_\phi}{\partial \bar{M}} = \frac{1}{2} \frac{\partial \ln Z_\phi}{\partial \ln \bar{M}}$$
Replacing, we find the Callan-Symanzik equation:
$$\left[\bar{M} \frac{\partial}{\partial \bar{M}} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n \gamma_\phi \right] G_r^{(n)} = 0$$

→ Note that the $G_r^{(n)}$ is the connected n-pt function, but not the 1PI. When measuring, we cannot avoid the legs. So the $n\gamma$ term is important

11.3 Computation of β and γ in $\lambda\phi^4$

→ As a first example, let us calculate the one-loop contribution to $\beta(\lambda)$ and $\gamma(\lambda)$ in $\lambda\phi^4$:

→ The renormalization conditions at $s = t = u = -\bar{M}^2$ are
$$-6i\lambda + V(s) + V(t) + V(u) - 6i\delta_\lambda \Big|_{s=t=u=-\bar{M}^2} \stackrel{!}{=} -i6\delta_\lambda$$

eq. p 45, (9.3)

In section 9.3, we had (with a $\neq$ of sign):
$$6i\delta_\lambda = 3V(p^2 = -\bar{M}^2) = -\frac{54 i\lambda^2}{(4\pi)^2} \int_0^1 dx \log\left[\frac{m^2 + \bar{M}^2 x(1-x)}{\Lambda^2}\right]$$
$$\simeq \frac{54 i \lambda^2}{(4\pi)^2} \log\left[\frac{\Lambda^2}{\bar{M}^2}\right] \quad \text{when } m^2 \ll \bar{M}^2 \ll \Lambda^2$$
$$\Rightarrow \delta_\lambda = \frac{9\lambda^2}{(4\pi)^2} \log\left[\Lambda^2/\bar{M}^2\right]$$

→ This means, at λ_b fixed, λ changes when $\bar{M}$ is moved because of $\delta_\lambda = \delta_\lambda(\bar{M}^2)$:

$\lambda_b = \lambda + \delta\lambda$. For bare quantities, $\bar{M} \cdot \frac{\partial \lambda_b}{\partial \bar{M}} = 0$

and
$$\beta(\lambda) = \frac{\bar{M}}{\delta \bar{M}} \delta\lambda = -\bar{M} \frac{\partial \delta\lambda}{\partial \bar{M}} = \frac{18\lambda^2}{(4\pi)^2}$$

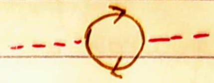
→ Reconsider the renormalization of $\lambda\phi^4$ term we had:

$$\lambda_b \phi_b^4 = \lambda_b Z_\phi^2 \phi_r^4 = \lambda_r \phi_r^4 + \delta_r \phi_r^4 \equiv \lambda_r Z_\lambda \phi_r^4$$

so that $\lambda_r = \frac{Z_\phi^2}{Z_\lambda} \cdot \lambda_b$. Writting $Z_\phi = 1 + \delta_\phi$ and $Z_\lambda = 1 + \delta_\lambda$, we have:

$$\beta(\lambda_r) = \bar{M} \frac{\partial}{\partial \bar{M}} \lambda_r = \left(2 \frac{\bar{M}}{Z_\phi} \frac{\partial Z_\phi}{\partial \bar{M}} - \frac{\bar{M}}{Z_\lambda} \frac{\partial Z_\lambda}{\partial \bar{M}}\right) \lambda_r$$

? $$= \left(2 \bar{M} \frac{\partial}{\partial \bar{M}} \delta_\phi - \bar{M} \frac{\partial}{\partial \bar{M}} \delta_\lambda\right) \lambda_r = \bar{M} \frac{\partial}{\partial \bar{M}} \left(2\delta_\phi - \delta_\lambda\right) \lambda_r$$

pTo → In Yukawa theory we had, in the limit  $\bar{M}^2 \gg m_\psi^2$

$$\delta_\phi \simeq + \frac{12 g^2}{(4\pi)^2} \int_0^1 dx \; x(1-x) \log\left(\frac{\bar{M}^2}{\Lambda^2}\right)$$

$$= \frac{2g^2}{(4\pi)^2} \log\left(\frac{\bar{M}^2}{\Lambda^2}\right) \rightsquigarrow \gamma = \frac{2y^2}{(4\pi)^2}$$

DEF | This γ is called anomalous dimension because it gives a correction to the classical dimension of the field.

→ Indeed, since $\gamma = \frac{1}{2} \frac{\partial \ln Z}{\partial \ln \bar{M}}$, then $\ln \frac{Z}{Z_0} = 2\gamma \ln \frac{\bar{M}}{\bar{M}_0}$

$$\Rightarrow Z = Z_0 \left(\bar{M}/\bar{M}_0\right)^{2\gamma} \text{ and so } \phi_r = Z^{-1/2} \phi_b = Z_0^{-1/2} \left(\frac{\bar{M}}{\bar{M}_0}\right)^{-\gamma} \phi_b$$

↳ It gives an effective scaling dimension to ϕ_r wich is no longer the classical one $[\phi_b] = 1$, but becomes $[\phi_r] = 1 - \gamma$

→ The Callan-Symanzik equation can be trivially generalized to theories with different fields: there will be a γ_{ϕ_i} related to each field ϕ_i and with different couplings. There will be a β_j for each coupling β_j.

We only need to pay a little attention to theories where the couplings are dimensionfull.

11.4 Generalization to $\lambda_n \phi^n$

→ Consider $\lambda_n \phi^n$ theory, in d dimensions. Recall that

(p40)

$$[\lambda_n] = d - n . \frac{d-2}{2}$$

DEF We define a dimensionless coupling $\bar{\lambda}_n$ by

$$\lambda_n \equiv \bar{\lambda}_n \bar{M}^{d - n . \frac{d-2}{2}}$$

↳ Now, $\bar{\lambda}_n$ and $\beta_{\bar{\lambda}}$ can appear in the Callan-Symanzik equation as the other couplings.

→ Note that there is a tree-level (semi-classical) contribution to the β-function. Indeed, for λ_n independent of $\bar{M}$, we have:

$$\beta_{\bar{\lambda}_n} = \bar{M} \frac{\partial}{\partial \bar{M}} \bar{\lambda}_n = M \frac{\partial}{\partial \bar{M}} \left(\lambda_n \bar{M}^{n . \frac{d-2}{2} - d} \right) = \left(n . \frac{d-2}{2} - d \right) \bar{\lambda}_n \text{ at}$$

tree level. ($\Leftarrow$ because $\beta_{\bar{\lambda}_n}$ linear in $\bar{\lambda}_n \Leftrightarrow$ tree level)

→ In all generality, for any field ϕ,

$$\gamma_\phi = \frac{1}{2} \frac{\partial \ln Z_\phi}{\partial \ln \bar{M}} = \frac{\bar{M}}{2} \frac{\partial}{\partial \bar{M}} \delta_{Z_\phi}$$

And for a coupling g evolving a number of fields φ:

$$\mathcal{L}_b \supset g_b \prod_i^n \varphi_{b,i} = g_b \prod_i^n Z_{\varphi_i}^{1/2} \varphi_i \equiv Z_g \, g \prod_i^n \varphi_i$$

$$\text{with } g \equiv g_b \prod_i^n \frac{\sqrt{Z_i}}{Z_g} = \left(1 + \delta_g + \frac{1}{2} \Sigma \delta_\varphi \right) g_b$$

$$Z_g \equiv 1 + \delta_g$$

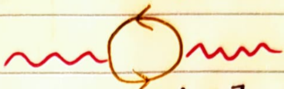
and, at lowest order,

$$\beta = \bar{M} \frac{\partial}{\partial \bar{M}} g = g \bar{M} \frac{\partial}{\partial \bar{M}} \left(- \delta_g + \frac{1}{2} \Sigma \delta_\varphi \right)$$

11.5 An application to QED

→ For the coupling e, we have the vertex with 1 A_μ and 2 ψ: $\mathcal{L}_{QED} \supset e A_\mu \bar{\psi} \gamma^\mu \psi$. From Ward id., we have $\delta_e = \delta_\psi$. Then:

$$\beta_e = e \bar{M} \frac{\partial}{\partial \bar{M}} \left(-\cancel{\delta_e} + \frac{1}{2} \delta_A + \cancel{\delta_\psi} \right) = \frac{1}{2} e \bar{M} \frac{\partial}{\partial M} \delta_A$$

→ From the QED computation of 

(PS5) $$\delta_A \simeq + \frac{8 e^2}{(4\pi)^2} \int_0^1 dx \, (x-1) \left[\frac{1}{\epsilon} - \gamma - \log \left[\frac{\bar{M}^2}{4\pi\mu^2} \right] \right]$$

so that

$$\beta_e \propto \frac{1}{2} e \frac{\partial}{\partial \log \bar{M}} \left\{ \frac{8 \cdot e^2}{(4\pi)^2} \cdot \frac{1}{6} \cdot \log \left[\frac{\bar{M}^2}{4\pi\mu^2} \right] \right\}$$

$$\beta_e = \frac{4 e^3}{3 (4\pi)^2}$$

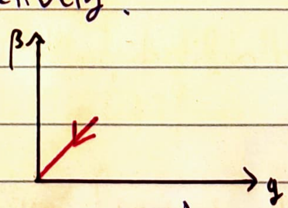
↳ Note that we have here $\beta_e > 0$, like for $\lambda \phi^4$ theory.

11.6 Renormalization group flow

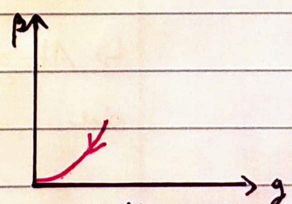
→ Let us explore the different behaviours of β, and what are the consequences for the running of the couplings.

→ We start by considering theories with a small coupling so we can treat them perturbatively.

→ If $\beta_g > 0$



$\beta \sim g$ at tree level



$\beta \sim g^n$ at 1-loop or higher

↳ this means $\bar{M} \frac{\partial}{\partial \bar{M}} g > 0$

↳ There will be a UV scale Λ at wich g ceases to be perturbatively small. It's the Landau pole.

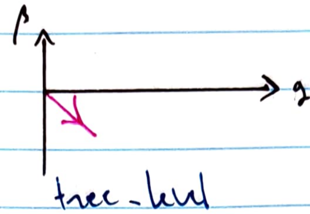
↳ At lower energies (IR), g decreases. If no threshold is met (massive particles), the theory is said to be IR free.

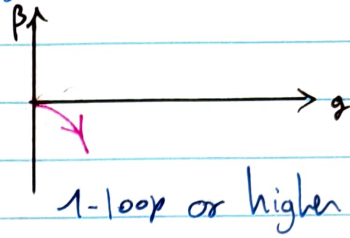
example : $\lambda\phi^4$. We had $\beta_\lambda \propto \lambda^2$

QED. We had $\beta_e \propto e^3$

$\lambda_n \phi^n$ with $[\lambda_n] < 0$, we had $\beta_{\tilde{\lambda}_n} \propto -[\lambda_n]\tilde{\lambda}_n$ even for a non renormalizable theory.

→ If $\beta_g < 0$





↳ This means $\bar{M} \frac{\partial g}{\partial \bar{M}} < 0$

↳ g decreases when $\bar{M}$ grows !

DEF | The asymptotic or UV freedom is the property of a decreasing coupling as energy increases.
The scale at wich the theory stops being perturbative is called the dynamical scale and is denoted by Λ

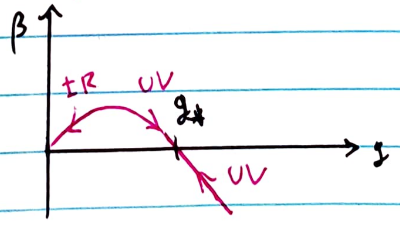
⊙ Fixed points:

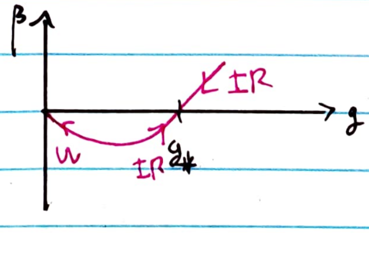
→ When $\beta > 0$ (or $\beta < 0$), the evolution of g converges in the IR (resp. in the UV) to zero coupling.

DEF | A fixed point of the RG flow is a point at wich $\beta = 0 = \bar{M} \frac{\partial g}{\partial \bar{M}}$

↳ At $g=0$, there is a fixed point. But it might not be the only one.

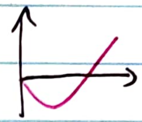
DEF | An interacting fixed point of the RG flow is a point at wich $\beta = 0$ and $g_* \neq 0$. We call g_* an attractor





Example: $\lambda\phi^4$ in $d = 4-2\epsilon$:

→ The contribution to β_λ at tree-level and 1-loop is $\beta_\lambda = -2\epsilon\lambda + \frac{18\lambda^2}{(4\pi)^2}$. We find $\beta_\lambda = 0$ at $\lambda = 0$ and at $\frac{\lambda^*}{(4\pi)^2} = \frac{\epsilon}{9}$ wich for small enough ϵ, is within the perturbatin regime.



DEF The Wilson-Fisher fixed point is $\lambda^* = \frac{16\pi^2}{9}\cdot\epsilon = \frac{16\pi^2}{9}(4-d)$

Qualitative picture of the evolution of couplings:

→ At a fixed point, $\beta = 0 \Rightarrow M\,\partial_M g = 0$: the coupling is no longer scale dependent. At the fixed point, the theory is scale invariant

→ Kenneth Wilson introduced an other approached of the evolution of the couplings: we consider the cutt-off to be physical. One defines the path integral as

$$Z = \int \mathcal{D}\phi\Big|_{|k|\leq\Lambda} e^{iS[\phi]}$$ when we explicitly cutoff to only modes with $|k| \leq \Lambda$

↳ To do this, we make a Wick rotation.

↳ Then we slice the modes on $\begin{cases} \phi_{low} & \text{with } |k| \leq \Lambda' < \Lambda \\ \phi_{high} & \text{with } \Lambda' < |k| \leq \Lambda \end{cases}$

DEF We can define an effectin field theory by

$$Z_\Lambda \equiv \int \mathcal{D}\phi_{low} \exp\{i S_{eff}[\phi]\}$$

$$\equiv \int \mathcal{D}\phi_{low}\, e^{iS[\phi_{low}]} \int \mathcal{D}\phi_{high}\, e^{i\tilde{S}[\phi_{high}, \phi_{low}]}$$

↳ the physics of the high energy modes is encoded in $S_{eff}[\phi_{low}]$: corrections to the couplings of $S[\phi]$ and additional non-1PI terms. One can do that by integrating out the high energy dof step by step. The evolution of coupling of S_{eff} is the same as the one given by the Callan-Symanzik equation and β_g.

→ When $\beta > 0$, then $E \searrow \Rightarrow g \searrow$: it is irrelevant in the IR

↳ It's the situation for a coupling that we call non-renormalizable.

→ When $\beta < 0$, then $E \searrow \Rightarrow g \nearrow$: it is relevant in the IR

→ When $\beta = 0$ at tree level, the coupling is marginal, and if $\begin{cases} \beta > 0 \\ \beta < 0 \end{cases}$ at loop-level, it is marginally $\begin{cases} \text{irrelevant} \\ \text{relevant} \end{cases}$, or truly marginal if $\beta = 0$ at loop-level too.

→ From Wilson's perspective, in the IR, all QFT's end up being renormalizable, since all non-renormalizable coupling sooner or later become irrelevant. This is the point of view of effective field theory

↳ Renormalizability and UV completeness still has a value, from the opposite point of view of the search for a theory of fundamental interactions.