10 RENORMALIZATION AND GAUGE SYMMETRY : QED

→ We consider renormalization in a theory wich possesses a non-trivial gauge symmetry. We take QED to investigate a few consequences of symmetries.

10.1 Counterterms and gauge symmetry

→ Consider the bare Lagrangian of QED:

$$\mathcal{L}_b^{QED} = -\frac{1}{4}F_b^{\,2} + \bar{\Psi}_b(i\not{D}_b - m_b)\Psi_b$$
$$= -\frac{1}{4}F_{b\,\mu\nu}F_b^{\mu\nu} + i\bar{\Psi}_b\not{\partial}\Psi_b + e_b A_{b\mu}\bar{\Psi}_b\gamma^\mu\Psi_b - m_b\bar{\Psi}_b\Psi_b$$

→ Field renormalization on A_μ and Ψ:

$A_{b\mu} \equiv \sqrt{Z_A}\,A_\mu$ and $\Psi_b \equiv \sqrt{Z_\Psi}\,\Psi$. The $\mathcal{L}$ turns into:

$$\mathcal{L}_b = -\frac{1}{4}Z_A F_{\mu\nu}F^{\mu\nu} + iZ_\Psi\bar{\Psi}\not{\partial}\Psi + e_b Z_\Psi\sqrt{Z_A}\,A_\mu\bar{\Psi}\gamma^\mu\Psi - m_b Z_\Psi\bar{\Psi}\Psi$$

→ Splitting bare quantities into renormalized ones and counterterms:

$Z_A \equiv 1+\delta_A$, $Z_\Psi \equiv 1+\delta_\Psi$, $e_b Z_\Psi\sqrt{Z_A} = e(1+\delta_e) = eZ_e$

and $m_b Z_\Psi = m + \delta_m$. We get:

$$\mathcal{L}_b = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\not{\partial}\Psi + eA_\mu\bar{\Psi}\gamma^\mu\Psi - m\bar{\Psi}\Psi}_{\mathcal{L}_r}$$
$$\underbrace{-\frac{1}{4}\delta_A F_{\mu\nu}F^{\mu\nu} + i\delta_\Psi\bar{\Psi}\not{\partial}\Psi + e\delta_e A_\mu\bar{\Psi}\gamma^\mu\Psi - \delta_m\bar{\Psi}\Psi}_{\mathcal{L}_{ct}}$$

⊙ <u>Constrain from gauge symmetry:</u>

→ Under the gauge symmetry $\Psi \mapsto e^{ie\alpha}\Psi$ and $A \mapsto A + \partial\alpha$, we need

$$\mathcal{L} \ni i\delta_\Psi\bar{\Psi}\not{\partial}\Psi + e\delta_e A_\mu\bar{\Psi}\gamma^\mu\Psi$$
$$\mapsto i\delta_\Psi\bar{\Psi}\not{\partial}\Psi + i^2 e\,\not{\partial}\alpha.\bar{\Psi}\Psi.e^{ie\alpha} + e\delta_\Psi\partial_\mu\alpha.\bar{\Psi}\gamma^\mu\Psi$$
$$= 0 \iff \delta_\Psi \overset{!}{=} \delta_e$$

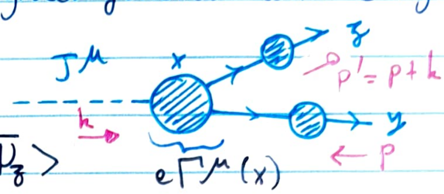
→ It actually follows from the Ward identities. Since it's already incoded in the QFT, we don't need to enforce it.

10.2 Counterterms and Ward identities

→ Recall for $U(1)_g$ rotation of $\psi, \bar\psi$, the Ward id. are:
$\langle \partial_\mu J^\mu(x)\, O_1(x_1)\cdots O_n(x_n)\rangle = i\delta(x-x_1)\langle \Delta O_1\cdots O_n\rangle + \dots + \langle O_1 \cdots \Delta O_n\rangle \cdot i\delta(x-x_n)$

→ Consider $J^\mu = e\bar\psi\gamma^\mu\psi$ at classical level. Under
$\delta\psi = ie\alpha\psi = \alpha\Delta\psi \Rightarrow \Delta\psi = ie\psi$ and $\Delta\bar\psi = -ie\bar\psi$, one has:
$\langle\partial_\mu J^\mu(x)\,\psi(y)\bar\psi(z)\rangle = -e\delta(x-y)\langle\psi_x\bar\psi_z\rangle + e\delta(x-z)\langle\psi_y\bar\psi_x\rangle$

→ We can see $\langle J^\mu_x \psi_y\bar\psi_z\rangle$ as a Green function with 2 legs and a vertex:
$\langle J^\mu_x\psi_y\bar\psi_z\rangle = e\langle\bar\psi_x\gamma^\mu\psi_x\psi_y\bar\psi_z\rangle$
$= \langle\psi_y\bar\psi_x\rangle \underbrace{e\Gamma^\mu_x}_{1PI}\langle\psi_x\bar\psi_z\rangle$



↳ From Ward id., we can write:
$\frac{\partial}{\partial x^\mu}\langle\psi_y\bar\psi_x\rangle\Gamma^\mu_x\langle\psi_x\bar\psi_z\rangle = -\delta(x-y)\langle\psi_x\bar\psi_z\rangle + \delta(x-z)\langle\psi_y\bar\psi_x\rangle$

Going in the momentum space, we have
$\langle\psi_y\bar\psi_x\rangle \to iD_F(p)$ $\quad\langle\psi_x\bar\psi_z\rangle \to iD_F(p') = iD_F(p+k)$ $\quad\Gamma^\mu_x \to \Gamma^\mu(k)$
$\Rightarrow -ik_\mu\, iD_F(p)\Gamma^\mu(k)\, iD_F(p') = -iD_F(p') + iD_F(p)$
$\Leftrightarrow k_\mu\Gamma^\mu(k) = D_F(p')^{-1} - D_F(p)^{-1}$

→ From $\mathcal{L}_{ct} \ni i\delta_\psi\bar\psi\not\partial\psi + e\delta_e A_\mu\bar\psi\gamma^\mu\psi$ we see that we have $i\delta_\psi\not p$ for $\langle\psi\bar\psi\rangle$ and $i\delta_e\gamma^\mu$ for 

→ Consider the 1PI 2-pt function of a fermion:
$$iD_F(\not p) = \frac{i}{\not p - m + \Sigma(p)} = \frac{i}{\not p - m}\left(1 - \frac{\Sigma}{\not p - m} + \dots\right) = \frac{i}{\not p - m} + \frac{i}{\not p - m}\overbrace{(i\Sigma)}^{1PI}\frac{i}{\not p - m} + \dots$$

$\Leftrightarrow D_F(\not p)^{-1} = \not p - m + \Sigma(p)$. Since for the 1-loop, we have:
$i\Sigma = i\Sigma_{finite} + i\delta_\psi\not p + \dots$, the divergent piece of Σ wich is linear in $\not p$ is indeed $\delta_\psi\cdot\not p$

↳ As for Γ^μ, we have $ie\Gamma^\mu = ie\Gamma^\mu_{finite} + ie\delta_e\gamma^\mu$ so eventually,
$\not k\,\delta_e = \not p'\delta_\psi - \not p\,\delta_\psi \Leftrightarrow \delta_e = \delta_\psi$ as expected.

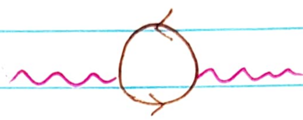
⊙ Consequences of $\delta_\psi = \delta_e$:

→ $\delta_\psi = \delta_e$ implies that $Z_\psi = Z_e$, so we have the following relation:

$$e = \frac{Z_\psi \sqrt{Z_A}\, e_0}{Z_e} = \sqrt{Z_A}\, e_0$$

↳ The renormalization of the coupling is completely encoded in the renormalization of the vector field itself.

10.3 One-Loop Structure of QED

→ We are going to compute  at $\mathcal{O}(e^2)$

DEF 1 We denote by $i\Pi_{\mu\nu}(p^2)$ the 1PI diagram 

↳ It should have the same structure as the kinetic term:

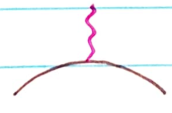
$$\Pi_{\mu\nu}(p) = (p^2 \eta_{\mu\nu} - p_\mu p_\nu)\, \Pi(p^2)$$

In this way, we're in a transverse gauge: $p^\mu \Pi_{\mu\nu}(p) = 0$.
Again, it's a consequence of the Ward id.

p332 Peskin ⊙ **Feynman rules for QED in RPT:**

$$= \frac{-i\, g_{\mu\nu} - p_\mu p_\nu / p^2}{p^2} \quad \text{(Landau gauge)}$$

$$= \frac{i}{\not p - m + i\epsilon}$$



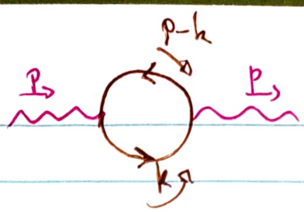
$$= -ie\gamma^\mu$$

$$= -i(g^{\mu\nu} p^2 - p^\mu p^\nu)\, \delta_A$$

$$= i(\not p\, \delta_\psi - \delta_m)$$

$$= -ie\gamma^\mu \delta_e$$

⊙ Computing NLO $\langle A_\mu A_\nu \rangle$:



→ From the path integral, we get:

$$\langle A_\mu A_\nu \rangle_{1\text{-loop}} = (ie)^2 \langle A_{\mu_1} A_{\nu_1} \int A_{\rho x} \bar{\Psi}_x \gamma^\rho \Psi_x \int A_{\sigma y} \bar{\Psi}_y \gamma^\sigma \Psi_y \rangle$$

$$= -(ie)^2 \int_x \int_y \langle A_{\mu_1} A_{\rho x} \rangle \gamma^\rho \langle \Psi_x \bar{\Psi}_y \rangle \gamma^\sigma \langle \Psi_y \bar{\Psi}_x \rangle \langle A_{\sigma y} A_{\nu_2} \rangle$$

$$\hookrightarrow i\Pi^{\mu\nu} = -(ie)^2 \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left\{ \gamma^\mu \frac{i}{\not{p} - \not{k} - m} \gamma^\nu \frac{i}{-\not{k} - m} \right\}$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\operatorname{tr}\{\gamma^\mu (\not{p} - \not{k} + m) \gamma^\nu (-\not{k} + m)\}}{[(p-k)^2 - m^2](k^2 - m^2)}$$

⊙ Transversality of $\Pi_{\mu\nu}$:

→ Using Ward identities, we have $\langle \partial_\mu J^\mu(x) J^\nu(y) \rangle = 0$ with no contact term since $\delta J^\mu = 0$. Now,

$$\langle J_x^\mu J_y^\nu \rangle = e^2 \langle \bar{\Psi}_x \gamma^\mu \Psi_x \bar{\Psi}_y \gamma^\nu \Psi_y \rangle$$

$$= -e^2 \operatorname{tr}\{\gamma^\mu \langle \Psi_x \bar{\Psi}_y \rangle \gamma^\nu \langle \Psi_y \bar{\Psi}_x \rangle \equiv i\Pi^{\mu\nu}_{1\text{-loop}}(x-y)$$

In Fourier space, we have:

$$p_\mu \Pi^{\mu\nu}(p) = 0$$

wich show that $\Pi^{\mu\nu}$ must be transverse.

p96

⊙ Back to $\langle A_\mu A_\nu \rangle$:

→ As usual, we use Feynman parametrization and make a shift $k \mapsto k + px$ to get:

$$i\Pi^{\mu\nu} = -e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\operatorname{tr}\{\gamma^\mu [\not{k} - \not{p}(1-x) - m] \gamma^\nu (\not{k} + \not{p}x - m)\}}{[k^2 + p^2 x(1-x) - m^2]^2}$$

$$= -4e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - 2p^\mu p^\nu x(x-1) - [k^2 - p^2 x(1-x) - m^2] g^{\mu\nu}}{[k^2 + px(1-x) - m^2]^2}$$

→ We notice that this integral, by Lorentz invariance:

$\displaystyle\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 - \Delta)^2} \stackrel{!}{=} g^{\mu\nu} I$ has to be $\propto$ to $\eta^{\mu\nu}$.

We find that $I = \frac{1}{4}\int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2-\Delta)^2}$

→ Computing a Wick rotation: $k^0 \mapsto ik_E^0$, we get:

$$i\Pi^{\mu\nu} = -4ie^2 \int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4} \frac{\frac{1}{2}k_E^2 \eta^{\mu\nu} + [p^2x(1-x)+m^2]\eta^{\mu\nu} - 2p^\mu p^\nu x(1-x)}{[k_E^2 + m^2 - px(1-x)]^2}$$

→ We introduce a momentum-cut-off, and get 2 problems:

① The quadratically divergent piece is only $\propto \eta_{\mu\nu}$
$\Rightarrow$ not transverse! It violates Ward id.

② It would evaluate to
$i\Pi^{\mu\nu} = -\frac{2ie^2}{(4\pi)^2}\Lambda^2\eta^{\mu\nu} + \dots$ wich gives $\Pi(p^2) = \frac{-e^2\Lambda^2}{(4\pi)^2p^2} + \dots$

signaling a large tachyonic mass for the photon!

⊙ Dimensional regulation:

Recall that $\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu\ell^\nu}{(\ell^2+\Delta)^2} = \frac{\eta^{\mu\nu}}{d}\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^2}$

$$\rightarrow i\Pi^{\mu\nu} = -4ie^2\int_0^1 dx \int \frac{d^dk_E}{(2\pi)^d} \frac{(1-\frac{d}{2})k_E^2\eta^{\mu\nu} + [p^2x(1-x)+m^2]\eta^{\mu\nu} - 2p^\mu p^\nu x(1-x)}{[k_E^2+m^2-p^2x(1-x)]^2}$$

We can use previous results:

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2} = \frac{\Delta^{\frac{d}{2}-2}}{(4\pi)^{d/2}}\Gamma(2-d/2) \text{ and}$$

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^2} = \frac{\Delta^{\frac{d}{2}-1}}{(4\pi)^{d/2}}\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right) = \frac{\Delta^{\frac{d}{2}-2}}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\Delta\cdot\frac{-1}{1-2/d}$$

So that

$$i\Pi^{\mu\nu} = -4ie^2\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\int_0^1 dx\, \Delta^{\frac{d}{2}-2}\left\{-(m^2-p^2x(1-x))\eta^{\mu\nu} + (m^2+p^2x(1-x))\eta^{\mu\nu} - 2p^\mu p^\nu x(1-x)\right\}$$

$$= \frac{-8ie^2}{(4\pi)^2}(p^2\eta^{\mu\nu} - p^\mu p^\nu)\int_0^1 dx\, \Gamma\left(2-\frac{d}{2}\right)\left(\frac{\Delta}{4\pi\mu^2}\right)^{\frac{d}{2}-2} x(1-x)$$

Taking $d = 4-2\epsilon$, We find

$$\Pi(p^2) = -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\frac{\Delta}{\mu^2}\right]$$

⊙ Normalization condition:

→ The renormalization condition is $\Pi(p^2=0) \stackrel{!}{=} 0$. The full correction reads:

$$\Pi(p^2) = -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(x-1)\log\left[\frac{m^2-p^2x(1-x)}{m^2}\right]$$

→ Consider the limit $-p^2 \equiv Q^2 \gg m^2$ (large spacelike momentum or short spacetime separation). Then, $\log \frac{m^2+Q^2x(1-x)}{m^2} \sim \log \frac{Q^2}{m^2}$ and since $\int_0^1 dx\, x(1-x) = 1/6$, we have:

$$\Pi(Q^2) = \frac{4}{3}\frac{e^2}{(4\pi)^2}\log\left[Q^2/m^2\right]$$

↳ That means: $Z_A = \frac{1}{1-\Pi} = \frac{-1}{1-\frac{4}{3}\frac{e^2}{(4\pi)^2}\log(Q^2/m^2)}$

Now, recall that $e = \sqrt{Z_A}\, e_L$ and we set $Z_A(Q^2=m^2) = 1$. We get

$$\frac{e^2}{(4\pi)^2} \stackrel{\circledast}{=} \frac{e_0^2}{(4\pi)^2}\cdot\frac{1}{1-\frac{4}{3}\frac{e_0^2}{(4\pi)^2}\log\left[Q^2/m^2\right]}$$

where e_0 is the value of the coupling when $Q^2=m^2$

Prop | We found that the coupling e depends on the energy scale. When $Q^2\downarrow$, then $e^2\downarrow$ and when $Q^2\nearrow$, then $e^2\nearrow$

→ $\circledast$ is a good approximation as long as $\frac{e^2}{(4\pi)^2}\log\frac{Q^2}{m^2} \ll 1$

But we can compute the variation of e when we vary Q.
Let's consider by simplicity $\frac{(4\pi)^2}{e^2} = \frac{(4\pi)^2}{e_0^2} - \frac{4}{3}\log\left(\frac{Q}{m}\right)^2$

We get: $Q^2\frac{d}{dQ^2}\left(\frac{(4\pi)^2}{e^2}\right) = -4/3$

⇒ The coupling increases with energy! It reaches infinity when
$\log\frac{Q^2}{m^2} = \frac{3}{4}\frac{(4\pi)^2}{e_0} = \frac{3}{4}\cdot\frac{4\pi}{\alpha} = 3\pi\cdot 137 \sim 1000 \Rightarrow Q \sim e^{500}\, m_e$

DEF | The scale at wich a coupling wich grows in the UV eventually diverges is called Landau pole.