QUANTUM FIELD THEORY II

Path integrals and Renormalization

→ Goal: introduce the path integral approach to QFT and discuss symmetries, radiative correction and renormalization, and eventually the coupling constant evolution.

1 PATH INTEGRAL FORMULATION OF QUANTUM MECANICS

1.1 Recall of QM

→ Consider a system with a coordinate q, a conjugate momentum p and a Hamiltonian $H(p,q)$.

Usually, we'll consider $H(p,q) = \frac{1}{2m} p^2 + V(q)$

DEF The evolution of the wave fct is given by the Schrödinger equation

$$i\hbar \partial_t |q(t)\rangle = \hat{H} |q(t)\rangle$$

where we take the Schrodinger picture (wave function evolves in time), and $\hat{H}$, $\hat{P}$ and $\hat{Q}$ are operators.

↳ We introduce the evolution operator U such that $|q(t)\rangle = U(t,t_0)\, |q(t_0)\rangle$ and it satisfies the S̈ equation.

→ If H doesn't depend explicitly on time, we have:

$$U(t,t_0) = \exp\left\{ \frac{-i}{\hbar} H (t-t_0) \right\}$$

1.2 Operators and representations

→ The operators $\hat{P}$ and $\hat{Q}$ are hermitians ($\hat{P}^+ = \hat{P}$ and $\hat{Q}^+ = \hat{Q}$) and satisfies $[\hat{Q}, \hat{P}] = i$

So, if $c = 1$, we have $[P] = M$ and $[Q] = M^{-1}$

→ We cannot diagonalize $\hat{P}$ and $\hat{Q}$ simultaneously. We define 2 complete sets of eigenstates:

→ $\hat{Q}|q\rangle = q|q\rangle$ with $\int dq\, |q\rangle\langle q| = \mathbb{1}$ and $\langle q'|q\rangle = \delta(q-q')$

→ $\hat{P}|p\rangle = p|p\rangle$ with $\int dp\, |p\rangle\langle p| = \mathbb{1}$ and $\langle p'|p\rangle = \delta(p-p')$

→ For a state $|\psi\rangle \in \mathcal{H}$, we have $\langle q|\psi\rangle = \psi(q)$

? On such wave function, $\langle q|\hat{P}|\psi\rangle = -i\,\partial_q \psi(q)$ ($\Leftarrow [\hat{Q},\hat{P}] = i$)

? Then, $\langle q|\hat{P}|p\rangle = -i\,\partial_q \langle q|p\rangle = p\langle q|p\rangle$

$$\Rightarrow \langle q|p\rangle = \alpha\, e^{ipq}$$

↳ Lets normalize to find α:

$$\langle q'|q\rangle = \delta(q'-q) = \int dp\, \langle q'|p\rangle\langle p|q\rangle = \int dp\, |\alpha|^2 e^{ipq'} e^{-ipq}$$

$$= |\alpha|^2 \int dp\, e^{ip(q'-q)} = |\alpha|^2\, 2\pi\, \delta(q'-q)$$

We find $\langle q|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipq}$

1.3 Amplitude

→ We evaluate the amplitude of going from q at t to q' at t':

$$\langle q'|\, U(t',t)\, |q\rangle$$

We consider an infinitesimal time step: $t' \equiv t + \delta t$. Then:

$$\langle q'|U(t',t)|q\rangle = \langle q'|e^{-i\hat{H}dt}|q\rangle \simeq \langle q'|(\mathbb{1} - i\hat{H}\,dt)|q\rangle$$

Generically, there is ordering ambiguity $\Rightarrow$ put all the $\hat{P}$'s on the right

$$= \int dp\, \langle p'|\mathbb{1} - i\hat{H}\,dt|p\rangle\langle p|q\rangle$$

$$= \int dp\, (1 - i H(p,q')\,dt)\,\langle q'|p\rangle\langle p|q\rangle$$

$$= \int dp\, (1 - i H(p,q')\,dt)\, \frac{1}{\sqrt{2\pi}} e^{iq'p} \frac{1}{\sqrt{2\pi}} e^{-ipq}$$

$$\simeq \int \frac{dp}{\sqrt{2\pi}}\, e^{ip(q'-q)}\, e^{-iH(p,q')\,dt}$$

→ Let $\Delta t \equiv t' - t$ be finite. Then we devide it in infinitesimal pieces $\Delta t = N\,dt$. We can write:

$$\langle q'|U(t',t)|q\rangle = \lim_{N\to\infty} \langle q'|U(t',t_N)\,U(t_N,t_{N-1})\cdots U(t_1,t)|q\rangle$$

$$= \lim_{N\to\infty} \int dq_N \cdots dq_1 \Big\{ \langle q'|U(t',t_N)|q_N\rangle\langle q_N|U(t_N,t_{N-1})|q_{N-1}\rangle \cdots$$

$$\cdots \underbrace{\langle q_2|U(t_2,t_1)|q_1\rangle}_{= \int \frac{dp_2}{2\pi} e^{ip_2(q_2-q_1)}\, e^{-iH(p_2,q_2)dt}} \langle q_1|U(t_1,t)|q\rangle \Big\}$$

$$= \int \prod_{k=1}^{N} dq_k \prod_{\ell=1}^{N+1} \frac{dp_\ell}{2\pi} \Big[\exp\{ i p_{N+1}(q'-q_N) - iH(p_{N+1},q')dt \}$$

$$\times \exp\Big\{ \sum_{\ell=1}^{N} \big(ip_\ell(q_\ell - q_{\ell-1}) - iH(p_\ell,q_\ell)\,dt \big) \Big\} \Big]$$

→ We integrate over all paths from q to q'.

→ Let's take the continuum limit: $q_k \equiv q(t_k)$, $p_k \equiv p(t_k)$ and $t \equiv t_0$ and $t_{N+1} = t'$. We have $q_k - q_{k-1} = \dot{q}_k \cdot dt$

$$\langle q'|U(t',t)|q\rangle = \int \prod_{k=1}^{N} dq(t_k) \prod_{k=1}^{N+1} \frac{dp(t_k)}{2\pi} \exp\Big\{ i \sum_{\ell=0}^{N+1} \Big[p(t_\ell)\big(q(t_\ell) - q(t_{\ell-1})\big)$$

$$- H(p(t_\ell), q(t_\ell))\,dt \Big] \Big\}$$

$$= \int \prod_{k=1}^{N} dq(t_k) \prod_{k=1}^{N+1} \frac{dp(t_k)}{2\pi} \exp\Big\{ i \sum_{\ell=1}^{N+1} dt \Big[p(t_k)\dot{q}(t_k) - H(p(t_k), q(t_k)) \Big] \Big\}$$

$$= \int_{\substack{q(t)=q \\ q(t')=q'}} \mathcal{D}q(\tau)\, \mathcal{D}\frac{p(\tau)}{2\pi} \exp\Big\{ i \int_t^{t'} d\tau \Big[p(\tau)\dot{q}(\tau) - H(p(\tau), q(\tau)) \Big] \Big\}$$

DEF We definine the Hamilton action S_H as

$$S_H = \int dt\, \{ p\dot{q} - H(p,q) \}$$ We then have:

$$\langle q'|U(t',t)|q\rangle = \int \mathcal{D}q\, \mathcal{D}\frac{p}{2\pi}\, e^{iS_H(p,q)}$$

1.4 Canonical Hamiltonian and gaussian integrals

→ We consider a canonical Hamiltonian : $H = \frac{p^2}{2m} + V(q)$

The integrals over dp are essentially gaussian integrals:

$$\int \frac{dp}{2\pi} \exp\{ i p\dot{q}\, dt - \frac{i}{2m} p^2 dt - i V dt \}$$

$$= \int \frac{dp}{2\pi} \exp\{ \frac{-i\,dt}{2m} (p - m\dot{q})^2 + \frac{i}{2} m\, dt\, \dot{q}^2 - i V dt \}$$

$$= \exp\{ i (\tfrac{1}{2} m \dot{q}^2 - V(q))\, dt \} \int \frac{d\tilde{p}}{2\pi} \exp\{ \frac{-i\,dt}{2m} \tilde{p}^2 \} \qquad \int e^{-\alpha u^2} = \sqrt{\pi/\alpha}$$

$$= \exp\{ i (\tfrac{1}{2} m \dot{q}^2 - V(q))\, dt \} \cdot \frac{1}{2\pi} \cdot \sqrt{\frac{2\pi m}{i\,dt}}$$

$$= \sqrt{\frac{m}{2\pi i\, dt}} \exp\{ i L(q,\dot{q})\, dt \} = \langle q + \delta q | U(t+\delta t, t) | q \rangle$$

The normalization factor is not relevant, we call it N

→ For the finite amplitude, we get:

$$\langle q' | U(t', t) | q \rangle = N \int \mathcal{D}q(\tau) \exp\{ i \int_t^{t'} d\tau \left(\tfrac{1}{2} m \dot{q}(\tau)^2 - V(q(\tau)) \right) \}$$

DEF We define the Lagrangian action S_L as

$$S_L \equiv \int d\tau\, L(q,\dot{q}) = \int d\tau \left(\tfrac{1}{2} m \dot{q}^2 - V(q) \right)$$

→ We can write: $\langle q' | U(t', t) | q \rangle = N \int \mathcal{D}q(\tau)\, e^{i S_L}$

Indeed:

$$\int \mathcal{D}q\, \mathcal{D}p\, \exp\{ i \int d\tau \left(p\dot{q} - \tfrac{1}{2m} p^2 - V(q) \right) \}$$

$$= \int \mathcal{D}q\, \mathcal{D}p\, \exp\{ \frac{-i}{2m} \int d\tau \left[(p - m\dot{q})^2 + i \left(\tfrac{1}{2} m\dot{q}^2 - V(q) \right) \right] \}$$

$$= \int \mathcal{D}q\, \exp\{ i \int d\tau \left(\tfrac{1}{2} m\dot{q}^2 - V(q) \right) \} \cdot \int \mathcal{D}\tilde{p}\, \exp\{ \frac{-i}{2m} \tilde{p}^2 \}$$

$$= \int \mathcal{D}q\, \exp\{ i S_L \} \cdot N$$

1.5 Operators under the path integral

→ Let's consider $\int_t^{t'} \mathcal{D}q \, e^{iS} \, O_1(t_1) \, O_2(t_2)$

We fix $t < t_1 < t_2 < t'$. We can break the integral:

$$\int \mathcal{D}q \, e^{iS} O_1(t_1) O_2(t_2) = \int \prod_{k=1}^{N} dq(t_k) \exp\left\{ i \sum_{\ell=k_2}^{N} \frac{1}{2} m \dot{q}_\ell^2 - V(q_\ell) \right\}$$

$$\times O_2(t_2) . \exp\left\{ i \sum_{\ell=k_1}^{k_2-1} \left(\frac{1}{2} m \dot{q}_\ell^2 - V(q_\ell) \right) \right\} . O_1(t_1) \exp\left\{ i \sum_{\ell=1}^{k_1} \left(\frac{1}{2} m \dot{q}_\ell^2 - V(q_\ell) \right) \right\}$$

$$\sim \langle q' | U(t', t_2) O_2(t_2) U(t_2, t_1) O_1(t_1) U(t_1, t) | q \rangle$$

↳ The path integral gives always the time-ordered correlation function of the several operators. We write, symbolically:

$$\langle q' | T[O_1(t_1) O_2(t_2) \dots] | q \rangle = N \int \mathcal{D}q \, (O_1(t_1) O_2(t_2) \dots) \, e^{iS(q)}$$

1.6 Towards field theory

→ For a large number of d.o.f., the path integral becomes:

$$\int \mathcal{D}q^i(t) \exp\left\{ i \int dz \sum_i \left(\frac{1}{2} m \dot{q}_i^2 - V(q_i) \right) \right\}$$

→ The transition from QM to QFT is made taking a continuous index $i \to x$ a dynamical variable at every space point, $q \to \varphi$ relabel, and implement Poincaré invariance. We get to

$$S[\varphi] = \int d^4x \, \mathcal{L}[\varphi] \quad \text{with} \quad \mathcal{L}[\varphi] = \frac{1}{2} (\partial_\mu \varphi)^2 - V(\varphi)$$

$$\hookrightarrow S[\varphi] = \int d^4x \left\{ \frac{1}{2} \dot{\phi}^2 \underbrace{- \frac{1}{2} (\partial_i \varphi)^2 - V(\varphi)}_{\text{"}-V(q)\text{"}} \right\}$$

This expression is Poincaré invariant, except for the boundary conditions.