→ The Hamiltonian, momentum and spin operator are:

$$\hat{H} = \int d^3k \; \omega_k \; \hat{a}^{i+}(k) \; \hat{a}^i(k)$$
$$\hat{\vec{P}} = \int d^3k \; \vec{k} \; \hat{a}^{i+}(k) \; \hat{a}^i(k)$$

→ Acting on a particle at rest by the angular momentum squared, we get

$$\hat{M}^2 \hat{a}^{i+}(0)|0\rangle = 2\hat{a}^{i+}(0)|0\rangle = s(s+1)\hat{a}^{i+}(0)|0\rangle \text{ for } s=1$$

6 FEYNMANN RULES FOR FERMIONS AND VECTORS

6.1 For fermions

① The expression of the S-matrix as a T-exponent remains valid because the interaction action S_I is bilinear in fermionic fields.

② Lorentz-invariance requires that we change the definition of the T-product.

DEF

$$T[\psi(x)\bar{\psi}(y)] \equiv \begin{cases} \psi(x)\bar{\psi}(y) & \text{at } x^0 > y^0 \\ -\bar{\psi}(y)\psi(x) & \text{at } y^0 > x^0 \end{cases}$$

③ the definition of the normal order also changes:

DEF

$$:a_k a_p a_q^+: \equiv (-1)^2 a_q^+ a_k a_p = (-1)^3 a_q^+ a_p a_k$$

every time we interchange the operators, there appears the sign "−"

4 With these new definitions, we have:

$$T[\psi(x)\bar{\psi}(y)] = :\psi(x)\bar{\psi}(y): + \overbrace{\psi(x)\bar{\psi}(y)}$$

where

DEF:

$$\overbrace{\psi(x)\bar{\psi}(y)} \equiv \begin{cases} \{\psi^+(x), \bar{\psi}^-(y)\} & \text{at } x^0 > y^0 \\ -\{\bar{\psi}^+(y), \psi^-(x)\} & \text{at } y^0 > x^0 \end{cases}$$

→ The propagator is given by:

$$\psi_\alpha(x)\,\bar{\psi}_\beta(y) \equiv S_F(x-y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\gamma^\mu p_\mu + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

→ The Wick theorem becomes:

$$T[\psi_1 \bar{\psi}_2 \psi_3 \ldots] = :\psi_1 \bar{\psi}_2 \psi_3 \ldots: + \text{(all possible contractions)}$$

⊙ Case when we have fermions in $|i\rangle$ and $|f\rangle$:

→ We can have 2 types of fermionic states:

$|a_k^s\rangle \equiv (2\pi)^{3/2}\sqrt{2\omega_k}\; a^{+s}(k)|0\rangle$ a fermion and

$|b_p{}^s\rangle \equiv (2\pi)^{3/2}\sqrt{2\omega_k}\; b^{+s}(k)|0\rangle$ an anti-fermion.

→ Because we have: $\psi_\alpha(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ u_\alpha^s(k)\, e^{-ikx}\, \hat{a}_\alpha^s(k) + v_\alpha{}^s(k)\, e^{ikx}\, \hat{b}_\alpha^{+s}(k) \right\}$

and $\bar{\psi}_\beta(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \left\{ \bar{v}_\beta^s(\bar{p})\, e^{-ipx}\, \hat{b}_\beta^s(\bar{p}) + \bar{u}_\beta^s(\bar{p})\, e^{ipx}\, \hat{a}_\beta^{s+}(\bar{p}) \right\}$,

the operator $\hat{a}^+$ does not anticommute with ψ, while b^+ doesn't with $\bar{\psi}$

→ Let's calculate:

$$\psi(x)\,|a_p^s\rangle = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\, u^i(k)\, e^{-ikx}\, \underbrace{\hat{a}^i(k),\ (2\pi)^{3/2}\sqrt{2\omega_p}\; a^{+s}(\bar{p})}_{\delta^{is}\,\delta^3(\bar{p}-\bar{k})}|0\rangle$$

$$= \int d^3k \sqrt{\omega_k/\omega_p}\; u^i(k)\, e^{-ikx} \left\{ a^i(k), \hat{a}^{+s}(\bar{p}) \right\} |0\rangle$$

$$= u^s(p)\, e^{-ipx}\,|0\rangle$$

→ We find the first rules:

$$\psi(x)\,|\hat{a}^s(p)\rangle \longrightarrow u^s(\bar{p})\, e^{-ipx}$$

$$\bar{\psi}(x)\,|b^s(p)\rangle \longrightarrow \bar{v}^s(\bar{p})\, e^{-ipx}$$

$$\langle a^s(p)|\,\bar{\psi}(x) \longrightarrow \bar{u}^s(p)\, e^{ipx}$$

$$\langle b^s(p)|\,\psi(x) \longrightarrow v^s(\bar{p})\, e^{ipx}$$

Particular case: Yukawa theory:

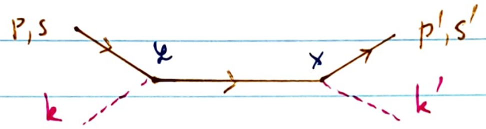
→ The Yukawa theory consists of a scalar, a fermion and their interaction $S_I = \lambda \int d^4x\, \varphi_x \bar{\Psi}_x \Psi_x$

→ We consider the fermion scalar scattering $\Psi \phi \to \Psi \phi$
The matrix element in the lowest order is given by:
$$\int d^4x\, d^4y\, \langle a^{s'}(p'), k' | \frac{1}{2!} (i\lambda)^2 T\left[\varphi_x \bar{\Psi}_{x_\alpha} \Psi_{x_\alpha} \varphi_y \bar{\Psi}_{y_\rho} \Psi_{y_\rho}\right] | a^s(p), k\rangle$$
→ There are 2 types of contractions:

$$\langle a_{p'}\, k' | T(\varphi_x \bar{\Psi}_x \Psi_x\, \varphi_y \bar{\Psi}_y \Psi_y | a_p\, k\rangle$$

$$= \int d^4x\, d^4y\, \frac{1}{2}(i\lambda)^2\, \bar{u}^{s'}_{\alpha}(p')\, e^{ip'x}\, e^{ik'x} \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q}+m)_{\alpha\rho}\, e^{-iq(x-y)}}{q^2-m^2+i\epsilon}\, e^{-iky}\, u^s_{\rho}(p)\, e^{-ipy}$$

$$= \frac{1}{2!}(i\lambda)^2\, \bar{u}^{s'}(p)\, \frac{i(\gamma(k+p)+m)}{(k+p)^2-m^2+i\epsilon}\, u^s(p)\,.\,(2\pi)^4\, \delta^4(k+p-k'-p')$$



→ Fermion lines have arrows wich distinguish fermions and antifermions in initial and final states.

→ There is a factor (-1) per each closed fermion loop.

→ Fermion lines are continuous.

6.2 Vectors

→ In the case of vectors, the definition of the T-product remains the same as for scalars:

DEF $T[A_\mu(x) A_\nu(y)] = :A_\mu A_\nu: + \overline{A_\mu(x) A_\nu(y)}$ where

$$\overline{A_\mu(x) A_\nu(y)} = \begin{cases} [A_\mu^+(x), A_\nu^-(y)] & \text{at } x^0 > y^0 \\ [A_\nu^+(y), A_\mu^-(x)] & \text{at } y^0 > x^0 \end{cases}$$

→ The propagator gives:

$$\overline{A_\mu(x) A_\nu(y)} = \int \frac{d^4p}{(2\pi)^4} \, i \sum_j \frac{e_\mu^j(p) e_\nu^j(p)}{p^2 - m^2 + i\epsilon} e^{-ip.(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(-g_{\mu\nu} + p_\mu p_\nu / m^2)}{p^2 - m^2 + i\epsilon} e^{-ip.(x-y)}$$

↳ Note that $p^\mu(-\eta_{\mu\nu} + p_\mu p_\nu / m^2) = -p_\nu + \frac{p^2}{m^2} p_\nu = 0$

The numerator is $\perp$ to p_μ.

→ Contraction with initial and final states produce factors:

$$\overline{A_\mu(x)|\ldots a^{i}}(p)\ldots\rangle \longrightarrow e^i_\mu(p)\, e^{-ipx}$$

$$\langle\ldots \overline{a^i(p)\ldots| A_\mu}(x) \longrightarrow e^{*i}_\mu(p)\, e^{ipx}$$

6.3 Decay of a vector into 2 fermions $A \mapsto \psi\bar\psi$

→ Let's consider the interaction $\mathcal{L}$ of the form:

$$\mathcal{L}_I = g A_\mu \bar\psi \gamma^\mu \psi$$

→ Diagrams:

→ Matrix element:

$$\langle a^{s_1}_{k_1} b^{s_2}_{k_2} | ig \int d^4x\, A^\mu(x)\, \bar\psi_\alpha(x)\, \gamma^\mu_{\alpha\beta}\, \psi_\beta(x) | p^i \rangle$$

$\hookrightarrow \mathcal{M} = -ig\, e^i_\mu(\bar p)\, \bar u^{s_1}_\alpha(k_1)\, \gamma^\mu_{\alpha\beta}\, v^{s_2}_\beta(k_2)$

→ If the boson is not polarized and the spin of final fermions are not measured, we have to

→ average over the vector boson polarization

→ sum over spins of fermions

$$\frac13 \sum_{i=1}^{3} \sum_{s_1,s_2}^{2} |\mathcal{M}|^2 = \overline{|\mathcal{M}|^2}$$

$$= \frac13 g^2 \sum_i \sum_{s_1 s_2} \underbrace{e^i_\mu(p)\, e^i_\nu(p)}_{=-\eta_{\mu\nu} + p_\mu p_\nu/m^2}\, \bar u^{s_1}_\alpha(k_1)\, \gamma^\mu_{\alpha\beta}\, \underbrace{v^{s_2}_\beta(k_2)\, \bar v^{s_2}_\rho(k_2)}_{(\gamma\cdot k_2 - m)_{\beta\rho}}\, \gamma^\nu_{\rho\sigma}\, u^{s_1}_\sigma(k_1)$$

$$= (\gamma^\mu(\gamma\cdot k_2 - m)\gamma^\nu)_{\alpha\sigma}$$

$$= \frac13 g^2 \left(-\eta_{\mu\nu} + p_\mu p_\nu/m^2\right)\left(\gamma^\mu(\gamma\cdot k_2 - m)\gamma^\nu\right)_{\alpha\sigma} (\gamma\cdot k_1 + m)_{\alpha\sigma}$$

$$= \frac13 g^2 \left(-\eta_{\mu\nu} + p_\mu p_\nu/m^2\right) \mathrm{Tr}\left[\gamma^\mu(k_2\gamma - m)\gamma^\nu(k_1\gamma - m)\right]$$

$$= k_{2\sigma} k_{1\rho}\, \underbrace{\mathrm{Tr}[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho]}_{=4(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\sigma\rho} + \eta^{\mu\rho}\eta^{\sigma\nu})} - m^2\, \mathrm{Tr}[\gamma^\mu\gamma^\nu]$$

$$= \frac13 g^2 \left(-\eta_{\mu\nu} + p_\mu p_\nu/m^2\right)\left(4\left(k_1^\mu k_2^\nu + k_2^\nu k_1^\mu - \eta^{\mu\nu}(k_1 k_2)\right) - 4m^2\eta^{\mu\nu}\right)$$

$$= g^2 \cdot \frac43 \left((k_1 k_2) + 2\frac{(k_1 p)(k_2 p)}{M^2} + 3m^2\right) \quad \text{where } p^\mu = k_1^\mu + k_2^\mu$$

$$\to M^2 = 2m^2 + 2(k_1\cdot k_2)$$

$\hookrightarrow (P\cdot k_1) = m^2 + \frac12 M^2 - m^2 = \frac12 M^2$

$$= \frac43 g^2\left(-\frac12 M^2 - m^2 + \frac12 M^2 + 3m^2\right) = \frac43 g^2 (M^2 + 2m^2)$$

$$\Rightarrow \Gamma = \frac{g^2 M}{12\pi}\left(1 + \frac{2m^2}{M^2}\right)\sqrt{1 - \frac{4m^2}{M^2}}$$