5 VECTOR FIELD

5.1 Construct a Lagrangian

→ Limits: quadratic in the vector field A_μ and not more than 2 derivatives. The the most general Lagrangian is:

$$S = \int d^4x \left\{ \frac{1}{2}\alpha\, \partial_\nu A_\nu \partial^\mu A^\nu + \frac{1}{2}\beta\, \partial_\mu A_\nu \partial^\nu A^\mu + \frac{1}{2} m^2 A_\mu A^\mu \right\}$$

↳ the corresponding EOM are:

$$-\alpha\, \partial_\nu \partial^\nu A_\mu - \beta\, \partial_\mu \partial_\nu A^\nu + m^2 A_\mu = 0$$

→ We need to choose α and β such that the Hamiltonian is positive-definite ($z^\dagger M z > 0\ \forall z$) for arbitrary functions A_0, A_i.

→ We can rewrite the Lagrangian as:

$$\mathcal{L} = \frac{1}{2}\alpha\left(\dot{A}_0^2 - (\partial_i A_0)^2 - \dot{A}_i^2 + (\partial_i A_j)^2\right)$$
$$+ \frac{1}{2}\beta\left(\dot{A}_0^2 - 2\dot{A}_i \cdot \partial_i A_0 + \partial_i A_j \partial_j A_i\right) + \frac{1}{2}m^2\left(A_0^2 - A_i^2\right)$$

→ The momentum $\pi_\mu = (\pi_0, \pi_i)$ is:

$$\pi_0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = \alpha \dot{A}_0 + \beta \dot{A}_0 \qquad \pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -\alpha \dot{A}_i - \beta \partial_i A_0$$

→ The Hamiltonian $H = p\dot{q} - \mathcal{L}$ becomes: $(p\dot{q} = \pi_0 \dot{A}_0 + \pi_i \dot{A}_i)$

$$H = \int d^3x\, (\alpha \dot{A}_0 + \beta \dot{A}_0)\dot{A}_0 + (-\alpha \dot{A}_i - \beta \partial_i A_0)\dot{A}_i$$
$$- \alpha/2\left(\dot{A}_0^2 - (\partial_i A_0)^2 - \dot{A}_i^2 + (\partial_i A_j)^2\right)$$
$$+ \beta/2\left(\dot{A}_0^2 - 2\dot{A}_i \cdot \partial_i A_0 + \partial_i A_j \partial_j A_i\right) + \frac{1}{2}m^2\left(A_0^2 - A_i^2\right)$$
$$= \frac{1}{2}\dot{A}_0^2(\alpha+\beta) - \frac{1}{2}\dot{A}_i^2\alpha + \frac{1}{2}(\partial_i A_0)^2\alpha - \frac{1}{2}(\partial_i A_j)^2\alpha$$
$$- \frac{1}{2}\partial_i A_j \partial_j A_i \cdot \beta + \frac{1}{2}m^2 A_i^2 - \frac{1}{2}m^2 A_0^2$$

For $A_0 \gg 1$ and $\partial A_0 \ll 1$, H becomes negative! ☹

→ There is a degenerate case: $\alpha + \beta = 0$.

→ In this case, $\pi_0 = 0$

→ From the EOM, we have:

$$-\alpha\left(\partial_0^2 - \partial_i^2\right)A_\mu + \underbrace{\alpha}_{-\beta}\, \partial_\mu\left(\partial_0 A_0 - \partial_i A_i\right) + m^2 A_\mu = 0$$

$\mu = 0 \Rightarrow \alpha \partial_i^2 A_0 - \alpha \partial_0 \partial_i A_i + m^2 A_0 = 0$

$\Leftrightarrow (\alpha \partial_i^2 + m^2) A_0 = \alpha \partial_0 \partial_i A_i$

We have 3 dynamical variables instead of 4.

$\mu = j \Rightarrow -\alpha(\partial_0^2 - \partial_i^2) A_j + \alpha \partial_j (\partial_0 A_0 - \partial_i A_i) + m^2 A_j = 0$

↳ From the EOM $(-\alpha \partial^2 A_\mu - \beta \partial_\mu (\partial A) + m^2 A_\mu)$, we have:

$\partial^\mu (\text{EOM}) = -\alpha \partial_\mu \partial^2 A_\mu - \beta \partial^2 \partial A + m^2 \partial A = 0$

$\Leftrightarrow -(\alpha + \beta) \partial^2 (\partial A) + m^2 \partial A = 0 \Rightarrow \partial_\mu A^\mu = 0$

↳ becomes: $(-\alpha \partial^2 + m^2) A_j = 0$

? ↳ We have to set $\alpha = -1$ and $\beta = 1$

→ The Hamiltonian becomes:

$$H = \int d^3x \left\{ \frac{1}{2} \dot{A}_i^2 - \frac{1}{2} (\partial_i A_0)^2 + \frac{1}{2} (\partial_i A_j) - \frac{1}{2} \partial_i A_j \partial_j A_i + \frac{1}{2} m^2 (A_i^2 - A_0^2) \right\}$$

$F_{0i}^2 = (\partial_0 A_i - \partial_i A_0)^2 = \dot{A}_i^2 + (\partial_i A_0)^2 - 2 \partial_0 A_i \cdot \partial_i A_0$

$$= \int d^3x \left\{ \frac{1}{2} F_{0i}^2 + \frac{1}{4} F_{ij}^2 + \frac{1}{2} m^2 A_i^2 + A_0 \underbrace{(\partial_i^2 A_0 - \partial_0 \partial_i A_i)}_{= m^2 A_0^2 \text{ from } \circledast} - \frac{1}{2} m^2 A_0^2 \right\}$$

$$= \int d^3x \left\{ \frac{1}{2} F_{0i}^2 + \frac{1}{4} F_{ij}^2 + \frac{1}{2} m^2 A_i + \frac{1}{2} m^2 A_0^2 \right\}$$ positive-definite!

→ The action is given by

$$S = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \right\}$$

with $F_{\mu\nu} \hat{=} \partial_\mu A_\nu - \partial_\nu A_\mu$

→ The EOM is the Proca equation: $\partial^\nu F_{\nu\mu} + m^2 A_\mu = 0$

? ↳ At $m = 0$, we have $\partial^\nu A_\nu = 0$

→ $\partial_\mu F^{\mu\nu} + m^2 A^\nu = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\nu = 0$

? $\Leftrightarrow \partial^2 A^\nu - \cancel{\partial_\mu \partial^\nu A^\mu} + m^2 A^\nu = 0$

$\Leftrightarrow (\partial^2 + m^2) A^\nu = 0$

5.2 Classical solutions

→ We saw that $(\partial^2 + m^2) A_\mu = (\partial_0^2 - \partial_i^2 + m^2) A_\mu = 0$

We may look for solution of the form:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ e_\mu(k)\, e^{-ikx} + \tilde{e}_\mu(k)\, e^{ikx} \right\}$$

With the condition $\partial A = 0 \Rightarrow k^\mu e_\mu(k) = 0$

↳ All solutions are labelled by vectors $e_\mu \perp k_\mu$.

DEF We introduce 3 vectors $e_\mu^i(k)$ according to:

$$e^\mu_{1,2}(k) \equiv (0, \vec{e}^{\,1,2}(k))$$

$$e_3^\mu(k) \equiv \left(\frac{|k|}{m}, \frac{\omega}{m} \frac{k}{|k|} \right)$$

with $\vec{e}_1 \cdot \vec{e}_2 = 0$ and $\bar{e}_{1,2} \cdot k = 0$

↳ Then, $k_\nu e_3^\nu = \frac{\omega_k . |k|}{m} - \frac{\omega_k k^2}{m|k|}$

→ The general solution to the Proca equation reads:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_{i=1}^{3} \left\{ a^i(k)\, e_\mu^i(k)\, e^{-i\omega t + ik\bar{x}} + a^{*i}(k)\, e_\mu^{*i}(k)\, e^{i\omega t - ik\bar{x}} \right\}$$

5.3 Canonical quantization

→ We had: $\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$ and $\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \partial_0 A_i - \partial_i A_0 = E_i$

↳ The system is constrained: it's not possible to express the velocity $\dot{A}_0$ in terms of the momentum π_0

→ We impose the commutation relations as follows:

$$[\pi_i(\bar{x}), A_j(\bar{y})] = -i\,\delta_{ij}\,\delta^3(\bar{x}-\bar{y})$$

$$\Rightarrow \quad [a^i_k, a^{j\dagger}_q] = \delta^{ij}\,\delta^3(k-q)$$