4 FREE DIRAC FIELD

→ Our field doesnt have to be a scalar, we need to consider all representations of the Lorentz group. Here we consider the spinor representation wich correspond to particles with half-integer spin.

4.1 Spinor representation

→ Spinor representation is constructed with the help of matrices γ_μ wich satisfy the property

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} = 2g_{\mu\nu}$$

They're called the gamma matrices.

→ From the gamma matrices, we define the spin matrices as

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

wich satisfy the commutation relation of the Lorentz algebra:

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho} S^{\mu\sigma} + g^{\mu\sigma} S^{\nu\rho} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho}\right)$$

→ The minimum size of γ-matrices in 4-D is 4. We're gonna use the chiral representation defined as:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} ; \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \Leftrightarrow \gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

with $\sigma^\mu \hat{=} (1, \vec{\sigma})$ and $\bar{\sigma}^\mu \equiv (1, -\vec{\sigma})$

↳ $S^{\mu\nu}$ are 6 matrices 4×4, acting on $\Psi_\alpha(x)$ where $\alpha \in \{1,2,3,4\}$ is a spinor index, and generate Lorentz transfo.

⊙ Lorentz transformation:

→ The finite Lorentz transformation is:

$$\Lambda_{1/2} = \exp\left\{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right\}$$

with $\omega_{\mu\nu}$ are the 6 parameters of transformations

↳ In our representation, the generators are:

$$S^{0i} = \frac{i}{4}[\gamma^0,\gamma^i] = \frac{i}{4}(\gamma^0\gamma^i - \gamma^i\gamma^0) = \frac{i}{4}\left\{\begin{pmatrix}0 & 1\!\!1\\ 1\!\!1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix} - \begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}\begin{pmatrix}0 & 1\!\!1\\ 1\!\!1 & 0\end{pmatrix}\right\}$$

$$= \frac{i}{4}\left\{\begin{pmatrix}-\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix} - \begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix}\right\} = \frac{-i}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix} \rightsquigarrow \text{boost}$$

$$S^{ij} = \frac{i}{4}[\gamma^i,\gamma^j] = \frac{i}{4}\left\{\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix} - (i\leftrightarrow j)\right\} = \frac{i}{4}\left\{\begin{pmatrix}-\sigma_i\sigma_j & 0\\ 0 & -\sigma_i\sigma_j\end{pmatrix} - (i\leftrightarrow j)\right\}$$

$$= \frac{i}{4}\begin{pmatrix}-\sigma_i\sigma_j + \sigma_j\sigma_i & 0\\ 0 & -\sigma_i\sigma_j + \sigma_j\sigma_i\end{pmatrix} = \frac{i}{4}\begin{pmatrix}-i2\epsilon_{ijk}\sigma_k & 0\\ 0 & -i2\epsilon_{ijk}\sigma_k\end{pmatrix} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} \rightsquigarrow \text{rotat}^{\circ}$$

Recall | The Pauli matrices are:

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\ \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix};\ \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \text{ and } \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

DEF | We can split our 4-component spinor into 2 parts:

$$\psi_\alpha = \begin{pmatrix}\psi_{Li}\\ \psi_{Ri}\end{pmatrix} \text{ with } i=1,2 \text{ called Weil spinors}$$

PROP | The Lorentz transformation does not mix the Weil spinors

DEMO:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \frac{i}{4}\left\{\begin{pmatrix}\sigma^\mu\bar\sigma^\nu & 0\\ 0 & \bar\sigma^\mu\sigma^\nu\end{pmatrix} - (\mu\leftrightarrow\nu)\right\}$$

$$= \frac{i}{4}\begin{pmatrix}\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0\\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\end{pmatrix} \rightsquigarrow \text{block-diagonal}$$

∎

→ Theories constructed only on ψ_L (or ψ_R) won't be invariant under space reflection (P)

→ The Lorentz transformation acts on spinors as follows:

$$\psi_\alpha(x) \longrightarrow \psi'_\alpha(x) = (\Lambda_{1/2})_{\alpha\beta}\, \psi_\beta(\Lambda^{-1}x)$$

where Λ is the matrix of Lorentz transformation in the vector representation, $\Lambda = \exp\left(\frac{-i}{2}\omega_{\mu\nu}T^{\mu\nu}\right)$ where $(T^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)$ are the corresponding generators.

4.2 Lagrangian of a free Dirac field

⊙ Construction of the Lagrangian:

→ We need to construct scalars. We could think of $\psi^+\psi$, but it's not a scalar.

Let's show that $\psi^+\psi$ is not a scalar:

→ $\sigma_i^+ = \sigma_i$; $\gamma^{0+} = \gamma^0$; $\gamma^{i+} = \begin{pmatrix} 0 & -\sigma_i^+ \\ \sigma_i^+ & 0 \end{pmatrix} = -\gamma^i$

→ Consider a boost:

$$(S^{0i})^+ = \left(\frac{i}{4}[\gamma^0,\gamma^i]\right)^+ = \frac{-i}{4}\left(\gamma^{i+}\gamma^{0+} - \gamma^{0+}\gamma^{i+}\right)$$

$$= \frac{+i}{4}\left(\gamma^i\gamma^0 - \gamma^0\gamma^i\right) = -\frac{i}{4}[\gamma^0,\gamma^i] = -S^{0i}$$

→ For a rotation: $(S^{ij})^+ = S^{ij}$

→ Thus, $(S^{\mu\nu})^+ \neq S^{\mu\nu}$ and therefore:

$$\psi^+\psi \underset{\Lambda}{\mapsto} \psi^+\Lambda^+\Lambda\psi \simeq \psi^+\left(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu +}\right)\left(\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi$$
$$\neq \psi^+\psi$$

→ We need to use the properties of γ-matrices.

→ Since $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$, $(\gamma^0)^2 = 1$.

↳ $\begin{cases} \gamma^0\gamma^0\gamma^0 = \gamma^0 \; ; \; \gamma^0(\gamma^0)^n\gamma^0 = (\gamma^0)^n \\ \gamma^0\gamma^i\gamma^0 = -\gamma^i \; ; \; \gamma^0(\gamma^i)^n\gamma^0 = \gamma^0\gamma^i(\gamma^0\gamma^0)\gamma^i \dots \gamma^i\gamma^0 = (-\gamma^i)^n \end{cases}$

↳ $\gamma^0 S^{0i}\gamma^0 = -S^{0i}$ and $\gamma^0 S^{ij}\gamma^0 = S^{ij}$

We found that: $\gamma^0 S^{0i+}\gamma^0 = S^{0i}$

$\gamma^0 S^{ij+}\gamma^0 = S^{ij}$

DEF | Let ψ be a Dirac spinor. Then its Dirac adjoint is defined as $\bar{\psi} = \psi^\dagger \gamma^0$

↳ Under Lorentz transformation, it transforms as follow:

$$\bar{\psi}\psi \xrightarrow{\Lambda} \psi^\dagger \left(\exp\left[\tfrac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}\right]\right)^\dagger \gamma^{0\dagger} \left(\exp\left[\tfrac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}\right]\right)\psi$$
$$= \psi^\dagger \gamma^0\gamma^0 \exp[\dots]^\dagger \gamma^0 \exp[\dots]\psi$$
$$= \bar{\psi}\psi$$

→ $\bar{\psi}\psi$ is a scalar.

→ More properties of the γ-matrices:

? → $\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\mu$ ⇝ in γ^μ, the index is a genuine vector index.

DEF → We define a 5^{th} γ-matrix: $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

PROP

1) $\gamma^5 = \frac{-i}{4!}\epsilon^{\mu\nu\lambda\rho}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

2) This matrix anticommutes with all γ^μ:
$\{\gamma^\mu, \gamma^5\} = 0$ because $\gamma^5 = \mathrm{diag}(-1, 1)$

3) $[\gamma^5, S^{\mu\nu}] = 0 \iff \Lambda_{1/2}^{-1}\gamma^5\Lambda_{1/2} = \gamma^5$

Indeed: $[\gamma^5, [\gamma^\mu, \gamma^\nu]] = \gamma^5\gamma^\mu\gamma^\nu - \gamma^5\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\gamma^5 + \gamma^\nu\gamma^\mu\gamma^5$
$= \gamma^\mu\gamma^\nu\gamma^5 - \gamma^5\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\gamma^5 + \gamma^5\gamma^\nu\gamma^\mu = 0$

→ We then have several ingredients for our Lagrangian:

$\bar{\psi}\psi \longrightarrow$ scalar
$\bar{\psi}\gamma^5\psi \longrightarrow$ pseudo-scalar
$\bar{\psi}\gamma^\mu\psi \longrightarrow$ vector
$\bar{\psi}\gamma^\mu\gamma^5\psi \longrightarrow$ pseudo-vector
$\bar{\psi}\gamma^\mu\gamma^\nu\psi \longrightarrow$ tensor

DEF | We define the Dirac Lagrangian $\mathcal{L}_{Dirac}$ as:

$$\mathcal{L}_{Dirac} \equiv \bar{\psi}(i\gamma^\mu\partial_\mu)\psi - m\bar{\psi}\psi$$
$$= \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

→ Matrix notation: $\psi^\dagger_a\, i\gamma^0_{ab}\gamma^\mu_{bc}\partial_\mu\psi_c - \psi^\dagger_a\gamma^0_{ab}\psi_b\, m$

↳ Let's check the hermiticity of $S = \int d^4x \{ \bar{\Psi}(i\not\partial - m)\psi \}$

→ $(\bar{\Psi}\psi)^\dagger = (\psi^\dagger \gamma^0 \psi)^\dagger = \psi^\dagger \gamma^{0\dagger} \psi = \psi^\dagger \gamma^0 \psi = \bar{\Psi}\psi$

→ $\left(\int d^4x \{ i\bar{\Psi}\gamma^\mu \partial_\mu \psi \} \right)^\dagger = \int d^4x \{ (-i) \partial_\mu \psi^\dagger \gamma^{\mu\dagger} \gamma^0 \psi \}$

$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$ $\quad = -\int d^4x \{ i\partial_\mu \bar{\Psi} \gamma^\mu \psi \} = \int d^4x \{ i\bar{\Psi}\gamma^\mu \partial_\mu \psi \}$

⊙ <u>Equations of motion:</u>

→ We vary the action with respect to $\bar{\Psi}$:

$S[\bar{\Psi} + \delta\bar{\Psi}] = \int d^4x \{ (\bar{\Psi} + \delta\bar{\Psi})(i\gamma^\mu\partial_\mu - m)\psi \}$

$\quad = \int d^4x \{ \bar{\Psi} i\not\partial \psi - m\bar{\Psi}\psi + \delta\bar{\Psi}(i\gamma^\mu\partial_\mu\psi - m\psi) \}$

$\delta S = \int d^4x \{ \delta\bar{\Psi}(i\not\partial - m)\psi \} = 0 \Rightarrow (i\not\partial - m)\psi = 0$

We find the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$

↳ We also find the conjugate eqs if we vary the action with respect to ψ, but it's not an independent one: $-i\partial_\mu \bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0$

4.3 Solutions to the free Dirac equation

⊙ <u>Relation with the Klein-Gordon equation:</u>

→ Each component of dirac spinor satisfies the K-G equation:

$0 = (i\not\partial + m)(i\not\partial - m)\psi$

$= (i\gamma^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\psi = (-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi$

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ $\quad = \left\{ -\frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu - m^2 \right\}\psi = \left(-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu - m^2 \right)\psi$

$= (\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2)\psi = (\partial^2 - m^2)\psi$

↳ All solution of Dirac eq. are linear superpositions of the plane waves $e^{\pm i\omega t \mp i k \cdot \bar{x}}$. By analogy with the scalar case, we guess:

$$\psi_\alpha(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \left\{ a^i_{\bar{p}}\, u^i_\alpha(\bar{p})\, e^{-i(\omega t - \bar{p}\cdot\bar{x})} + b^{*i}_{\bar{p}}\, v^i_\alpha(\bar{p})\, e^{i(\omega t - \bar{p}\cdot\bar{x})} \right\}$$

where $a^i_{\bar{p}}$, $b^{i*}_{\bar{p}}$ are arbitrary coefficients and $u^i_\alpha(p)$, $v^i_\alpha(p)$ spinors, with $i = 1, 2$.

→ In addition, the wave function has to satisfy the equations:

$$\begin{cases}(i\gamma^\mu\partial_\mu - m)\, u(\bar{p})\, e^{-i\omega t + i\bar{p}\bar{x}} = 0\\(i\gamma^\mu\partial_\mu - m)\, v(\bar{p})\, e^{i\omega t - i\bar{p}\bar{x}} = 0\end{cases}$$

↳ Considering the 1st one, we get:

$$(i\gamma^\mu\partial_\mu - m)_{\beta\alpha}\, u_\alpha(\bar{p})\, e^{-i\omega t + i\bar{p}\bar{x}}$$
$$= e^{-i\omega t + i\bar{p}\bar{x}}\,(\omega\gamma^0 - \bar{p}.\bar{\gamma} - m\,\mathbb{1}_4)\, u(\bar{p}) = 0$$

$$\Leftrightarrow \begin{pmatrix} -m & \omega_p - \bar{p}.\bar{\sigma} \\ \omega_p - \bar{p}.\bar{\sigma} & -m\end{pmatrix}\begin{pmatrix} u_L \\ u_R\end{pmatrix} = 0$$

$$\Leftrightarrow \begin{cases} -m\, u_L + (\omega_p - \bar{p}.\bar{\sigma})\, u_R = 0 \\ (\omega_p + \bar{p}.\bar{\sigma})\, u_L - m\, u_R = 0\end{cases} \quad \Rightarrow u_L = \frac{\omega_p - \bar{p}.\bar{\sigma}}{m}.\, u_R$$

Injecting, we get: $\frac{1}{m}(\omega_p + \bar{p}.\bar{\sigma})(\omega_p - \bar{p}.\bar{\sigma})\, u_R - m\, u_R = 0$

$$\Leftrightarrow \frac{1}{m}\left(\omega_p^2 - (\bar{p}.\bar{\sigma})(\bar{p}.\bar{\sigma})\right) u_R - m\, u_R = 0$$

And $p_i p_j \sigma_i \sigma_j = (\delta_{ij} + i\,\cancel{\epsilon_{ijk}\sigma_k})\, p_i p_j = p_j p_j = \bar{p}^2$

$$\Leftrightarrow \frac{1}{m}(m^2 + \bar{p}^2 - \bar{p}^2)\, u_R - m\, u_R = 0$$

It's always true! We have the freedom to pick the 1st solution (say for u_R) and the 2nd depends on the 1st by $u_L = \frac{1}{m}(\omega_p - \bar{p}.\bar{\sigma})\, u_R$

DEF We define $\xi^1 \triangleq \begin{pmatrix}1\\0\end{pmatrix}$ and $\xi^2 \triangleq \begin{pmatrix}0\\1\end{pmatrix}$ so we can write

$$u^i = \begin{pmatrix} u_L \\ u_R\end{pmatrix} = \begin{pmatrix} (\omega_p - \sigma.p)/m\,.\,\xi^i \\ \xi^i\end{pmatrix}$$

⊙ Normalization of our solutions:

→ Let's check the actual normalisation:

$$\bar{u}u = u^\dagger\gamma^0 u$$
$$= \left(\sigma.p/m\,.\,\xi^i,\ \xi^i\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\sigma p/m\,.\,\xi^j \\ \xi^j\end{pmatrix} = 2\,\xi^i\,\frac{\sigma.p}{m}\,\xi^j$$

$$= 2\,\frac{\sigma.p}{m}\, u_R^\dagger\, u_R$$

→ We remember that we could take any linearly independant spinors for u_R. We're gonna redefine u_R so the normalisation doesn't depend on p.

→ Since $\sigma.p = \begin{pmatrix} \omega + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & \omega - p_3 \end{pmatrix}$, we can compute:

$Det(\sigma.p) = \omega^2 - p_3^2 - p_1^2 - p_2^2 = m^2 > 0$

$Tr(\sigma.p) = 2\omega_p > 0$

We showed that $\sigma.p$ has positive eigenvalues. We can then define $\sqrt{\sigma.p}$ and $\sqrt{\bar\sigma.p}$

→ We define $u_R^i = \sqrt{\bar\sigma^\mu p_\mu}\, \xi^i$ with $\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$. Our solutions become:

$$u^i = \begin{pmatrix} \frac{\sigma p}{m} \sqrt{\bar\sigma p}\, \xi^i \\ \sqrt{\sigma p}\, \xi^i \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma p}\, \xi^i \\ \sqrt{\bar\sigma p}\, \xi^i \end{pmatrix} = \begin{pmatrix} u_L^i \\ u_R^i \end{pmatrix}$$

Indeed; $\sqrt{\sigma p} . \sqrt{\bar\sigma p} = \sqrt{\sigma p \bar\sigma p} = \left(\begin{pmatrix} \omega + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & \omega - p_3 \end{pmatrix} . \begin{pmatrix} \omega - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & \omega + p_3 \end{pmatrix} \right)^{\frac{1}{2}} = \sqrt{m^2 \mathbb{1}_2}$
$= m$

→ Normalisation of $\bar u^i u^j$:

$$\bar u^i u^j = \left(\xi^{i\dagger} \sqrt{\sigma p}\, , \xi^{i\dagger} \sqrt{\bar\sigma p} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\sigma p}\, \xi^j \\ \sqrt{\bar\sigma p}\, \xi^j \end{pmatrix}$$

$$= 2m \xi^{i\dagger} \xi^j = 2m\, \delta^{ij} \quad \text{Nice!}$$

⊙ Final solutions:

→ We have: $u^i = \begin{pmatrix} \sqrt{\sigma^\mu p_\mu}\, \xi^i \\ \sqrt{\bar\sigma^\mu p_\mu}\, \xi^i \end{pmatrix}$ with: $\bar u^i u^j = 2m\delta^{ij}$

$v^i = \begin{pmatrix} \sqrt{\sigma p}\, \eta^i \\ -\sqrt{\bar\sigma p}\, \eta^i \end{pmatrix}$ $\quad \bar v^i v^j = -2m\,\delta^{ij}$

$\bar u^i v^j = \bar v^i u^j = 0$

⊙ Usefull relations:

→ We have: $\sum_i u_\alpha^i(p)\, \bar u_\beta^i(p) = \sum_i \begin{pmatrix} \sqrt{\sigma p}\, \xi^i \\ \sqrt{\bar\sigma p}\, \xi^i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\xi^{\dagger i} \sqrt{\sigma p}, \xi^{\dagger i} \sqrt{\bar\sigma p} \right)$

? $= \sum_i \begin{pmatrix} \sqrt{\sigma p}\, \xi^i \xi^{i\dagger} \sqrt{\bar\sigma p} & \sqrt{\sigma p}\, \xi^i \xi^{\dagger} \sqrt{\sigma p} \\ \sqrt{\bar\sigma p}\, \xi^i \xi^{i\dagger} \sqrt{\bar\sigma p} & \sqrt{\bar\sigma p}\, \xi^i \xi^{i\dagger} \sqrt{\sigma p} \end{pmatrix} = \begin{pmatrix} m & \sigma p \\ \bar\sigma p & m \end{pmatrix}$

$= \gamma^\mu p_\mu + m = (\gamma^\mu p_\mu)_{\alpha\beta} + m\,\delta_{\alpha\beta}$

↳ In general, we have:

$$\sum_i u^i(p)\,\bar{u}^i(p) = \gamma^\mu p_\mu + m$$

$$\sum_i v^i(p)\,\bar{v}^i(p) = \gamma^\mu p_\mu - m$$

→ Other useful normalization relations:

$$v^{i\dagger}(p)\,v^j(p) = 2\omega_p\,\delta^{ij}$$
$$\bar{v}^i(p)\,u^j(p) = \bar{u}^i(p)\,v^j(p) = 0$$
$$u^{i\dagger}(\vec{p})\,v^j(-\vec{p}) = 0$$

4.4 Quantization

Classical expression:

→ Our action reads $S = \int d^4x \{ i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi \}$

→ The momentum is given by $i\psi^\dagger$ because $\psi^\dagger\gamma^0(\gamma^0\partial_0 + \gamma^i\partial_i)\psi = \underbrace{\psi^\dagger\partial_0\psi}_{\text{no } \gamma^0} + \dots$

→ The hamiltonian is $H = \int d^3x\, \bar{\Psi}(-i\vec{\gamma}.\vec{\nabla} + m)\psi$ and the impulsion: $P_j = \int d^3x\, \psi^\dagger(-i\partial_j)\psi$

→ The Noether currents and charges are:

$$J^\mu = \bar{\Psi}\gamma^\mu\psi \rightsquigarrow Q = \int d^3x\, \bar{\Psi}\gamma^0\psi = \int d^3x\, \psi^\dagger\psi$$

Commutation relations:

→ By analogy with the K-G equation, we impose:

$$[\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})] = \delta_{\alpha\beta}\,.\,\delta^3(\vec{x}-\vec{y})$$

and then decompose the Dirac field in $\hat{a}$ and $\hat{a}^\dagger$ operators:

$$\psi_\alpha(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \{ e^{-i\omega t + i\vec{k}\vec{x}}\, u_\alpha^i(k)\, a^i(k) + e^{i\omega t - i\vec{k}\vec{x}}\, v_\alpha^i(k)\, b^{\dagger i}(k) \}$$

The corresponding hamiltonian would read:

$$H = \int d^3p \sum_i \{ \omega_k \hat{a}_k^\dagger \hat{a}_k - \omega_k \hat{b}_k^\dagger \hat{b}_k \} \rightarrow \text{Not bounded from below!}$$

⊙ Anticommutation relations:

→ We interpret particles as excitations of a given mode, or oscillator. Several oscillator in the same mode are a more excited state of the same oscillator. If we want fermions, they must respect Pauli exclusion principle: 1 mode = 1 fermion max. This correspond to $|0\rangle$, $\hat{a}^+|0\rangle$, $\hat{a}^+\hat{a}^+|0\rangle \overset{!}{=} 0$

→ These properties correspond to the anticommutation relations of the creation and annihilation operators. We thus impose:

$$(*) \quad \{\psi_\alpha(\vec{x}), \psi_\beta^+(\vec{y})\} = \delta_{\alpha\beta}\, \delta^3(\vec{x}-\vec{y})$$

and $\{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} = \{\psi_\alpha^+(\vec{x}), \psi_\beta^+(\vec{y})\} = 0$

↳ Correspondingly, $\hat{a}$ and $\hat{a}^+$ should anticommute:

$$(**) \quad \{\hat{a}_{\vec{p}}^{i+}, \hat{a}_{\vec{q}}^{j}\} = \delta^{ij}\, \delta^3(\bar{p}-\bar{q})$$
$$\{\hat{b}_{\bar{p}}^{i+}, \hat{b}_{\bar{q}}^{j}\} = \delta^{ij}\, \delta^3(\bar{p}-\bar{q})$$

→ We need to check that $** \Rightarrow *$ and calculate $\hat{N}, \hat{P}, \dots$

⊙ Construction of $\hat{H}$, $\hat{P}_i$ and $\hat{Q}$:

→ Check of the anticommutation relations:

$$\to \psi_\alpha(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_i \left\{ \hat{a}_{\underline{k}}^i\, u_\alpha^i(\underline{k})\, e^{i\underline{k}\underline{x}} + \hat{b}_{\underline{k}}^{+i}\, v_\alpha^i(\underline{k})\, e^{-i\underline{k}\underline{x}} \right\}$$

? $$\to \psi_\beta^+(\vec{y}) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \sum_j \left\{ \hat{b}_{\bar{q}}^j\, v_\beta^{j+}(\bar{p})\, e^{+i\bar{p}.\bar{y}} + \hat{a}_{\bar{p}}^{j+}\, u_\beta^j(\bar{p})\, e^{-i\bar{p}.\bar{y}} \right\}$$

$$\to \{\psi_\alpha(\bar{x}), \psi_\beta(\bar{y})\} = \int \frac{d^3k\, d^3p}{(2\pi)^3\, 2\sqrt{\omega_k\omega_p}} \sum_{i,j} \cdot \Big\{ e^{i\underline{k}\bar{x} - i\bar{p}\bar{y}}\, u_\alpha^i(\underline{k}) \underbrace{\{\hat{a}_{\underline{k}}^i, \hat{a}_{\bar{p}}^{j+}\}}_{\delta^{ij}\delta^3(\underline{k}-\bar{p})} u_\beta^{j+}(\bar{p})$$
$$+ e^{-i\underline{k}\bar{x} + i\bar{p}\bar{y}}\, v_\alpha^i(\underline{k}) \underbrace{\{\hat{b}_{\underline{k}}^{i+}, \hat{b}_{\bar{q}}^j\}}_{\delta^{ij}\delta^3(\underline{k}-\bar{p})} v_\beta^j(\bar{q}) \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3\, 2\omega_p} \sum_i \left(e^{i\bar{p}(\bar{x}-\bar{y})}\, u_\alpha^i(\bar{p})\, u_\beta^{i+}(\bar{p}) + e^{-i\bar{p}(\bar{x}-\bar{y})}\, v_\alpha^i(\bar{p})\, v_\beta^{i+}(\bar{p}) \right)$$

Previously, we showed that $\sum_i u_\alpha^i u_\beta^{i+} = \sum_i u_\alpha \bar{u}_\mu \gamma^0_{\mu\beta} = \left[(\gamma^\mu p_\mu + m)\gamma^0\right]_{\alpha\beta}$ and $\sum_i v_\alpha^i v_\beta^{i+} = (\gamma.p - m)\gamma^0$

$\vec{p} \leftrightarrow -\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^3\, 2\omega_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left(\gamma^0 \omega_p - \vec{\gamma}.\vec{p} + m + \gamma^0 \omega_p + \vec{\gamma}.\vec{p} - m \right)_{\alpha\sigma} \gamma^0_{\sigma\beta}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{2\omega_p}{2\omega_p} . e^{i\vec{p}(\vec{x}-\vec{y})} . \gamma^0_{\alpha\sigma} \gamma^0_{\sigma\beta} = \delta_{\alpha\beta}\, \delta^3(\vec{x}-\vec{y})$$

→ (See notes for details) We find the Hamiltonian to be:

$$\hat{H} = \int d^3p\ \omega_p \left\{ \hat{a}^{i\dagger}_{\vec{p}}\, \hat{a}^{i}_{\vec{p}} + \hat{b}^{i\dagger}_{\vec{p}}\, \hat{b}^{i}_{\vec{p}} \right\}$$

Similarly, we have: $\hat{P}_i = \int d^3p\ p_i \left\{ \hat{a}^{i\dagger}_{\vec{p}}\, \hat{a}^{i}_{\vec{p}} + \hat{b}^{i\dagger}_{\vec{p}}\, \hat{b}^{i}_{\vec{p}} \right\}$

and $\hat{Q} = \int d^3p \left\{ \hat{a}^{i\dagger}_{\vec{p}}\, \hat{a}^{i}_{\vec{p}} - \hat{b}^{i\dagger}_{\vec{p}}\, \hat{b}^{i}_{\vec{p}} \right\}$

→ We interpret $a^\dagger_{\vec{p}}, b^\dagger_{\vec{p}}$ as creation operators of particles and antiparticles

4.5 Spin

→ We have 2 types of particles and antiparticles $i = 1, 2$. To understand, we need to consider the angular momentum operator.

Recall: the conserved current under a Lorentz transformation for a scalar field is:

$M^{\mu}_{\lambda\rho} = T^{\mu}{}_{\lambda} x_\rho - T^{\mu}{}_{\rho} x_\lambda + S^{\mu}{}_{\lambda\rho}$

→ In the rest frame of a particle ($\vec{p} = 0$), this operator reduces to

$$\vec{S} \mathrel{\hat{=}} \hat{\vec{J}}\Big|_{rest} = \int d^3x\, \psi^\dagger \frac{1}{2} \vec{\Sigma}\, \psi \quad \text{with } \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

DEF 1 The spin operator is $\Sigma_i = \frac{i}{4} \epsilon_{ijk} [\gamma^j, \gamma^k] = \epsilon_{ijk} S^{jk}$

→ We need to calculate the action of this operator on the state with zero momentum $\hat{a}^{i\dagger}(0)|0\rangle$

$$\vec{S}\, a_0^{+s}|0\rangle = \int \frac{d^3x\, d^3p\, d^3k}{(2\pi)^3\, 2\sqrt{\omega_p \omega_k}}$$

$$\times \left(e^{ikx}\, v^{+i}(k)\, \hat{b}^i(k) + e^{-ikx}\, u^{i+}(k)\, \hat{a}^{+i}(k)\right) \qquad \psi^+$$

$$\times\ \frac{1}{2}\vec{\Sigma}\left(e^{i\bar{p}x}\, u^j(\bar{p})\, \hat{a}^j(\bar{p}) + e^{-i\bar{p}x}\, v^j(\bar{p})\, \hat{b}^{j+}(p)\right) \hat{a}^{k+}(0)|0\rangle \qquad \vec{\Sigma}\psi$$

→ Since $\bar{J}|0\rangle = 0$, we have $\bar{J}\,\hat{a}^{+k}(0)|0\rangle = [\bar{J}, \hat{a}^{+k}(0)]|0\rangle$. We'll have term like $[ba, a^+]$, $[bb^+, a^+]$, $[a_p^+ b_p^+, a_0^+]$ all giving 0, and
$[a_p^+ a_p, a_0^+]|0\rangle = (a_p^+ a_p a_0^+ - a_0^+ a_p^+ a_p)|0\rangle = \hat{a}_0^+|0\rangle$
Thus, the integral become:

$$\hat{\vec{S}}\, a_0^{+s}|0\rangle = [\hat{\vec{S}}, a_0^{+s}]|0\rangle$$

$$= \int d^3x \frac{d^3k\, d^3q}{(2\pi)^3\, 2\sqrt{\omega_k \omega_q}} \Big[\left(b_k^i\, v^{+i}(k)\, e^{ik\bar{x}} + a_k^{i+}\, u^{+i}(k)\, e^{-ikx}\right)\frac{1}{2}\vec{\Sigma}$$

$$\left(a_q^j\, u^j(\bar{q})\, e^{i\bar{q}\bar{x}} + b_q^{+j}\, v^j(\bar{q})\, e^{-i\bar{q}\bar{x}}\right), \hat{a}_0^{+s}\Big]|0\rangle$$

$$= \int \frac{d^3k}{2\omega_k}\, u^{+i}(k) \cdot \frac{1}{2}\vec{\Sigma}\, u^j(k)\, \underbrace{[a_k^{+i}\, a_k^j, a_0^{+s}]|0\rangle}_{a_k^{+i}\,\delta^{is}\,\delta(k)|0\rangle}$$

$$= \int \frac{d^3k}{4\omega_k}\, u^{i+}(k)\, \vec{\Sigma}\, u^j(k)\, a_k^{+i}\, \delta^{is}\, \delta(k)|0\rangle$$

$$= \frac{1}{2m}\, \underbrace{u^{+i}(0)\, \frac{1}{2}\vec{\Sigma}\, u^s(0)}_{(\xi^{+i}\sqrt{m}, \xi^{+i}\sqrt{m})\begin{pmatrix}\bar{\sigma}/2 & 0\\ 0 & \bar{\sigma}/2\end{pmatrix}\begin{pmatrix}\sqrt{m}\,\xi^s\\ \sqrt{m}\,\xi^s\end{pmatrix} = 2m\,\xi^{i+}\frac{1}{2}\bar{\sigma}\,\xi^s}\, a_0^{+i}|0\rangle$$

$$= \left(\xi_\alpha^{i+}\, \frac{1}{2}\bar{\sigma}_{\alpha\beta}\, \xi_\beta^s\right) a_0^{i+}|0\rangle$$

Consider the 3rd component: $\qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$$S_3\, a_0^{+s}|0\rangle = \left(\xi^{i+}\, \frac{1}{2}\sigma_3\, \xi^s\right) a_0^{+i}|0\rangle$$

$$= \begin{cases} 1/2\ a_0^{+1}|0\rangle & \text{for } s=1 \\ -1/2\ a_0^{+2}|0\rangle & \text{for } s=2 \end{cases}$$

4.6 One-particle states

→ Let's show that states created by $\hat{a}^{i\dagger}(\bar{p})$ and $\hat{b}^{i\dagger}(\bar{p})$ have given energy and momentum:

$$\hat{H}\,\hat{a}^{i\dagger}(\bar{p})|0\rangle = \int d^3k\,\omega_k\left(\hat{a}^{j\dagger}_k\hat{a}^j_k + \hat{b}^{i\dagger}_k\hat{b}^i_k\right)\hat{a}^{i\dagger}_p|0\rangle$$

$$= \int d^3k\,\omega_k\,\hat{a}^{j\dagger}_k\hat{a}^j_k\hat{a}^{i\dagger}_p|0\rangle = \int d^3k\,\omega_k\,\hat{a}^{j\dagger}_k\underbrace{\{\hat{a}^j_k,\hat{a}^{i\dagger}_p\}}_{\delta^{ij}\delta^3(\bar{k}-\bar{p})}|0\rangle$$

$$= \omega_p\,\hat{a}^{i\dagger}_p|0\rangle$$

Analogously, for $\hat{P}$: $\hat{\vec{P}}\,\hat{a}^{i\dagger}(\bar{k})|0\rangle = \vec{k}\,\hat{a}^{i\dagger}(\bar{k})|0\rangle$

For the charges $\hat{Q}$: $\hat{Q}\,\hat{a}^{i\dagger}(\bar{k})|0\rangle = (+1)\,\hat{a}^{i\dagger}(\bar{k})|0\rangle$

$\hat{Q}\,\hat{b}^{j\dagger}(\bar{k})|0\rangle = (-1)\,\hat{b}^{j\dagger}(\bar{k})|0\rangle$

4.7 Statistics:

→ A general 2-particles state is written as:

$$\int d^3k\,d^3p\,f(k,p)\,\hat{a}^{i\dagger}(\bar{k})\,\hat{a}^{j\dagger}(\bar{p})|0\rangle$$

The wave function $f(k,p)$ has to be antisymmetric in $k \leftrightarrow p$ and $i \leftrightarrow j$ because of the anti-symmetry of the product $\hat{a}^{i\dagger}(\bar{k})\,\hat{a}^{j\dagger}(\bar{p})$

→ We're dealing with fermions

→ Energy of a 2 particles state:

$$\hat{H}\,a^{i\dagger}_p a^{s\dagger}_q|0\rangle = \int d^3k\,\omega_k\,a^{\dagger j}_k a^j_k a^{i\dagger}_p a^{s\dagger}_q|0\rangle$$

$$= \int d^3k\,\omega_k\,a^{\dagger j}_k\left(\delta^{ij}\delta(\bar{k}-\bar{p}) - a^{\dagger i}_p a^j_k\right)a^{s\dagger}_q|0\rangle$$

$$= \ldots = (\omega_p+\omega_q)\,a^{i\dagger}_p a^{s\dagger}_q|0\rangle$$