3 INTERACTIONS

→ If we want to describe interactions, we must add non-linear terms to the Lagrangian.

→ We must create **renormalizable** theories to avoid divergencies. It can be shown that theories that involve coupling constants of dimension $(\text{mass})^{-n}$ (mass in negative power) are not renormalizable. This limits the # of possible interactions to only a few.

→ For a scalar field :

$$[\varphi] = m \rightsquigarrow \alpha\phi^3 \Rightarrow [\alpha] = m^1$$
$$\lambda\phi^4 \Rightarrow [\lambda] = m^0$$
$$c\phi^5 \Rightarrow [c] = m^{-1} \; ⚡$$

↳ We only have 2 terms possible.

3.1 Interaction representation

⊙ Schrodinger and Heisenberg representation

→ In QM, in Schrodinger representation, the evolution of the system is described by the Schrodinger equation:

$$i\partial_t \psi(t) = \hat{H}\psi(t)$$

wich has the formal solution $\psi(t) \overset{(*)}{=} e^{-i\hat{H}t}\psi(0)$

→ The observables are the averages of operators:

$$\langle\hat{B}\rangle = \langle\psi(t)|\hat{B}|\psi(t)\rangle$$

→ Using the solution (*), we introduce the Heisenberg representation:

$$\langle\hat{B}\rangle = \langle\psi(0)|e^{i\hat{H}t}\hat{B}\,e^{-i\hat{H}t}|\psi(0)\rangle$$

$$\equiv \langle\psi(0)|B_H(t)|\psi(0)\rangle$$

⊙ Construction of the interaction representation:

→ Consider the Hamiltonian as a sum of the free part $\hat{H}_0$ and the interaction $\hat{H}_1$. The $\ddot{S}$ become:

$$i\partial_t \psi(t) = (\hat{H}_0 + \hat{H}_1)\psi(t)$$

$\hat{H}_0 = \int d^3k \{\omega_k a_k^\dagger a_k\}$

? → We consider solution of the form $\psi(t) \overset{*}{=} e^{-i\hat{H}_0 t} \psi_I(t)$

Inserting, we get:

$$i\partial_t \psi = H_0 e^{-i\hat{H}_0 t} \psi_I + i e^{-i\hat{H}_0 t} \partial_t \psi_I \overset{!}{=} \hat{H}_0 e^{-i\hat{H}_0 t} \psi_I + \hat{H}_1 e^{-i\hat{H}_0 t} \psi_I$$

$$\Leftrightarrow \quad i\partial_t \psi_I = e^{i\hat{H}_0 t} \hat{H}_1 e^{-i\hat{H}_0 t} \psi_I$$

DEF The Hamiltonian of interaction in the interaction representation is

$$H_1^I \equiv e^{i\hat{H}_0 t} \hat{H}_1 e^{-i\hat{H}_0 t}$$

↳ We then have $i\partial_t \psi_I(t) = H_I \psi_I$ $\quad$ / $\ddot{S}$

→ Example: consider $\hat{H}_1 \equiv \int d^3x \, \phi^4(0,\bar{x})$

Then $\hat{H}_I = \int d^3x \, e^{i\hat{H}_0 t} \phi^4(0,\bar{x}) e^{-i\hat{H}_0 t} = \int d^3x \, \phi_0(t,\bar{x})$

→ Matrix elements:

$$\langle B \rangle = \langle \psi(t) | \hat{B} | \psi(t) \rangle \overset{*}{=} \langle \psi_I(t) | e^{+i\hat{H}_0 t} \hat{B} e^{i\hat{H}_0 t} | \psi_I(t) \rangle$$

$$\langle B \rangle = \langle \psi_I | \hat{B}_I | \psi_I \rangle$$

↳ In interaction representation, operators are function of the free fields

⊙ Evolution operator:

→ We invert ⊛: $\psi_I(t_2) = e^{+i\hat{H}_0 t_2} \psi(t_2) = e^{+i\hat{H}_0 t_2} e^{-iHt_2} \psi(0)$

→ $\psi(0) = e^{+iHt_1} e^{-iH_0 t_1} \psi_I(t_1)$

⇒ $\psi_I(t_2) = e^{iH_0 t_2} e^{-iH(t_2 - t_1)} e^{-iH_0 t_2} \psi_I(t_1)$

$= U(t_1, t_2) \psi_I(t_1)$

This expression is not convenient for practical calculations

3.2 Evolution in interaction representation

→ We want to find the explicit expression for the evolution operator in the interaction representation.

⊙ Integral form of the Schrodinger equation:

→ We had $i\partial_t \psi_I = \hat{H}_I(t)\,\psi_I(t)$. In the form of an integral equation,

$$\psi_I(t) = \psi_I(t_0) - i\int_{t_0}^{t} dt'\,\hat{H}_I(t')\,\psi_I(t')$$

↳ We can can solve this iteratively. Let's substitute in the r.h.s:

$$\psi_I(t) = \psi_I(t_0) - i\int_{t_0}^{t} dt'\left\{\hat{H}_I(t)\,\psi_I(t_0)\right\} + (-i)^2\int_{t_0}^{t'} dt''\,\hat{H}_I(t'')\,\psi_I(t'')$$

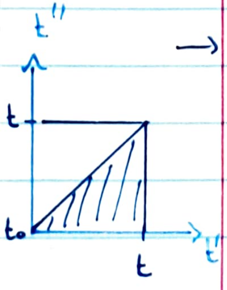
↳ One more time:

$$\psi_I(t) = \psi_I(t_0) - i\int_{t_0}^{t} dt'\left\{\hat{H}_I(t')\,\psi_I(t_0)\right\}$$
$$+ (-i)^2\int_{t_0}^{t'} dt''\left\{\hat{H}_I(t'')\,\psi_I(t_0)\right\}$$
$$+ (-i)^3\int_{t_0}^{t''} dt'''\left\{\hat{H}_I(t''')\,\psi_I(t_0)\right\} + \dots = U(t,t_0)\,\psi_I(t_0)$$

with $U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\,\hat{H}_I(t') + \dots + (-i)^n\int_{t_0}^{t} dt' \cdots \int_{t_0}^{t_n} dt_{n+1}\,\hat{H}(t_{n+1}) + \dots$

⊙ Complex analysis and time ordering:

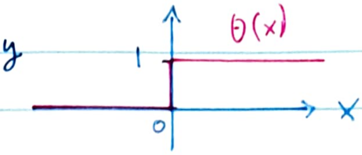
→ Consider individual terms of the series $U(t,t_0)$ (ex: 2nd term):



$$\underbrace{(-i)^2\int_{t_0}^{t} dt'\,\hat{H}_I(t')\int_{t_0}^{t'} dt''\,\hat{H}_I(t'')}_{t''\leq t'} = (-i)^2\int_{t_0}^{t} dt\,dt''\,\hat{H}_I(t')\,\hat{H}_I(t'')\,.\,\Theta(t'-t'')$$

DEF The Heaviside step function Θ is defined by

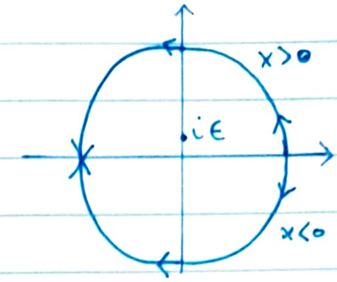
$$\begin{cases} \Theta(x) = 0 & \text{at } x < 0 \\ \Theta(x) = 1 & \text{at } x \geq 0 \end{cases}$$



Equivalently,

$$\Theta(x) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda - i\epsilon} e^{i\lambda x}$$

?



→ Check that $\partial_x \Theta = \delta(x)$:

$$\frac{d\Theta(x)}{dx} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} i\lambda \frac{d\lambda}{\lambda - i\epsilon} e^{i\lambda x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, e^{i\lambda x} = \delta(x)$$

DEF Let $\hat{A}(x^\mu)$, $\hat{B}(x^\mu)$ be 2 operators. The time-order product T is defined as follow:

$$T[\hat{A}(x)\hat{B}(y)] \equiv \begin{cases} \hat{A}(x)\hat{B}(y) & \text{at } x^0 > y^0 \\ \hat{B}(y)\hat{A}(x) & \text{at } x^0 < y^0 \end{cases}$$

Equivalently,

$$T[A(x)B(y)] = \Theta(x^0 - y^0)\hat{A}(x)\hat{B}(y) + \Theta(y^0 - x^0)\hat{B}(y)\hat{A}(x)$$

→ The 2^{nd} terms can be now expressed as follow:

$$(-i)^2 \int_{t_0}^{t} dt' dt'' \hat{H}_I(t')\hat{H}_I(t'') \cdot \Theta(t' - t'')$$

?

$$= (-i)^2 \frac{1}{2} \int_{t_0}^{t_0} dt' dt'' \left\{ \hat{H}_I(t')\hat{H}_I(t'')\Theta(t' - t'') + \hat{H}_I(t'')\hat{H}_I(t')\Theta(t'' - t') \right\}$$

$$= (-i)^2 \frac{1}{2} \int_{t_0}^{t} dt' dt'' \, T\left[\hat{H}_I(t')\hat{H}_I(t'')\right]$$

$$= \frac{(-i)^2}{2} T\left[\left(\int_{t_0}^{t} dt' \, \hat{H}_I(t')\right)^2\right]$$

→ Before going for the 3^{rd} term, let's calculate $T[A(t_1)B(t_2)C(t_3)]$:

$$T[A(t_1)B(t_2)C(t_3)] = ABC\,\Theta_{12}\Theta_{23} + ACB\,\Theta_{13}\Theta_{32}$$
$$+ BAC\,\Theta_{21}\Theta_{13} + BCA\,\Theta_{23}\Theta_{31}$$
$$+ CAB\,\Theta_{31}\Theta_{12} + CBA\,\Theta_{32}\Theta_{21}$$

→ Rewriting the 3rd term with the T-product:

$$(-i)^3 \int_{t_0}^{t} dt' \hat{H}_I(t) \int_{t_0}^{t'} dt'' \hat{H}_I(t'') \int_{t_0}^{t''} dt''' \hat{H}_I(t''')$$

$$= (-i)^3 \int_{t_0}^{t} dt' dt'' dt''' \cdot \hat{H}_I(t') \hat{H}_I(t'') \hat{H}_I(t''') \Theta(t'-t'') \Theta(t''-t''')$$

$$= \frac{(-i)^3}{6} \int_{t_0}^{t} dt' dt'' dt''' \, T\left[\hat{H}_I(t') \hat{H}_I(t'') \hat{H}_I(t''') \right]$$

$$= \frac{(-i)^3}{3!} T\left[\left(\int_{t_0}^{t} dt' \hat{H}_I(t') \right)^3 \right]$$

→ Generalization:

$$U(t,t_0) = 1 - i \int_{t_0}^{t} dt' H_I(t') + \frac{(-i)^2}{2!} T\left[\left(\int_{t_0}^{t} dt' \hat{H}_I(t') \right)^2 \right]$$

$$+ \frac{(-i)^3}{3!} T\left[\left(\int_{t_0}^{t} dt' H_I(t') \right)^3 \right] + \dots + \frac{(-i)^n}{n!} \left[\left(\int_{t_0}^{t} dt' H_I(t') \right)^n \right] + \dots$$

$$U(t,t_0) = T\left[\exp\left\{ -i \int_{t_0}^{t} \hat{H}_I(t') dt' \right\} \right]$$

DEF The evolution operator from $-\infty$ to $+\infty$ is called the S-matrix, defined as follow:

$$\hat{S} \equiv \hat{U}(\infty, -\infty) = T\exp\left\{ i \int_{-\infty}^{\infty} dt \, H_I(t) \right\}$$

↳ Notation! Can't take the integral out of the T-product

→ For the fermionic fields, the statistics modify the T-product:

$$T(\psi(x)\chi(y)) = \Theta(x^0 - y^0)\psi(x)\chi(y) - \Theta(y^0 - x^0)\chi(y)\psi(x)$$

3.3 Matrix elements of the S-matrix

→ Once we have an expression for the evolution operator (the S-matrix), we want to calculate amplitudes of physical processes involving evolution of one state into another (decay, scattering, ...).

DEF Let $|i\rangle$ and $|f\rangle$ be an initial and final state in the interaction representation. The amplitudes $\mathcal{A}_{f\leftarrow i}$ is given by

$$\mathcal{A}_{f\leftarrow i} \equiv \langle f|\hat{S}|i\rangle$$

→ Since the vacuum and particle states change when interaction is present, we can't construct these states with the $\hat{a}^\dagger_k$ from the free theory.

↳ We start with something simpler: the vacuum expectation values of T-product of Heisenberg fields, called the Green's functions.

⊙ Two-point Green's function:

→ We consider a two-point Green's function because it's the simplest case: $G(x,y) = \langle\Omega|T[\varphi_H(x)\,\varphi_H(y)]|\Omega\rangle$ with $|\Omega\rangle$ is the vacuum state of the theory ($\neq$ from $|0\rangle$)

→ (See note p. 69) One can show that

$$\langle\Omega|T[\phi_H(x)\,\phi_H(y)]|\Omega\rangle = \frac{\langle 0|T[\phi_I(x)\,\phi_I(y)\,S]|0\rangle}{\langle 0|S|0\rangle}$$

⊙ Generalization:

→ The difference between the true vacuum state $|\Omega\rangle$ and the perturbative vacuum $|0\rangle$ can be accounted for by dividing by the matrix element $\langle 0|S|0\rangle$

→ Amplitudes of processes are given by:

$$\mathcal{A} = \frac{\langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle}{\langle 0 | S | 0 \rangle} \times \left(\text{factors correcting for non-vacuum states} \right)$$

$$= \langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle \Big|_{\text{some diagrams removed}}$$

3.4 Calculation of the matrix element. (Wick thm)

→ Since the S-matrix contains the T-ordered products of the field operators, we need to know how to calculate expectation values of such operators:

$$\langle p_1 \cdots p_n | T[\varphi(x_1) \cdots \varphi(x_r)] | k_1 \cdots k_m \rangle$$

⊙ Green's function: vacuum average of $T[\cdot]$:

DEF The simplest Green's function is called the propagator D_F

$$D_F(x,y) \equiv \langle 0 | T[\varphi(x)\, \varphi(y)] | 0 \rangle$$

→ Remember the free field is:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ \hat{a}_k e^{-ikx} + \hat{a}_k^\dagger e^{ikx} \right\}$$

DEF We separate $\varphi(x)$ into the negative-frequency part $\varphi^-(x)$ and its positive-frequency part $\varphi^+(x)$:

$$\varphi^+(x) \equiv \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}_k e^{-ikx} \longrightarrow \varphi^+|0\rangle = 0$$

$$\varphi^-(x) \equiv \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}_k^\dagger e^{ikx} \longrightarrow \langle 0 | \varphi^- = 0$$

→ With this notation, let's compute:

→ $\varphi(x)\,\varphi(y) = (\varphi_x^+ + \varphi_x^-)(\varphi_y^+ + \varphi_y^-)$

$$= \varphi_x^+\varphi_y^+ + \varphi_x^+\varphi_y^- + \varphi_x^-\varphi_y^+ + \varphi_x^-\varphi_y^-$$

$$= \varphi_x^+\varphi_y^+ + \varphi_y^-\varphi_x^+ + \varphi_x^-\varphi_y^+ + \varphi_x^-\varphi_y^- + [\varphi_x^+, \varphi_y^-]$$

DEF: When the φ^- ($\hat{a}_k^\dagger$ parts) are before the φ^+ ($\hat{a}_k$ parts), it's called a normal order and noted $: \; :$

→ ex: $:\varphi_x\varphi_y: = \varphi_x^+\varphi_y^+ + \varphi_x^-\varphi_y^- + \varphi_x^-\varphi_y^+ + \varphi_y^-\varphi_x^+$

↳ Then we have: $\varphi(x)\varphi(y) = :\varphi(x)\varphi(y): + [\varphi^+(x), \varphi^-(y)]$

→ $\varphi(y)\varphi(x) = :\varphi(y)\varphi(x): + [\varphi^+(y), \varphi^-(x)]$

DEF: Let $\hat{A}, \hat{B}$ be two operators. We define their contraction $\sqcap$ as follow:

$$\overbracket{\hat{A}\,\hat{B}} \equiv \hat{A}\hat{B} - :\hat{A}\hat{B}:$$

↳ With this notation, we have:

$$\overbracket{\varphi_x\varphi_y} = \underbrace{[\varphi_x^+, \varphi_y^-]}_{\in \mathbb{C}} \quad \text{and} \quad \overbracket{\varphi_y\varphi_x} = [\varphi_y^+, \varphi_x^-] \in \mathbb{C}$$

→ We can then write:

$$T[\varphi(x)\varphi(y)] = :\varphi(x)\varphi(y): + \underbrace{\overbracket{\varphi(x)\varphi(y)}}_{\mathbb{C}\text{ number}}$$

→ Coming back to our propagator:

$$\langle 0|T[\varphi_x\varphi_y]|0\rangle = \langle 0|:\varphi(x)\varphi(y):|0\rangle + \overbracket{\varphi(x)\varphi(y)}$$

$$\Leftrightarrow D_F(x-y) = \overbracket{\varphi(x)\varphi(y)} = \begin{cases} [\varphi^+(x), \varphi^-(y)] & \text{at } x^0 > y^0 \\ [\varphi^+(y), \varphi^-(x)] & \text{at } y^0 > x^0 \end{cases}$$

↳ This object plays a fundamental role in perturbative calculations.

⊙ Study of the propagator $D_F(x-y)$:

→ $[\varphi^+(x), \varphi^-(y)]$ (case with $x^0 > y^0$)

$$= \int \frac{d^3k\, d^3q}{(2\pi)^3\, 2\sqrt{\omega_k \omega_q}} \left\{ \hat{a}_k e^{-i\bar{k}\bar{x}} a_{\bar{q}}^+ e^{i\bar{q}\bar{y}} - a_{\bar{q}}^+ e^{i\bar{q}\bar{y}} \hat{a}_k e^{-i\bar{k}\bar{x}} \right\}$$

$$= \int \frac{d^3k\, d^3q}{(2\pi)^3\, 2\sqrt{\omega_k \omega_q}} e^{-i\bar{k}\bar{x} + i\bar{q}\bar{y}} \underbrace{[\hat{a}_k, \hat{a}_q^+]}_{= \delta^3(\bar{k}-\bar{q})}$$

$$= \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} e^{-ik(x-y)}$$

x, y, k are 4 vectors:
$kx = k_0 x_0 - \bar{k}\bar{x} = k_0 x_0 - k_i x_i$

$$= \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \exp\{-ik_0(x_0 - y_0) + i\bar{k}(\bar{x}-\bar{y})\}$$

→ $[\varphi^+(y), \varphi^-(x)]$ (case with $y^0 > x^0$)

$$= \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \exp\{-ik_0(y^0 - x^0) + i\bar{k}(\bar{y}-\bar{x})\}$$

PROP We can write these 2 expressions as a single one valid all times:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)}$$

Proof:

$$D_F(x-y) = \int \frac{d^3k}{(2\pi)^4} \int_{-\infty}^{\infty} dk_0 \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik^0(x^0-y^0) + i\bar{k}(\bar{x}-\bar{y})}$$

$$= \int \frac{d^3k}{(2\pi)^4} i e^{i\bar{k}(\bar{x}-\bar{y})} \int_{-\infty}^{\infty} dk_0 \frac{1}{k_0^2 - \omega_k^2 + i\epsilon} e^{-ik^0(x^0-y^0)}$$

$k^2 - m^2 = k_0^2 - \bar{k}^2 - m^2$
$= k_0^2 - (\bar{k}^2 + m^2)$
$= k_0^2 - \omega_k^2$

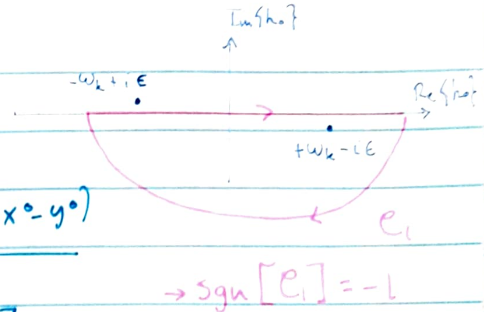
→ $\frac{1}{k_0^2 - \omega_k^2 + i\epsilon}$ has two poles: $k_0 = \pm\sqrt{\omega_k^2 - i\epsilon} = \pm\sqrt{1 - i\epsilon/\omega_k^2}\cdot\omega_k$

$\simeq \pm\omega_k(1 - i\epsilon/2\omega_k^2)$

$\simeq \pm\omega_k \mp i\frac{\epsilon}{2\omega_k^2}$

→ We gonna use the residue theorem to integrate:

$$\oint f(z)\,dz = 2\pi i \sum \mathrm{Res}(f, a_k)$$



→ Case at $x^0 > y^0$:

↳ $e^{-ik^0(x^0-y^0)} \ll 1$ when $\mathrm{Im}\{k_0\} \ll -1$

↳ $$\int_{-\infty}^{\infty} \frac{dk_0 \, e^{-ik_0(x^0-y^0)}}{(k_0-\omega_k+i\epsilon)(k_0+\omega_k-i\epsilon)} = (-2\pi i)\, \frac{e^{-i\omega_k(x^0-y^0)}}{2\omega_k}$$

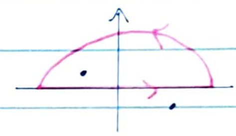
car $$\mathrm{Res}\left[\frac{e^{-ik_0(x^0-y^0)}}{(k_0-\omega_k+i\epsilon)(k_0+\omega_k-i\epsilon)},\ a_k = \omega_k - i\epsilon\right]$$

$$= \lim_{k_0\to\omega_k}\left[\frac{e^{-ik_0(x^0-y^0)}}{k_0+\omega_k-i\epsilon}\right] = \frac{e^{-i\omega_k(x^0-y^0)}}{2\omega_k}$$

↳ We then have $$D_F(x-y) = \int \frac{d^3k}{(2\pi)^4}\, i e^{i\vec{k}(\vec{x}-\vec{y})} \cdot (-2\pi i) \cdot \frac{e^{-i\omega_k(x^0-y^0)}}{2\omega_k}$$

$$= \int \frac{d^3k}{(2\pi)^3 \cdot 2\omega_k}\, e^{-i\omega_k(x^0-y^0)+i\vec{k}(\vec{x}-\vec{y})}$$

→ Case at $y^0 > x^0$:



↳ $$D_F(x-y) = \int \frac{d^3k}{(2\pi)^4}\, i e^{i\vec{k}(\vec{x}-\vec{y})} \int_{-\infty}^{\infty} \frac{dk_0}{(k_0-\omega_k+i\epsilon)(k_0+\omega_k-i\epsilon)}$$

$$= \int \frac{d^3k}{(2\pi)^4}\, i e^{i\vec{k}(\vec{x}-\vec{y})} \cdot (+2\pi i)\, \frac{e^{i\omega_k(x^0-y^0)}}{-2\omega_k}$$

? $$= +\int \frac{d^3k}{(2\pi)^3} \frac{-1}{2\omega_k}\, e^{i\omega_k(x^0-y^0)-i\vec{k}(\vec{x}-\vec{y})} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k}\, e^{-i\omega_k(y^0-x^0)+i\vec{k}(\vec{y}-\vec{x})}$$ ■

⓪ Wick theorem:

→ So far, we calculated $\langle 0|T[\varphi(x)\varphi(y)]|0\rangle = D_F(x-y)$.
How to calculate more complicated ones?

THM The Wick theorem states:

$$T[\varphi(x_1)\cdots\varphi(x_m)] = :\varphi(x_1)\cdots\varphi(x_m) + (\text{all possible contractions}):$$

→ ex: $T[\hat\varphi_1\hat\varphi_2\hat\varphi_3] = :\hat\varphi_1\hat\varphi_2\hat\varphi_3: + :\hat\varphi_1: \overbrace{\varphi_2\varphi_3} + :\hat\varphi_2: \overbrace{\varphi_1\varphi_3} + :\hat\varphi_3: \overbrace{\varphi_1\varphi_2}$

→ When the vacuum average is take, only the complete contractions survive (normal order operators vanish when sandwiched with vacuum)

→ ex: $\langle 0|T[\varphi_1\varphi_2\varphi_3\varphi_4]|0\rangle = 2\varphi_1\varphi_2\,\varphi_3\varphi_4 + 2\,\varphi_1\varphi_2\varphi_3\varphi_4 + 2\,\varphi_1\varphi_2\varphi_3\varphi_4$

Proof (of Wick thm) By induction: assume Wick thm for n-1 field. Without loss of generality, assume $x_1^0 \geq x_2^0 \geq \dots \geq x_n^0$

→ $T[\varphi_1\cdots\varphi_n] = \varphi_1\cdots\varphi_n = (\varphi_1^+ + \varphi_1^-) :\{\varphi_2\cdots\varphi_n + \text{contractions}\}:$

↳ $\varphi_1^+ :\varphi_2\cdots\varphi_n: = :\varphi_2\cdots\varphi_n:\varphi_1^+ + [\varphi_1^+, :\varphi_2\cdots\varphi_n:]$

$= :\varphi_1^+\varphi_2\cdots\varphi_n: + :[\varphi_1^+,\varphi_2^-]\varphi_3\cdots\varphi_n: + :\varphi_2[\varphi_1^+,\varphi_3^-]\varphi_4\cdots\varphi_n:$
$+\dots+ :\varphi_2\cdots\varphi_{n-1}[\varphi_1^+,\varphi_n^-]:$

$= :\varphi_1^+\varphi_2\cdots\varphi_n: + :\varphi_1\varphi_2\varphi_3\cdots\varphi_n: + \dots + :\varphi_2\cdots\varphi_{n-1}\varphi_1\varphi_n:$

→ $T[\varphi_1\cdots\varphi_n] = :(\varphi_1^+ + \varphi_1^-)\varphi_2\cdots\varphi_n + \text{contractions}:$ ∎

⊙ Graphical representation: an example:

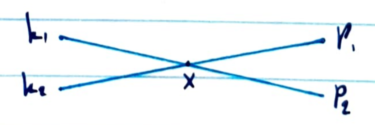
→ We consider the $\lambda\phi^4$ theory. We have:
$\langle p_1p_2|\hat{S}|k_1k_2\rangle = \langle p_1p_2|\left(1 + i\lambda\int d^4x\,\phi_x^4 + \frac{1}{2}(i\lambda)^2\int d^4x\,d^4y\,T(\phi_x^4\phi_y^4) + \dots\right.$
We'll treat the matrix element by order.

↳ Order 0: $\langle p_1p_2|1|k_1k_2\rangle$ just gives δ-functions

↳ Order 1: $\langle p_1p_2|i\lambda\int d^4x\,\varphi_x\varphi_x\varphi_x\varphi_x|k_1k_2\rangle$ contracted in all possible ways:

→ $\langle p_1p_2|\varphi_x\varphi_x\varphi_x\varphi_x|k_1k_2\rangle$

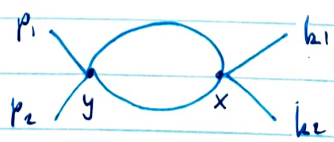
→ $\langle p_1p_2|\varphi_x\varphi_x\varphi_x\varphi_x|k_1k_2\rangle$



↳ gives $i\lambda\int d^4x\left\{e^{-ik_1x}e^{-ik_2x}e^{ip_1x}e^{ip_2x}\right\} = i\lambda(2\pi)^4\,\delta^4(k_1+k_2-p_1-p_2)$

↳ Order 2: $\frac{1}{2}(i\lambda)^2\int d^4x\,d^4y\,\langle p_1p_2|T(\varphi_x\varphi_x\varphi_x\varphi_x\varphi_y\varphi_y\varphi_y\varphi_y)|k_1k_2\rangle$

→ $\langle p_1p_2|\varphi_x\varphi_x\varphi_x\varphi_x\varphi_y\varphi_y\varphi_y\varphi_y|k_1k_2\rangle$



RJ

Rule:

→ Each internal line produces a factor

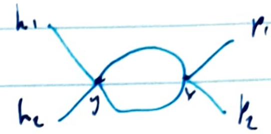
$$\phi(x_1)\,\phi(x_2) = \int \frac{d^4 q_{12}}{(2\pi)^4}\,\frac{i}{q_{12}^2 - m^2 + i\epsilon}\, e^{-iq_{12}(x_1-x_2)} = D_F(x_1-x_2)$$

→ Each external line produces a factor

$$\langle k|\varphi(x) \Longrightarrow e^{ikx} \quad \text{and} \quad \varphi(x)|p\rangle \longrightarrow e^{-ipx}$$

→ For our example, we have:

$\langle p_1 p_2|\, T(\varphi_x\varphi_x\varphi_x\varphi_x\varphi_y\varphi_y\varphi_y\varphi_y)\,|k_1 k_2\rangle$ that gives



$$\frac{1}{2!}(i\lambda)^2 \int d^4x\, d^4y \cdot 2\cdot 4\cdot 3\cdot 4\cdot 3\cdot 2\; e^{-ik_1 y}\, e^{-ik_2 y}\, e^{+ip_1 x}\, e^{+ip_2 x} \;\times$$

$$\cdot \int \frac{d^4 q_1}{(2\pi)^4}\,\frac{i}{q_1^2 - m^2 + i\epsilon}\, e^{-iq_1(x-y)} \cdot \int \frac{d^4 q_2}{(2\pi)^4}\,\frac{i}{q_2^2 - m^2 + i\epsilon}\, e^{-iq_2(x-y)}$$

$$= \frac{(4!)^2}{2!}(i\lambda)^2 \int \frac{d^4 q_1}{(2\pi)^4}\frac{d^4 q_2}{(2\pi)^4} \Big\{ (2\pi)^4 \delta^4(p_2+p_1-q_1-q_2)\cdot(2\pi)^4\delta^4(q_1+q_2-k_1-k_2)$$

$$\times \frac{i}{q_1^2 - m^2 + i\epsilon} \cdot \frac{i}{q_2^2 - m^2 + i\epsilon} \Big\}$$

$$= \frac{(4!)^2}{2}(i\lambda)^2 (2\pi)^4 \delta^4(p_2+p_1-k_1-k_2) \int \frac{d^4 q_1}{(2\pi)^4}\, \frac{i}{q_1^2 - m^2 + i\epsilon}\, \frac{i}{(k_1+k_2-q_1)^2 - m^2 + i\epsilon}$$

↳ It diverges!

Rules (Feynman)

① : Draw all possible diagrams

② : Vertex → $(-i\lambda)(2\pi)^4\,\delta^4(\sum k_i)$

③ : Each internal line: $\int d^4k/(2\pi)^4 \,.\; i/k^2 - m^2 + i\epsilon$

④ : Each external line: 1

3.5 Desintegration of a massive particle

⊙ From the S-matrix to probabilities:

→ Normalization of states $|k\rangle$: how many particles does one $|k\rangle$ contain?

$$\langle k|\hat{N}|k\rangle = \langle k|k\rangle$$
$$= \langle 0|(2\pi)^{3/2}\sqrt{2\omega_k}\,\hat{a}_k \cdot (2\pi)^{3/2}\sqrt{2\omega_k}\,\hat{a}_k^\dagger|0\rangle$$
$$= (2\pi)^3\, 2\omega_k\, \delta^3(k-k) = (2\pi)^3\, 2\omega_k\, \delta^3(0)$$

↳ Meaning of delta (0)?

$$\rightarrow \delta^3(k) = \int \frac{d^3x}{(2\pi)^3} e^{ikx}$$

$$\rightarrow \delta^3(0) = \int \frac{d^3x}{(2\pi)^3} = \frac{V}{(2\pi)^3}$$

↳ We get $\langle k|k\rangle = 2\omega_k V$, it mean that our states contain $2\omega_k V$ particles.

→ The state containing 1 particle is

$$|\varphi_k\rangle \equiv \frac{1}{\sqrt{2\omega_k}\sqrt{V}}|k\rangle$$

↳ We can create a state with 2 ≠ particles:

$$|\varphi_{p_1}\varphi_{p_2}\rangle = \frac{1}{\sqrt{2\omega_{p_1}V}}\frac{1}{\sqrt{2\omega_{p_2}V}}|p_1 p_2\rangle$$

→ From QM, the amplitude of $\varphi_k \rightarrow \varphi_{p_1} + \varphi_{p_2}$ is given by:

$$\langle \varphi_{p_1}\varphi_{p_2}|S|\varphi_k\rangle$$

and the probability is $|\langle\varphi_{p_1}\varphi_{p_2}|S|\varphi_k\rangle|^2$

↳ But we have a problem:

$$|\langle\varphi_{p_1}\varphi_{p_2}|S|\varphi_k\rangle|^2 = |\langle p_1 p_2|S|k\rangle|^2 \frac{(2\omega_{p_1})^{-1}}{V}\frac{(2\omega_{p_2})^{-1}}{V}\frac{(2\omega_k)^{-1}}{V} \xrightarrow[V\to\infty]{} 0$$

→ We need to look at the probability to decay in a state with momenta of particles in the intervals $d\vec{p}_1$ and $d\vec{p}_2$

From QM, the # of state in d^3p is:

$$\frac{d^3p \cdot V}{(2\pi)^3}$$

↳ The probability of decay in a state with momenta $[\bar{p}_1 ; \bar{p}_1 + \Delta\bar{p}_1]$, $[\bar{p}_2, \bar{p}_2 + \Delta\bar{p}_2]$ is

$$dP = |\langle \varphi_{p_1} \varphi_{p_2} | S | \varphi_k \rangle|^2 \cdot \frac{d\bar{p}_1 V}{(2\pi)^3} \cdot \frac{d\bar{p}_2 V}{(2\pi)^3}$$

$$= \frac{1}{2\omega_k \cdot V} \frac{d^3 p_1}{(2\pi)^3 2\omega_{p_1}} \cdot \frac{d^3 p_2}{(2\pi)^3 2\omega_{p_2}} |\langle p_1 p_2 | S | k \rangle|^2$$

→ Since we know that $|\langle p_1 p_2 | S | k \rangle|^2$ always contain the factor $(2\pi)^4 \delta^4(\Sigma k - \Sigma p)$ to ensure energy-momentum conservation, we can write:

$$\langle p_1 \cdots p_n | \hat{S} | k_1 \cdots k_n \rangle \equiv i \, (2\pi)^4 \delta^4(\Sigma k - \Sigma p) . \mathcal{M}(p_1 \dots k_n)$$

↳ We have then:

$$|\langle p_1 p_2 | \hat{S} | k \rangle|^2 = (2\pi)^8 \delta^4(k - p_1 - p_2) \delta^4(k - p_1 - p_2) |\mathcal{M}|^2$$

$$= (2\pi)^8 \delta^4(k - p_1 - p_2) \delta^4(0) |\mathcal{M}|^2$$

$\delta^3(0) = \frac{V}{(2\pi)^3}$

$$= (2\pi)^8 \delta^4(k - p_1 - p_2) \frac{V.T}{(2\pi)^4} |\mathcal{M}|^2$$

$$= (2\pi)^4 \, V.T. \, \delta^4(k - p_1 - p_2) |\mathcal{M}|^2$$

→ Finaly, we have:

PROP

$$d\Gamma \equiv \frac{dP}{T} = (2\pi)^4 \delta^4(k - \Sigma p) \frac{1}{2\omega_k} |\mathcal{M}|^2 \cdot \frac{d^3 p_1}{(2\pi)^3 2\omega_{p_1}} \frac{d^3 p_2}{(2\pi)^3 2\omega_{p_2}}$$

where Γ is the decay rate

⊙ Interpretation of Γ:

→ Γ is the probability of decay in a unit time.

→ If you take a large number of particules N, then their number depends on time according to:

$$N(t) = N_0 e^{-\Gamma t} = N_0 e^{-t/\tau}$$

↳ We see that

$$\Gamma = \frac{1}{\tau}$$

⊙ Total decay rate:

→ Setting $\omega_k = M$, we have:

$$\Gamma = \int \frac{dP}{T} = \frac{1}{8\pi^2 M} \int \frac{d^3p_1\, d^3p_2}{2\omega_{p_1}\, 2\omega_{p_2}} |\mathcal{M}|^2\, \delta^4(k - p_1 - p_2)$$

Using $\delta^4(k - p_1 - p_2) = \delta(M - \omega_{p_1} - \omega_{p_2}) \cdot \delta^3(\vec{p}_1 + \vec{p}_2)$, we integrate:

$$\Gamma = \frac{1}{32\pi^2 M} \int \frac{d^3p_1}{\omega_1 \omega_2} |\mathcal{M}|^2\, \delta(M - \omega_1 - \omega_2)$$

with $\omega_i = \sqrt{m_i^2 + p_i^2}$

→ We use $\int dx\, f(x)\, \delta[g(x)] = \sum_{\text{root}} \frac{f(x_i)}{|g'(x_i)|}$

Here, $g(p_1) = M - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}$

$\hookrightarrow \partial_{p_1} g = -\frac{p_1}{\omega_1} - \frac{p_1}{\omega_2} = -p_1 \frac{(\omega_1 + \omega_2)}{\omega_1 \omega_2}$

$$\frac{1}{|g'|} = \frac{\omega_1 \omega_2}{p_1(\omega_1 + \omega_2)} = \frac{\omega_1 \omega_2}{p_1 M}$$

Where did the $\delta(x)$ go? + Why Here $= |\langle f|S|i\rangle|^2$

→ Going in spherical coord., we have: $d^3p = p^2\, dp\, d\Omega$

$$d\Gamma = \frac{1}{32\pi^2 M} |\mathcal{M}|^2 \frac{p_1^2\, dp_1\, d\Omega}{\omega_1 \omega_2} \cdot \frac{\omega_1 \omega_2}{p_1 M}$$

$$d\Gamma = \frac{p_1\, d\Omega}{32\pi^2 M^2} |\mathcal{M}|^2$$

→ Case where $m_1 = m_2$, we have:

$M = 2\omega = 2\sqrt{m^2 + p_1^2} \Leftrightarrow M^2 = 4(m^2 + p_1^2) \Rightarrow p_1 = \sqrt{\frac{M^2}{4} - m^2}$

$\hookrightarrow d\Gamma = \frac{1}{32\pi^2 M^2} \cdot \sqrt{\frac{M^2}{4} - m^2}\, d\Omega\, |\mathcal{M}|^2$

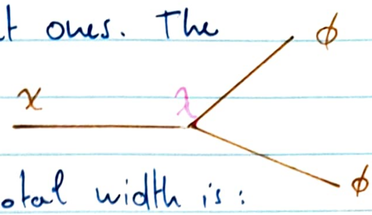
$$= \frac{1}{64\pi^2 M} \sqrt{1 - \frac{4m^2}{M^2}} |\mathcal{M}|^2\, d\Omega$$

$\hookrightarrow$ If $|\mathcal{M}|$ is not dependent on the angle, $\int d\Omega \to 4\pi$. We get

$$\Gamma = \frac{1}{16\pi M} \sqrt{1 - \frac{4m^2}{M^2}} |\mathcal{M}|^2$$

⊙ Particular examples:

→ Simplest case: heavy scalar into 2 light ones. The interaction term in this case is $\lambda \chi \phi^2$



↳ We have $|\mathcal{M}|^2 = \lambda^2$. Therefore, the total width is:

$$\Gamma = \frac{\lambda^2}{32\pi M_\chi} \sqrt{1 - \frac{4m_\phi^2}{M_\chi^2}} \simeq \frac{\lambda^2}{32\pi M_\chi}$$

3.6 Scattering and Cross section

DEF For a single target, the cross section σ is the area in the plane perpendicular to the incoming flux from wich the particles are scattered.

→ For a beam of particles with density n_A and velocity v_A, the # of event in a time interval T is

$$N_{ev} = n_A \cdot v_A \cdot \sigma T$$

$$\Leftrightarrow \quad \sigma = \frac{N_{ev}}{n_A \cdot v_A \cdot T}$$

→ Let's consider a normalized initial state $|\varphi_A \varphi_B\rangle$

$$|\varphi_A \varphi_B\rangle = \frac{1}{\sqrt{2\omega_A V}} \frac{1}{\sqrt{2\omega_B V}} |k_A k_B\rangle = \frac{(2\pi)^{3/2}}{\sqrt{V}} \frac{(2\pi)^{3/2}}{\sqrt{V}} a^\dagger_{k_A} a^\dagger_{k_B} |0\rangle$$

↳ $N_{ev} = |\langle f | \hat{S} | \varphi_A \varphi_B\rangle|^2 = \dfrac{|\langle f | \hat{S} | k_A k_B\rangle|^2}{2\omega_A V \cdot 2\omega_B V}$

↳ Since it's normalized (1 particle per volume V), $n_A = 1/V$

↳ We get: $\sigma = \dfrac{|\langle f | \hat{S} | k_A k_B\rangle|^2}{4\omega_A \omega_B V^2} \cdot \dfrac{V}{v_A \cdot T} = \dfrac{|\langle f | S | k_A k_B\rangle|^2}{4\omega_A \omega_B v_A T} \cdot \dfrac{1}{V}$ Not well defined.

→ We need to consider the probability of the transition into a state with momenta between $\vec{p}_i$ and $\vec{p}_i + d\vec{p}_i$. Then

$$d\sigma = \frac{|\langle p_1 \cdots p_n | \hat{S} | k_A k_B\rangle|^2}{2\omega_1 \cdots 2\omega_n \cdot 2\omega_A \cdot 2\omega_B} \cdot \frac{1}{v_A V T} \cdot \frac{d\vec{p}_1}{(2\pi)^3} \cdots \frac{d\vec{p}_n}{(2\pi)^3}$$

↳ Since our matrix element always contain a factor δ^4, we can write

$$\langle p_1 \cdots p_n | \hat{S} | k_A k_B \rangle = i \underbrace{(2\pi)^4}_{\text{choice}} \delta^4(\Sigma k - \Sigma p) \mathcal{M}(p_1 \cdots k_B)$$

$\delta^4(0) = \frac{VT}{(2\pi)^4}$

Then $|\langle p_1 \cdots p_n | \hat{S} | k_A k_B \rangle|^2 = (2\pi)^8 \delta^4(\Sigma k - \Sigma p)\, \delta^4(\Sigma k - \Sigma p)\, |\mathcal{M}|^2$

$$= (2\pi)^4 \delta^4(\Sigma k - \Sigma p)\, V.T\, |\mathcal{M}|^2$$

→ We find: $d\sigma = \frac{|\mathcal{M}|^2 (2\pi)^4 \delta^4(\Sigma p - \Sigma k)}{4 \omega_A \omega_B v_A} . \frac{d^3 p_1}{(2\pi)^3 2\omega_1} \cdots \frac{d^3 p_n}{(2\pi)^3 2\omega_n}$

→ In the frame when B is at rest:

$k_A = (\omega_A, \vec{k}_A)$; $k_B = (m_B, 0)$; $k_A k_B = \omega_A m_B = \omega_A \omega_B$

∇ Then, $(k_A k_B)^2 - m_A^2 m_B^2 = \omega_A^2 \omega_B^2 - m_A^2 m_B^2$

$$= m_B^2 (\omega_A^2 - m_A^2) = m_B^2 m_A^2 \left(\frac{1}{1 - v_A^2} - 1 \right)$$

$$= m_B^2 m_A^2 \frac{v_A^2}{1 - v_A^2} = m_B^2 E_A^2 v_A^2$$

We have

$$d\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(k_A k_B)^2 - m_1^2 m_2^2}} (2\pi)^4 \delta^4(\Sigma k - \Sigma p) \frac{d^3 p_1}{(2\pi)^3 2\omega_1} \cdots \frac{d^3 p_n}{(2\pi)^3 2\omega_n}$$