→ Let's consider now a vector field. A^μ

↳ We still have $M^{\mu}{}_{\lambda\rho} = \delta^\mu_\lambda x_\rho - \delta^\mu_\rho x_\lambda$

↳ $\delta A_\mu = \omega_{\mu\nu} A^\nu$

$$\equiv \sum_{\sigma<\rho} \Psi_{\mu\,\sigma\rho}\, \omega^{\sigma\rho}$$ ("$\mu\sigma\rho$" ↔ "ia")

? One reads: $\Psi_{\mu\sigma\rho} = \eta_{\mu\sigma} A_\rho - \eta_{\mu\rho} A_\sigma$

Thus,

$$M^\mu{}_{\lambda\rho} = M^{(0)\mu}{}_{\lambda\rho} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \Psi_{\sigma\lambda\rho}$$

$$= \underbrace{M^{(0)\mu}{}_{\lambda\rho}}_{\text{orbital momentum}} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \left(\eta_{\sigma\lambda} A_\rho - \eta_{\sigma\rho} A_\lambda\right)$$

DEF The spin is $S^\mu{}_{\lambda\rho} = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \left(\eta_{\sigma\lambda} A_\rho \eta_{\sigma\rho} A_\lambda\right) \right\}$

or $S_{\lambda\rho} = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial \dot{A}^\rho} A_\lambda - \frac{\partial \mathcal{L}}{\partial \dot{A}^\lambda} \cdot A_\rho \right\}$

And the vector of spin is $\lambda_i \equiv \epsilon_{ijk} S_{jk}$

$\boxed{2}$ QUANTIZATION OF A FREE SCALAR FIELD

2.1 Reminder: quantization in QM

→ We use the Lagrangian of an harmonic oscillator:

$$L = \frac{1}{2} m \dot{\tilde{q}}^2 - \frac{1}{2} k \tilde{q}^2$$

$\begin{cases} q - \text{position} \\ \dot{q} - \text{velocity} \end{cases}$

↳ $S = \int L\,dt = \int dt \left\{ \frac{1}{2} m \dot{\tilde{q}}^2 - \frac{1}{2} k \tilde{q}^2 \right\}$

↳ We rescale our variables: $q \equiv \sqrt{m}\, \tilde{q}$ and write $\omega^2 = k/m$

Then: $$S = \int dt \left\{ \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2 \right\}$$

→ Equations of motion (classical):

$$\delta S = 0$$

$$\delta S = \int dt \{ \dot{q}\,\delta\dot{q} - \omega^2 q\,\delta q \} = \int dt \{ \ddot{q} - \omega^2 q \} \delta q = 0$$

We find $-\ddot{q} - \omega^2 q = 0$ with ω a frequency. Solutions given by: $q = A e^{i\omega t} + A^* e^{-i\omega t} \in \mathbb{R}$

③ <u>Quantization:</u>

→ We have, classically: $p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$

We introduce operators $\hat{p}$ and $\hat{q}$ wich satisfy the following relation: $[\hat{q}, \hat{p}] = i \quad (\hbar = 1)$

These operators act in the Hilbert space

↳ Energy operator: $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2 \hat{q}^2$

↳ Schrödinger équation:

$$i\partial_t \psi = \hat{H}\psi$$

So wave functions are $|\psi(t)\rangle = U(t)|\psi(0)\rangle$
$= \exp(-i\hat{H}t)|\psi(0)\rangle$

→ We introduce the following operators:

$$\hat{a} \equiv \frac{\omega\hat{q} + i\hat{p}}{\sqrt{2\omega}} \qquad \hat{a}^+ \equiv \frac{\omega\hat{q} - i\hat{p}}{\sqrt{2\omega}}$$

↳ $[\hat{a}, \hat{a}^+] = \frac{1}{2\omega}\left(-i\omega(+i) + i\omega(-i)\right) = 1$

↳ Inversely, $\hat{q} = \frac{\hat{a} + \hat{a}^+}{\sqrt{2\omega}}$ and $\hat{p} = \frac{\hat{a} - \hat{a}^+}{i}\sqrt{\frac{\omega}{2}}$. The Hamiltonian is $\hat{H} = \omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$

→ The states of the system:

$|0\rangle$: $\hat{a}|0\rangle = 0 \qquad E = \omega/2$

$\hat{a}^+|0\rangle = |1\rangle \qquad E = (\omega + 2\omega)/2$

$\frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle = |n\rangle \rightarrow n\text{-particule state}$

→ Heisenberg picture: $\hat{q}_H = e^{i\hat{H}t}\hat{q}_S e^{-i\hat{H}t}$

↳ $\hat{q}(t) = \frac{1}{\sqrt{2\omega}}\left(\hat{a}e^{-i\omega t} + \hat{a}^+ e^{i\omega t}\right) \qquad // \; q(t) = A e^{-i\omega t} + A^* e^{i\omega t}$

↳ $\hat{p}(t) = \frac{1}{\sqrt{2\omega}}\left(-i\omega\hat{a}e^{-i\omega t} + i\omega\hat{a}^+ e^{i\omega t}\right)$

2.2 Quantization of a real scalar field

→ The action is $S = \int d^4x \, 1/2 \{ \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \}$. The eqs of motion are $-\partial_\mu \partial^\mu \phi - m^2 \phi = 0$

↳ The energy-momentum tensor:

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \eta_{\mu\nu} L \quad \text{with } \eta = \text{diag}(1,-1,-1,-1)$$

The energy is

$$P_0 = \int d^3x \, T_{00} = \int d^3x \{ \dot\phi^2 - L \} = \int d^3x \{ \partial_0 \phi \partial_0 \phi - \tfrac{1}{2} \partial_\mu \phi \partial_\mu \phi + \tfrac{1}{2} m^2 \phi^2 \}$$
$$= \int d^3x \{ \tfrac{1}{2} (\partial_0 \phi)^2 + \tfrac{1}{2} (\partial_i \phi)^2 + \tfrac{1}{2} m^2 \phi^2 \}$$

↳ Canonical formalism: $H = \int d^3x \left\{ \frac{1}{2} \pi^2(x) + \frac{1}{2} (\partial_i \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$

⊙ Canonical quantization:

→ We introduce the operators $\hat\pi(x)$, $\hat\phi(x)$, with relations

$$[\hat\pi(x), \hat\phi(x)] = -i \, \delta^3(x - x')$$

↳ We recall that $\delta^3(\bar x - \bar x') = \int d^3k / (2\pi)^3 \, . \, e^{ik(\bar x - x')}$

We consider the classical eqs of motion, and try to solve them through Fourier transformation.

$$\to \phi(\bar x, t) = \int d^3k \, e^{ik\bar x} \tilde\phi(k, t) \quad \text{in } (-\partial_0^2 + \partial_i^2 - m^2)\phi = 0$$

$$\int d^3k \, e^{ik\bar x} \{ -\ddot{\tilde\phi} - (k^2 + m^2) \tilde\phi \} \rightsquigarrow \tilde\phi \propto e^{\pm i\omega t}$$
$$\tilde\phi = A_k e^{-i\omega t} + A^*(k) e^{i\omega t}$$

$$\to \phi(\bar x, t) = \int \underbrace{\frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}}}_{} \{ e^{-i\omega t + ik\bar x} a_k + e^{i\omega t - ik\bar x} a_k^* \}$$

constante pour la relation de commutation (voir ci-dessous)

→ We guess the following expression:

$$\boxed{\begin{aligned} \hat\phi(\bar x) &= \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \{ e^{ik.\bar x} \hat a(k) + e^{-ik\bar x} \hat a^+(k) \} \\ \hat\pi(\bar x) &= \int \frac{d^3k}{(2\pi)^{3/2}} (-i) \sqrt{\frac{\omega_k}{2}} \{ e^{ik\bar x} \hat a(k) - e^{-ik\bar x} \hat a^+(k) \} \end{aligned}}$$

↳ We need to check the commutation relation for $\hat\pi$ and $\hat\phi$ assuming $[\hat a(k'), \hat a^+(k)] = \delta^3(k - k')$

$$[\hat{\phi}(\bar{x}), \hat{\pi}(\bar{y})] = \int \frac{d^3k\, d^3q\, (-i\omega_q)}{(2\pi)^3\sqrt{2\omega_k\, 2\omega_q}} \left[\hat{a}_k e^{i\bar{k}\bar{x}} + \hat{a}_k^{\dagger} e^{-i\bar{k}\bar{x}}, \hat{a}_q e^{i\bar{q}\bar{y}} - \hat{a}_q^{\dagger} e^{-i\bar{q}\bar{y}} \right]$$

$$= \int \frac{d^3k\, d^3q\, (i\omega_q)}{(2\pi)^3\sqrt{2\omega_k\, 2\omega_q}} \left\{ -\underbrace{[\hat{a}_k, \hat{a}_q]}_{\delta^3(\bar{k}-\bar{q})} e^{i\bar{k}\bar{x} - i\bar{q}\bar{y}} + \underbrace{[\hat{a}_k^{\dagger}, \hat{a}_q]}_{-\delta^3(\bar{k}-\bar{q})} e^{-i\bar{k}\bar{x} + i\bar{q}\bar{y}} \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3 \cdot \cancel{2}} (+i) \underbrace{\sqrt{\frac{\omega_k}{\omega_k}}}_{1} \cdot \cancel{2}\, e^{i\bar{k}(\bar{x}-\bar{y})} = +i\, \delta^3(\bar{x}-\bar{y})$$ ok!

$\phi = \int d\mu \{ e^{ikx} \cdot \hat{a}_k + e^{-ikx} \cdot \hat{a}_k^{\dagger} \}$

$\hat{H} = \int d^3x \{ \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\partial_i \phi)^2 + \frac{1}{2} m^2 \phi^2 \}$:

→ Calculation of the hamiltonian

$$\hat{H} = \int \frac{1}{2} d^3x \frac{d^3k\, d^3p}{(2\pi)^3\, 2\sqrt{\omega_k \omega_p}} \Big\{$$

$$\cancel{\hat{a}_k \hat{a}_p \left((-i\omega_k)(-i\omega_p) - \vec{k}\cdot\vec{p} + m^2\right)} e^{i\bar{k}\bar{x} + i\bar{p}\bar{x}} \quad \delta(\bar{k}+\bar{p}) \to \underbrace{-\omega_k^2 + k^2 + m^2}_{=0}$$

$$+ \cancel{\hat{a}_k^{\dagger} \hat{a}_p^{\dagger} \left((-i\omega_k)(-i\omega_p) - \vec{k}\cdot\vec{p} + m^2\right)} e^{-i\bar{k}\bar{x} - i\bar{p}\bar{x}} \quad \delta(\bar{k}+\bar{p}) \to \underbrace{-\omega_k^2 + k^2 + m^2}_{=0}$$

$$+ \hat{a}_k \hat{a}_p^{\dagger} \left(\omega_k \omega_p + \bar{k}\bar{p} + m^2 \right) e^{i\bar{k}\bar{x} - i\bar{p}\bar{x}} \quad \delta(\bar{k}-\bar{p})$$

$$+ \hat{a}_k^{\dagger} \hat{a}_p \left(\omega_k \omega_p + \bar{k}\bar{p} + m^2 \right) e^{-i\bar{k}\bar{x} + i\bar{p}\bar{x}} \quad \delta(\bar{k}-\bar{p}) \Big\}$$

$$= \int d^3k \frac{1}{4\omega_k} \cdot 2\omega_k^2 \left\{ \hat{a}_k \hat{a}_k^{\dagger} + \hat{a}_k^{\dagger} \hat{a}_k \right\}$$

$$= \int d^3k \frac{\omega_k}{2} \left\{ 2 \hat{a}_k^{\dagger} \hat{a}_k + [\hat{a}_k, \hat{a}_k^{\dagger}] \right\}$$ $= 1$, ignore it for now

DEF We find $\hat{H} = \int d^3k\, \omega_k\, \hat{a}_k^{\dagger} \hat{a}$

→ Calculation of the momentum operator:

$\hat{P}_i = \int d^3x\, \hat{T}_{0i} = \int d^3x\, \partial_0 \phi\, \partial_i \phi = \int d^3x\, \hat{\pi}(x)\, \partial_i \hat{\phi}(x)$

Skipping the algebra, we find:

DEF $\hat{\vec{P}} = \int d^3k \cdot \vec{k} \cdot \hat{a}_k^{\dagger} \hat{a}_k$

⊙ Interpretation of the results:

① : Hamiltonian = "sum" of Hamiltonians of harmonic oscillators

② : The vaccum state is defined such that:

$$\hat{a}_k |0\rangle = 0 \quad \forall k$$

We then find that $\hat{H}|0\rangle = \hat{\vec{P}}|0\rangle = 0$

③ Consider the state $|k\rangle \equiv \hat{a}^\dagger_k |0\rangle$. Then:

$$\begin{aligned}\hat{H}|k\rangle &= \int d^3p\, \omega_p\, \hat{a}^\dagger_p \hat{a}_p \hat{a}^\dagger_k |0\rangle \\ &= \int d^3p\, \omega_p\, \hat{a}^\dagger_p [\hat{a}_p, \hat{a}^\dagger_k] |0\rangle + 0 \\ &= \int d^3p\, \omega_p\, \hat{a}^\dagger_p\, \delta^3(\vec{p}-\vec{k}) |0\rangle \\ &= \omega_k \hat{a}^\dagger_k |0\rangle = \omega_k |k\rangle\end{aligned}$$

Similarly, $\hat{\vec{P}}|k\rangle = \vec{k}|k\rangle$

Prop:
↳ Operator $\hat{a}^\dagger_k$ creates a state of given energy $\omega_k = \sqrt{k^2+m^2}$ and momentum $\vec{k}$

DEF ④ We define the Heisenberg operator of field by

$$\hat{\phi}(x,t) \equiv e^{i\hat{H}t} \phi(x) e^{-i\hat{H}t} = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ e^{-i\omega t + i\vec{k}\vec{x}} \hat{a}_k + e^{i\omega t - i\vec{k}\vec{x}} \hat{a}^\dagger_k \right\}$$

Pf: Let's show it by step:

→ $[\hat{H}, \hat{a}_k] = -\omega_k \hat{a}_k = \hat{H}\hat{a}_k - \hat{a}_k \hat{H}$ $\qquad \hat{H} = \int d^3k\, \omega_k \hat{a}^\dagger_k \hat{a}_k$

$\Leftrightarrow \quad \hat{H}\hat{a}_k = \hat{a}_k(\hat{H} - \omega_k)$

→ $\hat{H}^n a_k = \hat{a}_k (\hat{H}-\omega_k)^n \rightsquigarrow f(\hat{H})\hat{a}_k = \hat{a}_k f(\hat{H}-\omega_k)$

→ $e^{i\hat{H}t} \hat{a}_k e^{-i\hat{H}t} = a_k e^{i(\hat{H}-\omega_k)t} e^{i\hat{H}t} = a_k e^{-i\omega_k t}$ ▨

→ The hamitonian generates time translations

$$e^{i\hat{H}t} \hat{\phi}(\vec{x}) e^{-i\hat{H}t} = \phi(\vec{x}, t)$$

→ The momentum generates space translations:

$$e^{i\bar{a}\hat{P}} \hat{\phi}(x) e^{-i\bar{a}\hat{P}} = \hat{\phi}(\bar{x} + \bar{a})$$

2.3 Complex scalar field

→ The action of the free complex scalar field reads:

$$S = \int d^4x \left\{ \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right\}$$

→ The equations of motion become:

$$-\partial_\mu \partial^\mu \phi - m^2 \phi = 0$$ 2 eqs, 1 for real, 1 for imaginary

→ General solution:

$$\phi(\bar{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ a_k e^{-i\omega_k t + i\bar{k}\bar{x}} + b_k^* e^{+i\omega_k t - i\bar{k}\bar{x}} \right\}$$

with a_k and b_k not conjugated! ⇝ More freedom.

↳ Unlike the real scalar field, $(a_k)^* \neq b_k^*$ ⇒ 4 real functions of k (as it should be for 2 2nd-order differential equations).

⊙ Canonical quantization:

Find momenta conjugate to canonical coordinates $\phi(x)$ and $\phi^*(x)$, then impose commutation relations.

→ We define $\pi(\bar{x}) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}(\bar{x})} = \dot{\phi}^*(x)$ and $\pi^* = \frac{\delta \mathcal{L}}{\delta \dot{\phi}^*(\bar{x})} = \dot{\phi}(x)$

→ Commutation relations are:

$[\hat{\phi}(\bar{x}), \hat{\pi}(\bar{y})] = i\,\delta^3(\bar{x}-\bar{y})$ $\quad [\hat{\phi}, \hat{\pi}^*] = [\hat{\phi}^*, \hat{\pi}] = 0$

$[\hat{\phi}^*(\bar{x}), \hat{\pi}^*(\bar{y})] = i\,\delta^3(\bar{x}-\bar{y})$ $\quad [\hat{\phi}, \hat{\phi}^*] = [\hat{\pi}, \hat{\pi}^*] = 0$

→ We introduce creation and annihilation operators:

$$\hat{\phi}(\bar{x}) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ e^{i\bar{k}\bar{x}} \hat{a}_k + e^{-i\bar{k}\bar{x}} \hat{b}_k^+ \right\}$$

$$\hat{\phi}^*(\bar{x}) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ e^{i\bar{k}\bar{x}} \hat{b}_k + e^{-i\bar{k}\bar{x}} \hat{a}_k^+ \right\}$$

For the conjugate momenta, we start from

$$\pi(\bar{x}) = \dot{\phi}^*(\bar{x}) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ i\omega\, e^{i\omega t - i\bar{k}\bar{x}} a_k^* + (-i\omega)\, e^{-i\omega t + i\bar{k}\bar{x}} b_k \right\}$$

We find:

$$\hat{\pi}(\bar{x}) = \int \frac{d^3k \cdot i\omega_k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ -e^{i\bar{k}\bar{x}}\, \hat{b}_k + e^{-i\bar{k}\bar{x}}\, \hat{a}_k^\dagger \right\}$$

$$\hat{\pi}^*(\bar{x}) = \int \frac{d^3k \cdot i\omega_k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left\{ -e^{i\bar{k}\bar{x}}\, \hat{a}_k + e^{-i\bar{k}\bar{x}}\, \hat{b}_k^\dagger \right\}$$

→ We have 2 sets of creation and annihilation operators, such that

$$[\hat{a}_{\bar{k}}, \hat{a}_{\bar{q}}^\dagger] = \delta^3(\bar{k}-\bar{q}) \qquad [\hat{b}_{\bar{k}}, \hat{b}_{\bar{q}}^\dagger] = \delta^3(\bar{k}-\bar{q})$$

(All the others are equal to 0).

→ We have 2 types of particules!

→ Hamiltonian and momentum:

$$\hat{H} = \int d^3k\, \omega_k \left\{ \hat{a}_k^\dagger \hat{a}_k + \hat{b}_k^\dagger \hat{b}_k \right\}$$

$$\hat{\vec{P}} = \int d^3k\, \vec{k} \left\{ \hat{a}_k^\dagger \hat{a}_k + \hat{b}_k^\dagger \hat{b}_k \right\}$$

→ ex: $\hat{H}\, \hat{a}_k^\dagger |0\rangle = \omega_k\, \hat{a}_k^\dagger |0\rangle$; $\hat{H}\, \hat{b}_k^\dagger |0\rangle = \omega_k\, \hat{b}_k^\dagger |0\rangle$

$\hat{\vec{P}}\, \hat{a}_k^\dagger |0\rangle = \vec{k}\, \hat{a}_k^\dagger |0\rangle$; $\hat{\vec{P}}\, \hat{b}_k^\dagger |0\rangle = \vec{k}\, \hat{b}_k^\dagger |0\rangle$

⊙ Interpretation of $\hat{a}_k^\dagger$ and $\hat{b}_k^\dagger$:

→ Recall: $\mathcal{L}$ is invariant under transformations $\phi \mapsto e^{i\alpha}\phi$ ($U(1)$)

The current J_μ and the charge Q associated with this symmetry are: $J_\mu = -i\,(\partial_\mu \phi^* \cdot \phi - \phi^* \partial_\mu \phi)$

and

$$Q = \int d^3x\, J_0 = -i \int d^3x \,(\dot{\phi}^* \cdot \phi - \phi^* \dot{\phi})$$

$$= -i \int d^3x \,(\pi \phi - \phi^* \pi^*)$$

→ In terms of operators, we compute:

$$\hat{Q} = -i\int d^3x \; \frac{d^3k \; d^3q}{(2\pi)^3 \, 2\sqrt{\omega_k \omega_q}} \times$$

$\pi\phi$

$$\Big\{ i\omega_k \left(-e^{ik\cdot x}\hat{b}_k + e^{-ik\cdot x}\hat{a}_k^+\right)\left(e^{iq\cdot x}\hat{a}_q + e^{-iq\cdot x}\hat{b}_q^+\right)$$

$-\phi^*\pi^*$

$$-i\omega_q \left(e^{ik\cdot x}\hat{b}_k + e^{-ik\cdot x}\hat{a}_k^+\right)\left(-e^{iq\cdot x}\hat{a}_q + e^{-iq\cdot x}\hat{b}_q^+\right)\Big\}$$

$$= -i\int d^3x \; \frac{d^3k \; d^3q}{(2\pi)^3 \, 2\sqrt{\omega_k\omega_q}} \Big\{ i\omega_k \Big(-\hat{b}_k\hat{a}_q e^{+i(k+q)x} - \hat{b}_k\hat{b}_q^+ e^{+i(k-q)x}$$

$$+ \hat{a}_k^+\hat{a}_q e^{-i(k-q)x} + \hat{a}_k^+\hat{b}_q^+ e^{-i(k+q)x}\Big) - i\omega_q\Big(-\hat{b}_k\hat{a}_q e^{i(k+q)x}$$

$$+ \hat{b}_k\hat{b}_q^+ e^{i(k-q)x} - \hat{a}_k^+\hat{a}_q e^{-i(k-q)x} + \hat{a}_k^+\hat{b}_q^+ e^{-i(k+q)x}\Big)\Big\}$$

$$\frac{1}{(2\pi)^3}\int d^3x \, e^{i(k-q)x} = \delta^3(k-q)$$

$$= \int \cancel{d^3x} \; \frac{d^3k \; d^3q}{\cancel{(2\pi)^3} \, 2\sqrt{\omega_k\omega_q}} \Big\{\Big(-\hat{b}_k\hat{a}_q \delta^3(k+q) - \hat{b}_k\hat{b}_q^+ \delta^3(k-q) + \hat{a}_k^+\hat{a}_q \delta^3(k-q)$$

$$+ a_k^+ b_q^+ \delta^3(k+q)\Big)\omega_k - \omega_q\Big(-\hat{b}_k\hat{a}_q \delta^3(k+q) + \hat{b}_k\hat{b}_q^+ \delta^3(k-q) - a_k^+ a_q \delta^3(k-q)$$
$$+ a_k^+ b_q^+ \delta(k+q)\Big)\Big\}$$

?

$$= \int \frac{d^3k}{2\omega_k} \cdot \omega_k \left\{ 2a_k^+ a_k - 2b_k^+ b_k \right\}$$

$$\hat{Q} = \int d^3k \left(\hat{a}_k^+ \hat{a}_k - \hat{b}_k^+ \hat{b}_k\right)$$

→ $\hat{a}^+$ creates a particle of charge +1, and $\hat{Q}\cdot\hat{a}_k^+|0\rangle = \hat{a}_k^+|0\rangle$

$\hat{b}^+$ creates a particle of charge −1 $\hat{Q}\,\hat{b}_k^+|0\rangle = -\hat{b}_k^+|0\rangle$

↳ We have particles and antiparticles