1

INTRODUCTION TO QFT

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→ Remark on units:

We're going to set $\hbar = c = 1$ (natural units)

ex: $[L] = [T] = eV^{-1}$ $\quad [LT^{-1}] = eV^0$

$[M] = [LT^{-2}] = eV$

1 CLASSICAL FIELDS

→ example: electromagnetic fields $\bar{E}(\bar{x},t)$, $\bar{B}(\bar{x},t)$. They're given by the potentials $A^\mu(\bar{x},t) \doteq (\varphi, A^i)$. It's then a vector field.

→ Constructing Lagrangians:

→ Hermitean $A^\dagger = A \Leftrightarrow \langle \vec{u}, A\vec{v}\rangle = \langle A\vec{u}, \vec{v}\rangle$

A square matrix is Hermitian $\Leftrightarrow$ it is unitarily diagonalizable with real eigenvalues $A = UDU^\dagger$ with $UU^\dagger = \mathbb{1}$

→ Lorentz invariance $A^{\mu\nu}A_{\mu\nu}$ is a Lorentz scalar for ex.

→ eqs not higher the 2^{nd} order in time

→ energy bounded from below

1.1 Consider a scalar field

→ In general, the Lagrangian density is the following:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$$

where $V(\phi) = \sum_i \kappa^i \phi^i(x)$ (Lorentz invariant)

⊙ Equations of motion:

→ We have to solve $\delta S = 0$ with $S \equiv \int dt\, L = \int d^4x\, \mathcal{L}$

→ $\delta S = S[\phi + \delta\phi] - S[\phi]$

$$= \int d^4x \left\{ \frac{1}{2}\partial_\mu(\phi+\delta\phi)\partial^\mu(\phi+\delta\phi) - V(\phi+\delta\phi) - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + V(\phi)\right\}$$

$$= \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \partial_\mu\phi\,\partial^\mu\delta\phi - V(\phi) - \delta\phi\frac{\partial V}{\partial\phi} - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + V(\phi)\right\} + \mathcal{O}(\delta\phi^2)$$

$$= \int d^4x \left\{ \partial_\mu\phi\,\partial^\mu\delta\phi - V'(\phi)\,\delta\phi\right\}$$

$$= \int d^4x \left\{ -\partial_\mu\partial^\mu\phi \cdot \delta\phi - V'(\phi)\,\delta\phi\right\} + \int d^4x \left\{ \cancel{\partial_\mu}\left[\partial_\mu\phi\,\delta\phi\right]\right\}$$

$$\Leftrightarrow \quad \partial_\mu\partial^\mu\phi + V'(\phi) = 0$$

↳ By choosing $V = \frac{1}{2} m^2 \phi^2$, we get the Klein-Gordon eq.:

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

(Similar to eq. for harmonic oscillator $\partial_0^2 x + \omega^2 x = 0$)

↳ Free field: which satisfy linear equations (no interaction)

1.2 Symmetries and conservation laws: Noether thm

→ Conserved qties play an important role. There is a deep relation between those and symmetries of the theory.

⊙ Example: complex scalar field:

→ Let the action be: $S = \int d^4x \{ \partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi) \}$

with $\phi \triangleq \phi_1 + i \phi_2$

↳ The action is invariant under $\phi \mapsto \phi e^{i\alpha}$. This is a global transformation because α doesn't depend on x^μ.

↳ Proof of the invariance:

$$\phi^* \mapsto \phi^* e^{-i\alpha} \; ; \; \phi^* \phi \mapsto \phi^* e^{-i\alpha} \phi e^{i\alpha} = \phi^* \phi$$

$$\partial_\mu \phi^* \partial^\mu \phi \mapsto \partial_\mu \phi^* e^{-i\alpha} \partial^\mu \phi e^{i\alpha} = \partial_\mu \phi^* \partial^\mu \phi \quad \blacksquare$$

DEF This is the $U(1)$ symmetry.

↳ Lets consider a local transformation $\phi \mapsto \phi e^{i\alpha(x)} = \phi(1 + i\alpha(x)) + \mathcal{O}(\alpha^2)$

$$\rightarrow \delta S = \int d^4x \tfrac{1}{2} \{ \partial_\mu (\phi^* e^{-i\alpha}) \partial^\mu (\phi e^{i\alpha}) - \tfrac{1}{2} \partial_\mu \phi \partial^\mu \phi \}$$

$$= \int d^4x \tfrac{1}{2} \partial_\mu (1 - i\alpha) \phi^* \partial^\mu (1 + i\alpha) \phi - \tfrac{1}{2} \partial_\mu \phi^* \partial^\mu \phi \}$$

$$= \frac{1}{2} \int d^4x \{ (\partial_\mu \phi^* - i\alpha \partial_\mu \phi^* - i \partial_\mu \alpha . \phi^*)$$
$$(\partial_\mu \phi + i\alpha \partial_\mu \phi + i \partial_\mu \alpha . \phi) \} - \frac{1}{2} \int d^4x \partial_\mu \phi^* \partial^\mu \phi \}$$

$$\Rightarrow \int d^4x \{ \cancel{i\alpha \partial_\mu \phi^* \partial^\mu \phi} + i \partial_\mu \phi^* \partial^\mu \alpha . \phi$$
$$\cancel{- i\alpha \partial_\mu \phi^* \partial^\mu \phi} - i \partial_\mu \alpha . \phi^* . \partial^\mu \phi \}$$

$$\delta S = \int d^4x \{ i \partial^\mu \alpha (\partial_\mu \phi^* . \phi - \phi^* \partial_\mu \phi) \}$$

We then set $J_\mu \equiv -i (\partial_\mu \phi^* . \phi - \phi^* \partial_\mu \phi)$ and we get

$$\delta S = - \int d^4x \{ \partial^\mu \alpha . J_\mu \}$$

In general, $J_\mu = J_\mu (\phi, \partial\phi, \partial^2 \phi, ...)$

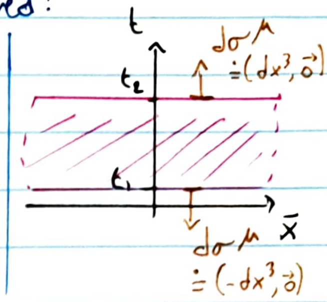
→ δS has to vanish if fields satisfy eqs of motion:

$$\delta S = - \int d^4x \{ \partial^\mu \alpha . J_\mu \}$$
$$= + \int d^4x \, \alpha(x) . \partial^\mu J_\mu = 0$$
$$\Leftrightarrow \quad \partial^\mu J_\mu = 0$$

→ The conservation of J^μ implies the existence of the conserved charge.

Let $Q \equiv \int d^3x \; J_0$ and prove it's conserved:

We consider the volume between 2 planes $t=t_1$ and $t=t_2$.



↳ $0 = \int dV \; \partial^\mu J_\mu$

$= \int_{\partial V} J^\mu . d\sigma^\mu$ $\quad (J^\mu . \vec{n} = J_0)$

$= \int d^3x \; J_0(\bar{x}, t_2) - \int d^3x \; J_0(t_1) = Q(t_2) - Q(t_1)$

$\Leftrightarrow \quad \dot{Q} = 0$

1.3 Energy-momentum tensor

→ Consider a system with the Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$

Thus $S = \int d^4x \; \mathcal{L}(\phi, \partial_\mu \phi)$

↳ Eqs of motion:

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} . \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \right\} \delta\phi = 0$$

DEF We've derived the Euler-Lagrange eqs:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] = 0$$

→ We do the calculation of $T_{\mu\nu}$ in 2 parts:

① $\underbrace{\partial_\mu \mathcal{L}}_{①} = \frac{\partial \mathcal{L}}{\partial \phi} . \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} . \partial_\mu (\partial_\nu \phi)$

$$= \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right] . \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} . \partial_\mu \partial_\nu \phi$$

$$= \underbrace{\partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi \right]}_{②}$$

(we've used the $\mathcal{L}$ doesn't depend on x^μ explicitly).

② ① - ② = 0 $\Leftrightarrow \partial_\mu \mathcal{L} - \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi \right] = 0$

Recall: $\partial_\mu \mathcal{L} - \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \cdot \partial_\mu \phi \right] = 0$

$$\Leftrightarrow \partial_\nu(\delta^\nu{}_\mu \mathcal{L}) - \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \cdot \partial_\mu \phi \right] = 0$$

$$\Leftrightarrow \partial_\nu \left[\delta^\nu{}_\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \cdot \partial_\mu \phi \right] = 0$$

$$\Leftrightarrow \partial^\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\nu \phi)} \partial_\mu \phi - \eta_{\mu\nu} \mathcal{L} \right] = 0$$

DEF

We define the energy-momentum tensor $T_{\mu\nu}$ as:

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\nu \phi)} \partial_\mu \phi - \eta_{\mu\nu} \mathcal{L}$$

By construction, $\partial^\nu T_{\mu\nu} = 0$. It's the conservation of the energy-impulsion.

→ By the divergence theorem, we've 4 conserved quantities:

DEF

The 4-vector energy-momentum P_μ is defined by

$$P_\mu \equiv \int d^3x \, T_{\mu 0}$$

They are integrals of motion. We have:

$E = \int d^3x \, T_{00}$ (then T_{00} is the energy density) and

$P_i = \int d^3x \, T_{i0}$ is the momentum.

② Noether theorem.

THM

Let the action of a system be invariant under a continuous set of transformations with N parameters ω_a such that for small ω_a:

$$x^\mu \to x'^\mu = x^\mu + \mathcal{X}^\mu_a \, \omega_a + \mathcal{O}(\omega_a^2)$$

$$\phi_i(x) \to \phi'_i(x') = \phi_i(x) + \Psi_{ia} \, \omega_a + \dots$$

Then the following N currents are conserved:

$$J^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} (\partial_\nu \phi_i \cdot \mathcal{X}^\nu_a - \Psi_{ia}) - \mathcal{X}^\mu_a \cdot \mathcal{L}$$

→ Example:

① Space-time translations:

$x'^\mu = x^\mu + \omega^\mu$; $\phi'_i(x') = \phi_i(x)$ (scalar field)

We've: $\mathcal{X}^\mu{}_\nu = \delta^\mu{}_\nu$ and $\Psi_{ia} = 0$

The conserved current is:

$$J^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta^\mu{}_\nu \mathcal{L} \equiv T^\mu{}_\nu$$

5.

② Angular momentum and spin:

→ Consider Lorentz transformations: $x'^\mu = x^\mu + \omega^{\mu\nu} x_\nu$

↳ Recall: Lie algebra of $SO(1,3)$, developped around the identity element of the Lorentz group:

$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \mathcal{O}(\omega^2)$ with

$\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \rightarrow \eta_{\mu\nu} = \eta_{\alpha\beta} (\delta^\alpha{}_\mu + \omega^\alpha{}_\mu)(\delta^\beta{}_\nu + \omega^\beta{}_\nu) + \mathcal{O}(\omega^2)$

$\Rightarrow \eta_{\mu\nu} = \eta_{\mu\nu} + \eta_{\mu\beta}\,\omega^\beta{}_\nu + \eta_{\alpha\nu}\,\omega^\alpha{}_\mu = \eta_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu}$

$\Rightarrow \omega_{\mu\nu} = -\omega_{\nu\mu}$. Since $\omega_{\mu\nu}$ is antisymmetric 4×4 matrix $n(n-1)/2\big|_{n=4} = 6$ independent parameters of transformation.

↳ Our parameters are $\omega^{\mu\nu}$ s.t. $\mu > \nu$

$\delta x^\mu = \omega^{\mu\nu} x_\nu = \mathcal{X}^\mu{}_a\, \omega^a$

$= \sum_{\lambda<\rho} \mathcal{X}^\mu{}_{\lambda\rho}\, \omega^{\lambda\rho}$ with $\lambda < \rho$ $\quad a = (\mu\nu), \mu<\nu$

↳ $= \sum_{\lambda<\rho} \omega^{\lambda\rho} (\delta^\mu_\lambda x_\rho - \delta^\mu_\rho x_\lambda)$

? factor $\frac{1}{2}$?

We find: $\mathcal{X}^\mu{}_{\lambda\rho} = \delta^\mu_\lambda x_\rho - \delta^\mu{}_\rho x_\lambda$

↳ Because the field is scalar:

$\delta\phi_i = 0 \rightarrow \Psi_{i(\lambda\rho)} = 0$

↳ The conserved current is then:

$$M^\mu{}_{\lambda\rho} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i (\delta^\nu_\lambda x_\rho - \delta^\nu_\rho x_\lambda) - (\delta^\mu_\lambda x_\rho - \delta^\mu_\rho x_\lambda)\mathcal{L}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\lambda \phi_i\, x_\rho - \delta^\mu{}_\lambda x_\rho \mathcal{L} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\rho x_\lambda - \delta^\mu_\rho x_\lambda \mathcal{L} \right)$$

$$= T^\mu{}_\lambda x_\rho - T^\mu{}_\rho x_\lambda$$

→ $M^\mu{}_{\lambda\rho}$ are 6 conserved currents, and therefore we've 6 charges: $\int d^3x\, M^0{}_{\lambda\rho}$. For $\lambda,\rho = i,j$, the conserved qty is the orbital momentum: $\int d^3x \{P_i x_j - P_j x_i\} = I_k = \int \epsilon_{ijk} M^0_{ij}\, dx^3$

DEF

→ Let's consider now a vector field. A^μ

↳ We still have $M^\mu{}_{\lambda\rho} = \delta^\mu_\lambda x_\rho - \delta^\mu_\rho x_\lambda$

↳ $\delta A_\mu = \omega_{\mu\nu} A^\nu$

$\equiv \sum_{\sigma<\rho} \Psi_{\mu\sigma\rho}\, \omega^{\sigma\rho}$ ("$\mu\sigma\rho$" ↔ "ia")

? One reads: $\Psi_{\mu\sigma\rho} = \eta_{\mu\sigma} A_\rho - \eta_{\mu\rho} A_\sigma$

Thus,

$$M^\mu{}_{\lambda\rho} = M^{(0)\mu}{}_{\lambda\rho} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \Psi_{\sigma\lambda\rho}$$

$$= \underbrace{M^{(0)\mu}{}_{\lambda\rho}}_{\text{orbital momentum}} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \left(\eta_{\sigma\lambda} A_\rho - \eta_{\sigma\rho} A_\lambda\right)$$

DEF The spin is $S^\mu{}_{\lambda\rho} = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \left(\eta_{\sigma\lambda} A_\rho - \eta_{\sigma\rho} A_\lambda\right) \right\}$

or $S_{\lambda\rho} = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial \dot{A}^\rho} A_\lambda - \frac{\partial \mathcal{L}}{\partial \dot{A}^\lambda} \cdot A_\rho \right\}$

And the vector of spin is $s_i \equiv \epsilon_{ijk} S_{jk}$

2 QUANTIZATION OF A FREE SCALAR FIELD

2.1 Reminder: quantization in QM

→ We use the Lagrangian of an harmonic oscillator:

$$L = \frac{1}{2} m \dot{\tilde{q}}^2 - \frac{1}{2} k \tilde{q}^2$$

{ $\tilde{q}$ - position, $\dot{\tilde{q}}$ - velocity

↳ $S = \int L\, dt = \int dt \left\{ \frac{1}{2} m \dot{\tilde{q}}^2 - \frac{1}{2} k \tilde{q}^2 \right\}$

↳ We rescale our variables: $q \equiv \sqrt{m}\, \tilde{q}$ and write $\omega^2 = k/m$

Then: $$S = \int dt \left\{ \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2 \right\}$$

→ Equations of motion (classical):

$$\delta S = 0$$