4 CARTAN FORMULATION

4.1 Mathematical reminders

4.1.1 Linear algebra :

Conv — For any invertible matrix $e^a{}_\mu$ we use the convention that the inverse matrix $(e^{-1})^\mu{}_a \equiv e_a{}^\mu$, so that

$$e^a{}_\mu e_c{}^\mu = \delta^a_c \iff e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$$

Thm — (Spectral) If $S_{\mu\nu} \in \mathcal{M}_{n\times n}(\mathbb{C})$, there exists an orthonormal matrix $E^a{}_\mu$ such that

$$S = E^T D E \iff S_{\mu\nu} = E^a{}_\mu D_{ab} E^b{}_\nu \quad \text{with } D_{ab} \text{ diagonal}$$

Law — (Sylvester's law of inertia) Let S, S' be 2 symmetric matrices of the same size, with the same number of positive, negative and zero eigenvalues. This is equivalent to have P an invertible matrix such that $S' = P S P^T$

Thm — (Polar decomposition) An invertible matrix $e_a{}^\mu$ always admits a unique polar decomposition of the form

$$e_a{}^\mu = O_a{}^b P_b{}^\mu \iff e^a{}_\nu = O^a{}_b P^b{}_\nu$$

where O is an orthogonal matrix, while P est a positive definite matrix symmetric : $O^c{}_a \delta_{cd} O^d{}_b = \delta_{ab}$ (Euclidian space)

→ We can define a metric $g_{\mu\nu}$. We have

$$g_{\mu\nu} = e^a{}_\mu \delta_{ab} e^b{}_\nu = P^a{}_\mu \delta_{ab} P^b{}_\nu$$

Thm — (QR decomposition) Locally in spacetime, there exists a unique decomposition of the frame of the form

$$e_a{}^\mu = O_a{}^b U_b{}^\mu$$

where $O \in O(n)$ and U is an upper triangular matrix with positive elements on the diagonal. For a positive definite symmetric matrix, (eg metric), it admits a unique Cholesky decomposition :

$$g_{\mu\nu} = U^a{}_\mu \delta_{ab} U^b{}_\nu$$

4.1.2 Elements of differential geometry:

⊙ Manifolds and functions:

DEF A topological space (X, Θ) consist of a point set X when $\Theta = \{U_i\}_{i\in I}$ when I is an index set labelling members of a collection of open sets X such that:

1. All union of open sets are open
2. Finite intersections of open sets are open
3. X and $\emptyset$ are open

DEF A basis for a topology is a subset of all possible open sets, wich by intersections and unions, can generate all possible open set.

DEF An open cover $\{U_i\}$ of X is a collection of open sets such that every points $x \in X$ is contained in at least one U_i

DEF X is compact if every open cover has a finite sub-cover.

DEF X is Hausdorff if every pair of disjoint points is contained in a disjoint pair of open sets

DEF A function $f: X \to Y$ between 2 topological space is continous if the inverse image of any open set is open.
$f \in \mathcal{C}^0$

DEF 2 topological spaces are homeomorphic if there is a 1-1 map $\phi: X \to Y$ (ϕ a bijection) such that $\phi, \phi^{-1} \in \mathcal{C}^0$
Then, $X \simeq Y$

→ If 2 topological spaces are homeomorphic, they are usually thought to be the same.

DEF A smooth n-manifold with a smooth atlas of charts as
① A topological space X
② An open cover $\{U_i\}$ called patches
③ A set (atlas) of maps $\phi_i : U_i \to \mathbb{R}^n$ called charts wich are injective, homeomorphism onto their images and whose images are open in $\mathbb{R}^n$ such that
④ If $U_i \cap U_j \neq \emptyset$, then $\phi_j \circ \phi_i^{-1} \in \mathcal{C}^\infty(\mathbb{R}^n \to \mathbb{R}^n)$
$\phi_i \circ \phi_j^{-1} \in \mathcal{C}^\infty(\mathbb{R}^n \to \mathbb{R}^n)$

→ $\phi(x) \equiv x^\mu$ is a local coordinate on X. Usually, we denote $\phi_i \circ \phi_j^{-1} \equiv \tilde{x}^\mu(x^\nu)$ and $\phi_j \circ \phi_i^{-1} \equiv x^\mu(\tilde{x}^\nu)$

→ Such a manifold, with a complete atlas, is detoted M.

DEF We can define a real valued, smooth function $f : M \to \mathbb{R}$ such that
$$f \circ \phi^{-1} \equiv f(x^\alpha) \in \mathcal{C}^\infty$$

PROP The set of all smooth functions $C^\infty(M)$ on manifold forms a commutative ring, trough pointwise addition and multiplication of the values.

Prop A manifold M is said orientable $\Leftrightarrow \det\left(\frac{\partial x^\alpha}{\partial \tilde{x}^\beta}\right) > 0 \ \forall$ overlaps

⊙ Tangent vectors and vector fields

DEF A smooth curve c in M is a smooth map $c : \mathbb{R} \mapsto M$. In local coord., $c : \begin{pmatrix} I \subset \mathbb{R} \to M \\ \lambda \mapsto x^\mu(\lambda) \end{pmatrix}$ The curve is closed if $c : S^1 \to M$ and simple if it's a bijection.
→ A path is the image of a curve. If M is a spacetime, a path is called a worldline.

→ If we orient $\mathbb{R}$, we can orient the paths: $x^\mu(t)$, $x^\mu(t')$ have the same orientation if $\frac{dt'}{dt} > 0$

→ Given a curve c and a function f we can compose them to get a map $f\circ c : \mathbb{R} \to \mathbb{R} : \lambda \mapsto f(x^\alpha(\lambda))$. We can write
$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\alpha} \cdot \frac{dx^\alpha}{d\lambda}$$, and evaluate at some point $p \in M$.

DEF A tangent vector at p T is a map
$$T : \begin{pmatrix} \mathcal{C}^\infty(M) \longrightarrow \mathbb{R} \\ f \longmapsto Tf \equiv \left.\frac{df}{d\lambda}\right|_{\lambda=0} \quad \text{where } x^\alpha(0)=p \end{pmatrix}$$
wich satisfies:

① $T(f+g) = Tf + Tg$ linearity

② $T(fg) = Tf \cdot g + f \cdot Tg$ Leibniz

DEF Since T is a linear operator $((T_1+T_2)f = T_1 f + T_2 f)$ and since we can multiply it by constants, the space of tangent vectors at $p \in M$ is a vector space, called vector space T_pM

→ $\dim(T_pM) = n$. Since $f(x) = f(p) + x^\alpha \partial_\alpha|_p f + \dots$, we have
$$Tx^\alpha = T^\alpha \quad \text{and} \quad Tf = T^\alpha \partial_\alpha f|_p$$

DEF The basis $\{\partial_\mu\}$ is called the coordinate basis: we have
$$T = T^\alpha \partial_\alpha = \tilde{T}^\alpha \tilde{\partial}_\alpha$$

→ If T is a tangent vector of a curve, $v^\mu = \left.\frac{dx^\mu}{dt}\right|_{t=0}$. In a new coord. system, we have $v^\mu \frac{\partial}{\partial x^\mu} = \tilde{v}^\mu \frac{\partial}{\partial \tilde{x}^\mu}$

$$\Leftrightarrow \tilde{v}^\mu \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \cdot \frac{\partial}{\partial x^\nu} \Rightarrow \tilde{v}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} v^\nu$$: contravariant vector

DEF A vector field is a continuous assignment of $v(p) \in T_pM$ $\forall p \in M$. In local coord., $V \equiv V^\mu(x^\alpha)\partial_\mu$. The set of all vector field on M is denoted by $\mathcal{X}(M)$

DEF Given $V \in \mathcal{X}(M)$, locally, the integral curves is the solution to
$$\frac{dx^\mu}{d\lambda} = V^\mu(x)$$
whose tangent vectors coincide with $V(p)$ $\forall p \in M$.

DEF A tangent bundle TM is the space of all possible vectors at all possible points:

$$TM \equiv \bigcup_{p\in M} T_pM$$

→ $\dim(TM) = 2n$. Local coord. of TM: (x^μ, v^ν) where $V = V^\nu \partial_\nu$.
A vector field is a n-dim surface in TM.

DEF A vector field commutator $[\,,]$ is an operator

$$[\,,] : \left(\begin{array}{ccc} \mathcal{X}(M)\times\mathcal{X}(M) & \to & \mathcal{X}(M) \\ (v,w) & \mapsto & v(w)-w(v) \end{array} \right)$$

↳ $[v,w]f = v(w(f)) - w(v(f))$

→ $[\,,]$ gives a vector field because it is a 1^{st} order differential operator. In a coord. basis, one has

$$[v,w]f = v^\mu\partial_\mu(w^\nu\partial_\nu f) - w^\mu\partial_\mu(v^\nu\partial_\nu f)$$
$$= \{v^\mu\partial_\mu w^\nu - w^\mu\partial_\mu\}\partial_\nu f$$

→ It has the Jacobi identity $[u[v,w]] + \text{cyclic}(u,v,w) = 0$
↳ A vector field commutator define an ∞-dim Lie algebra.

⊙ Non coordinate basis:

DEF We introduce a non-coord. basis for T_pM $\{e_a\}_{a=1}^n$, also called a tetrad, vielbein, tetrad, moving frame.

→ We write $T = v^a e_a$ and $e_a = e_a{}^\mu(x)\dfrac{\partial}{\partial x^\mu}$

where $\begin{cases} \mu : \text{world index} \\ a : \text{tangent space index} \end{cases}$

↳ We are now allowed not only coord. transfo. but also position dependant changes of the bases e_a: $e_a \mapsto \tilde{e}_a = \Lambda_a{}^b(x)\, e_b$

$\Rightarrow v^a \mapsto \tilde{v}^a = \Lambda^a{}_b(x)\, v^b$ such that $\Lambda^a{}_b \Lambda_c{}^b = \delta^a_c$

→ In matrix notation: $v^a = \Big(\;\Big)$ $e_a = \overline{\quad\quad}$

We can write: $\Lambda^a{}_b = (\Lambda)^a{}_b$, $\Lambda_c{}^b = (\Lambda^{-1})^b{}_c$. Then,

$$v' = \Lambda v \quad\text{and}\quad e' = e\Lambda^{-1}, \quad \Lambda \in GL(n,\mathbb{R})$$

→ For comparison, a coord. transformation induces a change of coord. basis with $\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ and $\Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}$. It's a particular case of frame rotation satisfying

$$\partial_\rho \Lambda^\mu{}_\nu - \partial_\nu \Lambda^\mu{}_\rho = 0$$

→ In term of components, combined change of frame and coord. transfo:

$$e'_a{}^\mu(x') = \Lambda_a{}^b(x)\, e_b{}^\nu(x) \frac{\partial x'^\mu}{\partial x^\nu}$$

⊙ Covectors:

DEF Let V be a vector space of dimension n. We define its dual space V^* as the space of linear maps $V \to \mathbb{R}$

↳ For $u, v \in V$, $\omega \in V^*$, we have

→ $\omega(u) = \langle \omega, u \rangle \in \mathbb{R}$ form

→ $\omega(\alpha u + \beta v) = \alpha\, \omega(u) + \beta\, \omega(v)$ linear

↳ $\dim(V^*) = n = \dim(V)$

Prop $(V^*)^* \simeq V$

→ Basis of the dual space: given $\{e_a\}$ a basis of V, we denote the dual basis $\{\overset{*}{e}{}^a\}$ such that

$$\langle \overset{*}{e}{}^a, e_b \rangle = \delta^a_b$$

↳ In this basis, $\omega = \omega_a \overset{*}{e}{}^a$

⊙ Change of basis:

Prop Under a change of basis,

$$\overset{*}{e}{}^a \mapsto \overset{*}{e}{}'^a = \Lambda^a{}_b \overset{*}{e}{}^b \quad / \quad e_a \mapsto e'_a = \Lambda_a{}^b e_b$$

$$\omega_a \mapsto \omega'_a = \Lambda_a{}^b \omega_b = (\Lambda^{-1})^b{}_a \omega_b$$

DEF At every $p \in M$, the cotangent space T_p^*M is defined as the dual of T_pM. A covector field is one for wich

$$\langle \omega, g v \rangle = g \langle \omega, v \rangle \quad \forall g \in \mathcal{C}^\infty(M)$$

25

→ Example of a co-vector: a differential 1-form.

→ Let $f: M \to \mathbb{R}$. Its exterior derivative is defined by

$$\langle df, v \rangle = vf \quad \forall v \in \mathcal{X}(M)$$

→ We call $\Omega^1(M)$ the space of 1-form and $\underbrace{\Omega^0(M)}_{= C^\infty(M)}$ the real-valued function on M. Then, we can write

$$d : C^\infty(M) \to \Omega^1(M) \text{ such that } d(fg) = f\,dg + g\,df$$

↳ In coord. basis:

$$df = \partial_\mu f(x)\, dx^\mu \quad \text{with} \quad \langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle = \delta^\mu_\nu$$

↳ We can write $df = e_a(f) e^a$ where $e^a \equiv e^a{}_\mu dx^\mu$ and such that

$$e_a{}^\mu e^b{}_\mu = \delta_a^b \quad \text{and} \quad e_a{}^\nu e^a{}_\mu = \delta^\nu_\mu$$

↳ Since $\omega = \omega_\mu dx^\mu = \omega'_\mu dx'^\mu = \omega'_\nu \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu$; $\omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu$

→ Not all 1-form are differential. A necessary condition is that

$$\partial_\mu \omega_\nu - \partial_\nu \omega_\mu = 0$$

↳ Geometrical interpretation:

→ $f(x^\alpha) = cst = a$ define a $(n-1)$ dim Σ_a surface on wich the function takes constant value a: it's a level sets of f

→ Along a curve $c(t)$ intersecting Σ_a, we have:

$$\frac{df}{dt} = \frac{\partial f}{\partial x^\alpha} \cdot \frac{dx^\alpha}{dt} = v(f) = \langle df, v \rangle \quad \text{for } v^\mu = \frac{dx^\mu}{dt}$$

↳ If $c(t)$ lies in Σ_a, $v \in T\Sigma_a$ and $\langle df, v \rangle = 0$. Then, $n_\alpha = \frac{\partial f}{\partial x^\alpha}$ is the co-normal of the surface Σ_a.

DEF We define a $2n$-dim co-tangent bundle T^*M as

$$T^*M = \bigcup_{p \in M} T_p^*M$$

↳ Also called phase space or momentum space space of M. It has local coord. (x^α, p_β), where p_β are the component of an arbitrary 1-form $p \equiv p_\beta dx^\beta$.

⊙ Tensor algebra:

DEF A co-tensor Ω of rang q at $p\in M$ is a multilinear map from

$$\Omega : \underbrace{T_pM \times \cdots \times T_pM}_{q \text{ factors}} \to \mathbb{R} : (v_1, \ldots, v_q) \mapsto \Omega(v_1, \ldots, v_q)$$

↳ For a co-tensor field, the linearity is over functions:

$$\Omega(f_1 v_1, \ldots, f_q v_q) = f_1 \cdots f_q \, \Omega(v_1, \ldots, v_q)$$

↳ In a basis, we have

$$\Omega = \Omega(e_{a_1}, \cdots e_{a_q}) \overset{*}{e}{}^{a_1} \otimes \cdots \otimes \overset{*}{e}{}^{a_q} = \Omega_{a_1 \ldots a_q} \overset{*}{e}{}^{a_1} \otimes \cdots \otimes \overset{*}{e}{}^{a_q}$$

↳ In a coord. basis, we have

$$\Omega(\partial_{\mu_1}, \ldots \partial_{\mu_q}) = \Omega_{\mu_1 \ldots \mu_q} = \Omega_{a_1 \ldots a_q} e^{a_1}{}_{\mu_1} \cdots e^{a_q}{}_{\mu_q}$$

DEF Similarly, we can define contravariant tensor of rank p and mixed tensor field of rank $\binom{p}{q}$

→ Operations on tensor:

→ contraction $\binom{p}{q} \mapsto \binom{p-1}{q-1}$ (partial trace)

→ (anti) symmetrization:

$$\Omega_{(a_1 \ldots a_r) a_{r+1} \ldots a_q} = \sum_{\sigma \in P(a_1, \ldots, a_r)} \frac{1}{r!} \Omega_{a_{\sigma(1)} \ldots a_{\sigma(r)} a_{r+1} \ldots a_q}$$

$$\Omega_{[a_1 \ldots a_r] a_{r+1} \ldots a_q} = \sum_{\sigma \in P(a_1, \ldots, a_r)} \operatorname{sgn}(\sigma) \frac{1}{r!} \Omega_{a_{\sigma(1)} \ldots a_{\sigma(r)} a_{r+1} \ldots a_q}$$

⊙ Metric:

DEF A inner product or metric on a vector space V is a real valued, non-degenerate, symmetric bilinear form $g : V \times V \to \mathbb{R}$ such that

① $g(u,v) = g(v,u) \quad \forall u, v \in V$ symmetry

② $g(fu, gv) = fg\, g(u,v) \quad \forall u, v \in V, \forall f, g \in \mathbb{R}$ linearity

③ $g(u,v) = 0 \ \forall v \Leftrightarrow u = 0$ non-degeneracy

→ In the basis $\{e_a\}$, we can write

$g(e_a, e_b) = g_{ab}$ and $g^{-1}(\overset{*}{e}{}^a, \overset{*}{e}{}^b) = g^{ab}$

and $g_{ab} = g_{ba}$ such that $g_{ab} g^{bc} = \delta_a{}^c$

$|g_{ab}| \neq 0$

→ Since g is symmetric, and non degenerate, we can diagonalise it. It will have s positive eigenvalues and t negative one. By rescaling, we find a pseudo-orthonormal basis in which $g_{ab} = \text{diag}(\underbrace{+1,\dots,+1}_{s}, \underbrace{-1,\dots,-1}_{t}) = \eta_{ab}$

↳ The signature of a metric is (s,t). We have $\dim V = s+t$.

Prop By Sylvester inertia theorem, the metric is $|s-t|$ invariant.

→ In coord. basis, we have $g_{\mu\nu}(x) = g(\partial_\mu, \partial_\nu)$ with

$g_{\mu\nu} = e^a{}_\mu \, g_{ab} \, e^b{}_\nu$ where $e_a{}^\mu(x)$ is a pseudo-ortho. frame.

→ $SO(s,t)$-rotations of the basis $\{e_a\}$ that preserve η_{ab} will give the same metric:

$$\eta'_{ab} = \Lambda_a{}^c \, \eta_{cd} \, \Lambda_b{}^d = \eta_{ab} \Leftrightarrow \Lambda \in \underbrace{SO(s,t)}_{\text{Lorentz group!}}$$

⊙ Exterior or Grassmann algebra:

DEF Given a vector space V, a p-form ω is a totally antisym. multilinear map $V \times \dots \times V \to \mathbb{R}$. We call $\Omega^p(V)$ the vector space of p-forms, with $\dim\{\Omega^p(V)\} = \dfrac{n!}{p!(n-p)!}$. By convention, $\Omega^0(V) = \mathbb{R}$.

We denote by $\Omega^*(V)$ the direct sum of the p-forms spaces:

$$\Omega^*(V) \equiv \bigoplus_{p=0}^{n} \Omega^p(V)$$

↳ $\dim \Omega^* = \sum_{p=0}^{n} \binom{n}{p} = 2^n$

DEF We create the algebra $(\Omega^*, \wedge)$ by defining the exterior product $\wedge$: $\quad \wedge : \Omega^p(V) \times \Omega^q(V) \to \Omega^{p+q}(V)$ such that

① $(\ell\alpha)\wedge\beta = \ell(\alpha\wedge\beta)$ linearity

② $(\alpha+\beta)\wedge\gamma = \alpha\wedge\gamma + \beta\wedge\gamma$ distributivity

③ $(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$ associativity

④ $\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p$ graded commutativity

→ $\Omega^*(V)$ is a graded vector space: splits into $\oplus$ of vector spaces.

DEF A non-holonomic basis or anholonomic is a basis $\{e_a\}$ such that $[e_a, e_b] = D_a{}^c{}_b\, e_c$ and $e_a \equiv e_a{}^{\mu}(x)\partial_\mu$ where $D_a{}^c{}_b$ are the position dependent structure functions

Prop For an anholonomic basis $\{e_a\}$ we have:

$$e_{[c}\, D_a{}^{l}{}_{b]} + D_{[a}{}^{d}{}_{b}\, D_{c]}{}^{l}{}_{d} = 0$$

→ We can write:

$$T = v^\mu \partial_\mu = v^a\, e_a{}^{\mu}(x)\, \partial_\mu = v^a \partial_a \quad \text{and} \quad (e^a{}_\mu)^{-1} = e^\mu{}_a$$

→ Let $\omega \in \Omega^p$. Then:

$$\omega = \frac{1}{p!}\, \omega_{[\mu_1 \cdots \mu_p]}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = \frac{1}{p!}\, \omega_{[a_1 \cdots a_p]}\, \overset{*}{e}{}^{a_1} \wedge \cdots \wedge \overset{*}{e}{}^{a_p}$$

DEF The exterior derivative d is a map $d : \Omega^p(M) \to \Omega^{p+1}(M)$ s.t.

1. $d(\alpha + \beta) = d\alpha + d\beta$ — linearity
2. $d(\alpha_p \wedge \beta_q) = (d\alpha_p) \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q$ — graded Leibnizian
3. $d^2 \alpha = 0$
4. df is the usual gradient of a function
5. $d\alpha$ is local: depends only on the 1^{st} ∂ of the components of α.

→ In components: $(d\omega^p)_{\nu \mu_1 \cdots \mu_p} = \frac{(p+1)!}{p!}\, \partial_{[\nu}\, \omega^p_{\mu_1 \cdots \mu_p]}$

That is: $d\omega = \partial \wedge \omega$

→ We can write $d = dx^\mu \wedge \dfrac{\partial}{\partial x^\mu}$ and $d\omega = \overset{*}{e}{}^a \wedge e_a\, \omega$

→ $$\begin{aligned} d\overset{*}{e}{}^a &= d(e^a{}_\mu\, dx^\mu) = \partial_\nu e^a{}_\mu\, dx^\nu \wedge dx^\mu \\ &= -e^a{}_\sigma\, \partial_\nu e_c{}^\sigma\, e^c{}_\mu\, dx^\nu \wedge dx^\mu \\ &= -\tfrac{1}{2} e^a{}_\sigma\, \partial_\nu e^\sigma{}_c\, e_d{}^\nu\, \overset{*}{e}{}^d \wedge \overset{*}{e}{}^c \\ &= -\tfrac{1}{2} e^a{}_\sigma (e^\sigma{}_{c,\nu}\, e^\nu{}_d - e^\sigma{}_{d,\nu}\, e^\nu{}_c)\, e^d \wedge e^c \\ &= -\tfrac{1}{2} e^a{}_\sigma [e_d, e_c]^\sigma\, e^d \wedge e^c = -\tfrac{1}{2} D_d{}^a{}_c\, \overset{*}{e}{}^d \wedge \overset{*}{e}{}^c \end{aligned}$$

$\partial_\nu M^{-1} = -M^{-1} \partial_\nu M\, M^{-1}$

$dx^\mu \to \overset{*}{e}{}^a$

Prop $d\overset{*}{e}{}^a = -\frac{1}{2} D_d{}^a{}_c\, \overset{*}{e}{}^d \wedge \overset{*}{e}{}^c \iff \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu = -D_\mu{}^a{}_\nu$

⊙ Affine connection:

→ We want to define a structure that will allow us to parallely propagate a vector.

DEF(∇_v) We say that v is parallely propagat along a curve c if
$$\nabla_u v = f(t)\, v \quad \text{where} \quad u \equiv \frac{dx^\mu}{dt} \text{ tangent to } c$$
and where $\nabla_u : TM^\ell \to TM$ such that

① $\nabla_{fu+gw} v = f \nabla_u v + g \nabla_w v$ — linearity

② $\nabla_u (f v) = u(f)\, v + f \nabla_u v$ — Leibniz

③ ∇_u commutes with contraction

④ ∇_u commutes with tensor product

DEF (∇_c) We define the covariant derivative operator ∇ as
$$\nabla : TM \to T^*M \otimes TM \text{ such that } \langle \nabla v, u \rangle = \nabla_u v$$
and $\nabla f v = df \otimes v + f \nabla v$

→ Evaluate on elements of the basis $\{e_a\}$, one has:
$$\nabla e_a = \Gamma_a{}^b{}_c\, e_b\, {}^*e^c \iff \nabla_a e_b = \Gamma_b{}^c{}_a\, e_c$$

↳ $\nabla (v^a e_a) = dv^a . e_a + v^a \nabla e_a$
$$= dv^a . e_a + v^a \Gamma_a{}^b{}_c\, e_b\, {}^*e^c$$

↳ $(\nabla v)^a{}_b = \partial_b v^a + \Gamma_c{}^a{}_b \equiv v^a{}_{;b}$

Indeed, $df = e_a(f)\, {}^*e^a = e_a{}^\mu \partial_\mu f\, e^a{}_\nu dx^\nu = \partial_\mu f\, dx^\mu$

↳ For covariant vectors we have:
$$(\nabla \omega)_{ab} := \nabla_b \omega_a = \partial_b \omega_a - \Gamma_a{}^c{}_b\, \omega_c$$

→ In matrix notation, $\nabla v = dv + \Gamma v$
$$\nabla \omega = d\omega - \Gamma \omega$$
with $\Gamma^a{}_b \equiv \Gamma_b{}^a{}_c\, {}^*e^c = \Gamma_b{}^a{}_c\, e^c{}_\mu\, dx^\mu = \Gamma_b{}^a{}_\mu\, dx^\mu$

→ **Change of basis:**

→ We had $\begin{cases} e'_a = \Lambda_a{}^b e_b \\ v'^a = \Lambda^a{}_b v^b \end{cases}$ with $\Lambda_a{}^b \Lambda^c{}_b = \Lambda_a{}^b (\Lambda^{-1})_b{}^c = \delta_a^c$ where $\Lambda \in GL(n,\mathbb{R})$

We see that $v' = \Lambda v$, thus $\nabla' v' \overset{!}{=} \Lambda \nabla v$

(We want ∇v to transform like a vector)

→ To get ∇v behave like a vector, we need:

$\nabla' v' = \Lambda \nabla v \Leftrightarrow (d' + \Gamma')\Lambda v = \Lambda(d+\Gamma) v$

↳ $d' = \overset{*}{e}{}'^a \cdot e'_a = \overset{*}{e}{}^a \Lambda^{-1}\Lambda e_a = \overset{*}{e}{}^a \cdot e_a = d$

↳ $d\Lambda . v + \Lambda d v + \Gamma' \Lambda v \overset{!}{=} \Lambda d v + \Lambda \Gamma v$

$\Leftrightarrow \Gamma' \Lambda = \Lambda \Gamma - d\Lambda \Leftrightarrow \Gamma' = \Lambda \Gamma \Lambda^{-1} - d\Lambda . \Lambda^{-1}$

Prop: The law of transformation of Γ under a change of basis is

$$\Gamma \mapsto \Gamma' = \Lambda \Gamma \Lambda^{-1} + \Lambda d \Lambda^{-1}$$

↳ In components:

$$\Gamma'^a{}_b = \Lambda^a{}_c \Gamma^c{}_d \Lambda_b{}^d + \Lambda^a{}_c d\Lambda_b{}^c$$

→ **Infinitesimal change of basis:**

$\Lambda_a{}^b = \delta_a^b + \omega_a{}^b$ with $\omega_a{}^b$ the n^2 generators of $GL(n,\mathbb{R})$

why $+ d\omega$ not $- d\omega$

↳ $\delta_\omega \Gamma^a{}_b = d\omega_b{}^a + \Gamma^a{}_c \omega_b{}^c - \Gamma^c{}_b \omega_c{}^a$

→ **Transformation in a coord. basis:**

$$\Gamma'^\mu{}_{\nu\lambda} = \frac{\partial x'^\mu}{\partial x^\rho} \Gamma^\rho{}_{\sigma\tau} \frac{\partial x^\tau}{\partial x'^\lambda} \frac{\partial x^\sigma}{\partial x'^\nu} + \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\lambda \partial x'^\nu}$$

→ **Remarks:**

→ $\Gamma_\beta{}^\alpha{}_\gamma$ not the components of a tensor, but $\Gamma_{[\beta}{}^\alpha{}_{\gamma]}$ are, and $\Gamma^a{}_b - \tilde{\Gamma}^a{}_b$ also (of a $\binom{1}{2}$ tensor field)

→ We have $(\nabla - \tilde{\nabla})(f v) = f(\nabla - \tilde{\nabla}) v$

⊙ Torsion tensor of an affine connection.

DEF We define the torsion tensor of the affine connection $\mathcal{T}$ as

$$\mathcal{T} : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$
$$(u, v) \mapsto \nabla_u v - \nabla_v u - [u, v] = \mathcal{T}(u, v)$$

↳ linear on the functions, antisymmetric

→ Basis: we have

$\partial_{[a} e_{b]} = 0$

$$\mathcal{T}(e_a, e_b) = \nabla_a e_b - \nabla_b e_a - [e_a, e_b]$$
$$= \{(\Gamma_b{}^c{}_a - \Gamma_a{}^c{}_b) - D_a{}^c{}_b\} e_c$$
$$= T_a{}^c{}_b \, e_c$$

$$\Rightarrow T_a{}^c{}_b = 2\Gamma_{[b}{}^c{}_{a]} + D_b{}^c{}_a$$

↳ In a coord. basis, it reduces to $T_\mu{}^\rho{}_\nu = 2\Gamma_{[\nu}{}^\rho{}_{\mu]}$

↳ Since we found that $d\overset{*}{e}{}^a = -\frac{1}{2} D_b{}^a{}_c \, \overset{*}{e}{}^b \wedge \overset{*}{e}{}^c$, we have:

$$d\overset{*}{e}{}^a = -\frac{1}{2}\left(T_b{}^a{}_c - 2\Gamma_{[b}{}^a{}_{c]}\right) \overset{*}{e}{}^b \wedge \overset{*}{e}{}^c$$

$$= -\frac{1}{2} T_b{}^a{}_c \, \overset{*}{e}{}^b \wedge \overset{*}{e}{}^c + \Gamma_{[b}{}^a{}_{c]} \, \overset{*}{e}{}^b \wedge \overset{*}{e}{}^c$$

Thm By setting $T^a \equiv \frac{1}{2} T_b{}^a{}_c \, \overset{*}{e}{}^b \wedge \overset{*}{e}{}^c$ we get the Cartan's First Structural Equation

$$T^a = d\overset{*}{e}{}^a + \Gamma^a{}_b \wedge \overset{*}{e}{}^b$$

where $\Gamma^a{}_b \equiv \Gamma_b{}^a{}_c \, \overset{*}{e}{}^c$ is the affine spin connection 1-form

→ On a function $f \in F(M)$, one has:

$$[\nabla_a, \nabla_b] f = \partial_a \partial_b f - \Gamma_b{}^a{}_c \nabla_c f - (a \leftrightarrow b)$$
$$= \partial_a (e_b{}^\nu \partial_\nu f) - \Gamma_b{}^a{}_c \nabla_c f - (a \leftrightarrow b)$$
$$= \partial_a e_b{}^\nu \cdot \partial_\nu f + e_b{}^\nu e_a{}^\mu \partial_\mu \partial_\nu f - \Gamma_b{}^a{}_c \nabla_c f - (a \leftrightarrow b)$$
$$= D_a{}^c{}_b \nabla_c f - 2\Gamma_{[b}{}^c{}_{a]} \nabla_c f = -T_a{}^c{}_b \nabla_c f$$

↳ $[\nabla_a, \nabla_b] f = -T_a{}^c{}_b \nabla_c f$

↳ $[\nabla_\mu, \nabla_\nu] f = T^\rho{}_{\mu\nu} \partial_\rho f$

⇒ For a connection with torsion, covariant differentiation fails to commute even on scalars.

→ For a torsion free connection, we get

$$d\overset{*}{e}{}^a + \Gamma_b{}^a \wedge \overset{*}{e}{}^b = 0 \iff 2\Gamma_{[b}{}^c{}_{a]} = D_{[a}{}^c{}_{b]}$$

$$\iff \Gamma^{\rho}{}_{[\mu\nu]} = 0$$

→ For a symmetric connection, any p-form ω has

$$d\omega = \partial \wedge \omega = \nabla \wedge \omega$$

⊙ Levi-Civita connection:

DEF The geodesic equation is the curve such that $\nabla_u u = 0$.
It is given by

$$\frac{d^2x^\alpha}{d\tau^2} + \left\{{}^{\alpha}_{\gamma\delta}\right\} \frac{dx^\gamma}{d\tau} \cdot \frac{dx^\delta}{d\tau} = 0$$

with $\left\{{}^{\alpha}_{\gamma\delta}\right\}$ the Christoffel symbols given by

$$\left\{{}^{\alpha}_{\gamma\delta}\right\} = \frac{1}{2} g^{\alpha\sigma}\left(g_{\sigma\delta,\gamma} + g_{\sigma\gamma,\delta} - g_{\gamma\delta,\sigma}\right)$$

Thm (Fundamental Theorem of Differential geometry)
The Levi-Civita connection $\left\{{}^{\alpha}_{\beta\gamma}\right\}$ is the unique affine connection such that:

① $\Gamma_{[\beta}{}^{\alpha}{}_{\gamma]} = 0$ ② $\nabla_\alpha g_{\gamma\beta} = 0$ ($\rightsquigarrow$ metricity)

$\boxed{\text{DEMO}}$ Let's write the metricity condition 3 times:

① $\nabla_\alpha g_{\beta\gamma} = g_{\beta\gamma,\alpha} - \Gamma_\beta{}^\sigma{}_\alpha g_{\sigma\gamma} - \Gamma_\gamma{}^\sigma{}_\alpha g_{\beta\sigma}$

② $\nabla_\beta g_{\gamma\alpha} = g_{\gamma\alpha,\beta} - \Gamma_\gamma{}^\sigma{}_\beta g_{\sigma\alpha} - \Gamma_\alpha{}^\sigma{}_\beta g_{\gamma\sigma}$

③ $\nabla_\gamma g_{\alpha\beta} = g_{\alpha\beta,\gamma} - \Gamma_\alpha{}^\sigma{}_\gamma g_{\sigma\beta} - \Gamma_\beta{}^\sigma{}_\gamma g_{\alpha\sigma}$

Now take (1) − (2) − (3). We get

$$g_{\beta\gamma,\alpha} - g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma} + 2\Gamma_\alpha{}^\sigma{}_\gamma g_{\beta\sigma} = 0 \quad \| \, g^{\alpha\beta}$$

$$\iff \Gamma_\beta{}^\alpha{}_\gamma = \frac{1}{2} g^{\alpha\sigma}\left(g_{\sigma\beta,\gamma} + g_{\sigma\gamma,\beta} - g_{\beta\gamma,\sigma}\right)$$

→ In general, we'll work in the anholonomic basis, and with a connection with torsion. This will add extra terms to the expression of the connection.

⊙ Lorentz connection:

→ We suppose a metric g_{ab} (such that $\det g_{ab} \neq 0$) and impose metricity:

(1) $\nabla_a g_{bc} = 0$. Computing it 3 times, we get:

(1) $\nabla_a g_{bc} = g_{bc,a} - \Gamma_b{}^l{}_a\, g_{lc} - \Gamma_c{}^l{}_a\, g_{bl} = g_{bc,a} - \Gamma_{bca} - \Gamma_{cba} = 0$

(2) $\nabla_c g_{ab} = g_{ab,c} - \Gamma_a{}^l{}_c\, g_{lb} - \Gamma_b{}^l{}_c\, g_{al} = g_{ab,c} - \Gamma_{abc} - \Gamma_{bac} = 0$

(3) $\nabla_b g_{ca} = g_{ca,b} - \Gamma_c{}^l{}_b\, g_{la} - \Gamma_a{}^l{}_b\, g_{cl} = g_{ca,b} - \Gamma_{cab} - \Gamma_{acb} = 0$

Let's compute (1) - (2) - (3):

$$-g_{bc,a} + g_{ab,c} + g_{ca,b} = -\Gamma_{bca} - \Gamma_{cba} + \Gamma_{abc} + \Gamma_{bac} + \Gamma_{cab} + \Gamma_{acb}$$
$$= 2\Gamma_{b[ac]} + 2\Gamma_{c[ab]} + 2\Gamma_{a(bc)}$$

Now, remembering that $T_{abc} = -2\Gamma_{a[bc]} - D_{abc}$, we get

$$\Gamma_{abc} = \Gamma_{a(bc)} + \Gamma_{a[bc]}$$
$$= \left(g_{ab,c} + g_{ca,b} - g_{bc,a} + T_{bac} + D_{bac} + T_{cab} + D_{cab}\right)\frac{1}{2} + \Gamma_{a[bc]}$$

$$= \frac{1}{2}\left(g_{ab,c} + g_{ca,b} - g_{bc,a} + T_{bac} + T_{cab} - T_{abc} + D_{bac} + D_{cab} - D_{abc}\right)$$

DEF The Lorentz connection Γ_{abc} is defined as follow:

$$\Gamma_{abc} \equiv \{abc\} + K_{abc} + r_{abc}$$ where we also define:

→ $K_{abc} \equiv \frac{1}{2}\left(T_{bac} + T_{cab} - T_{abc}\right)$ the contorsion tensor

→ $r_{abc} \equiv \frac{1}{2}\left(D_{bac} + D_{cab} - D_{abc}\right)$ the rotation coefficient

DEF When we don't have metricity, we define the nonmetricity tensor

$\Xi_{abc} \equiv -\nabla_c g_{ab} \Leftrightarrow \Xi^{ab}{}_c = \nabla_c g^{ab}$

We also define another usefull quantity: the cometricity Π as

$$\Pi_{abc} = \frac{1}{2}\left(\Xi_{abc} + \Xi_{acb} - \Xi_{bca}\right)$$

→ A general connection can be written as:

$$\Gamma_{abc} = \underbrace{\{abc\} + \Pi_{abc}}_{\text{sym in } (b,c)} + \underbrace{K_{abc} + r_{abc}}_{\text{skew in } (a,b)}$$

Prop We can always set $g_{\mu\nu} = e^a{}_\mu\, \eta_{ab}\, e^b{}_\nu$ with $\eta_{ab} = \mathrm{diag}(-1, 1, \dots, 1)$

↳ "$e^a{}_\mu$ is the square root of the metric"

Eguchi p246

⊙ Levi-Civita spin connection:

→ In Cartan's method, le Levi-Civita spin connection is obtain by restricting the affine spin connection $\Gamma^a{}_b = \Gamma_c{}^a{}_b \overset{*}{e}{}^c$ with:

① metricity : $\Gamma_{ab} = -\Gamma_{ba}$

② No torsion : $T^a = d\overset{*}{e}{}^a + \Gamma^a{}_b \wedge \overset{*}{e}{}^b = 0$

PROP : $\Gamma^a{}_{b\mu} = \overset{*}{e}{}^a{}_\nu \nabla_\mu e_b{}^\mu = \overset{*}{e}{}^a{}_\nu (\partial_\mu e_b{}^\nu + \Gamma^\nu_{\lambda\mu} e_b{}^\lambda)$

$$\Leftrightarrow \Gamma_{abc} \overset{\circledast}{=} \overset{*}{e}_{a\nu} \partial_c e_b{}^\nu + e_a{}^\nu e_b{}^\rho e_c{}^\mu \Gamma_{\nu\rho\mu}$$

$\boxed{\text{DEMO}}$ Indeed:

$$\{abc\} + \cancel{K_{abc}} + \gamma_{abc} = \overset{*}{e}_{a\nu} \partial_c e_b{}^\nu + e_a{}^\nu e_b{}^\rho e_c{}^\mu (\{\nu\rho\mu\} + \cancel{K_{\nu\mu\rho}})$$

$$\Leftrightarrow g_{ac,b} + g_{ab,c} - g_{bc,a} + D_{bac} + D_{cab} - D_{abc} = 2 e_{a\nu} \partial_c e_b{}^\nu$$
$$+ e_a{}^\nu e_b{}^\rho e_c{}^\mu \{\partial_\rho(\overset{*}{e}{}^d{}_\nu g_{df} \overset{*}{e}{}^f{}_\mu) + \partial_\mu(\overset{*}{e}{}^d{}_\nu g_{df} \overset{*}{e}{}^f{}_\rho) - \partial_\nu(\overset{*}{e}{}^d{}_\rho g_{df} \overset{*}{e}{}^f{}_\mu)\}$$

$$\Leftrightarrow g_{ac,b} + g_{ab,c} - g_{bc,a} + e_{b\lambda}(\partial_a e_c{}^\lambda - \partial_c e_a{}^\lambda) + e_{c\lambda}(\partial_a e_b{}^\lambda - \partial_b e_a{}^\lambda)$$
$$- e_{a\lambda}(\partial_b e_c{}^\lambda - \partial_c e_b{}^\lambda) = 2 e_{a\lambda} \partial_c e_b{}^\lambda + \partial_b(e^d{}_\nu g_{df} e^f{}_\mu) e_a{}^\nu e^\mu{}_c$$
$$+ \partial_c(e^d{}_\nu g_{df} e^f{}_\rho) e_a{}^\nu e_b{}^\rho - \partial_a(e^d{}_\rho g_{df} e^f{}_\mu) e_b{}^\rho e_c{}^\mu - \partial_b e_a{}^\nu e_{c\nu}$$
$$- \partial_b e_c{}^\mu e_{a\mu} - \partial_c e_a{}^\nu e_{b\nu} - \partial_c e_b{}^\rho - \partial_c e_b{}^\rho e_{a\rho} + \partial_a e_b{}^\rho e_{c\rho} + \partial_a e_c{}^\mu e_{b\mu}.$$

☒

→ One can eith // transport in the coord. basis with $\Gamma^\mu{}_{\nu\rho}$ or in the moving frame basis with $\Gamma^a_b{}_c$, and that the result will be the same.

→ One could also have:

$$e^\mu{}_b (\nabla_\mu v^\nu) e^a{}_\nu = D_b v^a$$

$$\Leftrightarrow e^\mu{}_b (\partial_\mu v^\nu + \Gamma^\nu_{\alpha\mu} v^\alpha) e^a{}_\nu = e^\mu{}_b \partial_\mu (\underbrace{e^a{}_\nu v^\nu}_{v^a}) + \Gamma^a{}_{cb} \underbrace{e^c{}_\nu v^\nu}_{v^c}$$

$$\Leftrightarrow e^\mu{}_b \Gamma^\alpha_{\nu\mu} e^a{}_\alpha = e_b{}^\mu \partial_\mu e^a{}_\nu + \Gamma_c{}^a{}_b e^c{}_\nu$$

$$\Leftrightarrow \Gamma_c{}^a{}_b = e_c{}^\nu (e_b{}^\mu \Gamma^\alpha_{\nu\mu} e^a{}_\alpha - e_b{}^\mu \partial_\mu e^a{}_\nu)$$
$$= e^a{}_\alpha e_c{}^\nu e_b{}^\mu \Gamma^\alpha{}_{\nu\mu} + \partial_b e_c{}^\nu e^a{}_\nu \Leftrightarrow \circledast$$

⊙ Curvature:

→ Parallel transport is non-commutative. A measure of this is provided by the curvature tensor, called the Riemann tensor in the case of the Levi-Civita connection.

DEF The curvature tensor is a tensor field $\binom{1}{3}$ R:

$$R : \left| \begin{array}{l} T(M) \otimes T(M) \wedge T(M) \to T(M) \\ (u, v, w) \mapsto R(u,v)w \equiv [\nabla_u, \nabla_v] w - \nabla_{[u,v]} w \end{array} \right.$$

with linearity over function in all arguments

$$R(fu, gv)hw = fgh\, R(u,v)w$$

→ One may think of the curvature as a matrix valued 2-form since

$$R(u,v)w = -R(v,u)w$$

→ In components the definition reads

$$R^\alpha{}_{\sigma\mu\nu} w^\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) w^\alpha + T_\mu{}^\sigma{}_\nu \nabla_\sigma w^\alpha$$

In a general basis, we compute:

$$R(e_a, e_b)e_c = \nabla_a \nabla_b e_c - \nabla_b \nabla_a e_c - D_a{}^d{}_b \nabla_d e_c$$
$$= \nabla_a (\Gamma_c{}^d{}_b e_d) - \nabla_b (\Gamma_c{}^d{}_a e_d) - D_a{}^d{}_d \Gamma_c{}^f{}_d e_f$$
$$= (\partial_a \Gamma_c{}^f{}_b - \partial_b \Gamma_c{}^f{}_a + \Gamma_d{}^f{}_a \Gamma_c{}^d{}_b - \Gamma_d{}^f{}_b \Gamma_c{}^d{}_a) e_f$$
$$\equiv R^f{}_{cab} e_f$$

We found:

$$R^f{}_{cab} \equiv \partial_a \Gamma_c{}^f{}_b + \Gamma_d{}^f{}_a \Gamma_c{}^d{}_b - \partial_b \Gamma_c{}^f{}_a - \Gamma_d{}^f{}_b \Gamma_c{}^d{}_a - D_a{}^d{}_b \Gamma_c{}^d{}_d$$

DEF We define the connection 1-form $\Gamma^a{}_b$ by

$$\Gamma^a{}_b \equiv \Gamma_b{}^a{}_c \overset{*}{e}{}^c = \Gamma_b{}^a{}_\mu dx^\mu$$

→ We can compute:

$\Gamma_c{}^f{}_d$ R ?

$$d\Gamma^f{}_c + \Gamma^f{}_d \wedge \Gamma^d{}_c = \overset{*}{e}{}^a \partial_a \Gamma_c{}^f{}_d \overset{*}{e}{}^d + \Gamma_d{}^f{}_c (T^d - \Gamma^d{}_b \overset{*}{e}{}^b)$$
$$+ \Gamma_d{}^f{}_a \Gamma_c{}^d{}_b \overset{*}{e}{}^a \wedge \overset{*}{e}{}^b = \overset{*}{e}{}^a \wedge \overset{*}{e}{}^b (\partial_{[a} \Gamma_{|c|}{}^f{}_{b]} + \Gamma_c{}^f{}_d (\Gamma_{[b}{}^d{}_{a]} + \tfrac{1}{2} D_b{}^d{}_a$$
$$- \Gamma_{[b}{}^d{}_{a]}) + \Gamma_d{}^f{}_{[a} \Gamma_{|c|}{}^d{}_{b]}) = \tfrac{1}{2} R^f{}_{cab} \overset{*}{e}{}^a \wedge \overset{*}{e}{}^b$$

DEF | We define the curvature 2-form $R^a{}_b$ by

$$R^a{}_b \equiv \frac{1}{2} R^a{}_{bcd}\, \overset{*}{e}{}^c \wedge \overset{*}{e}{}^d$$

Prop | The Cartan's second structural equation reads

$$R^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b$$

→ One may, thinking of matrices, simply write

$$R = d\Gamma + \Gamma \wedge \Gamma = d\Gamma + \frac{1}{2}[\Gamma \wedge \Gamma] = D\Gamma$$

Prop | Under a frame rotation, R transforms as

$$R' = \Lambda R \Lambda^{-1} \Leftrightarrow R'^a{}_b = \Lambda^a{}_c R^c{}_d \Lambda^d{}_b$$

Note that under a coord. transformation, R is a scalar.

PROP | We have $[\nabla_a, \nabla_b] v^c = R^c{}_{fab} v^f - T_a{}^d{}_b \nabla_d v^c$

DEMO Indeed,

$$[\nabla_a, \nabla_b] v^c = \partial_a(\nabla_b v^c) - \Gamma_b{}^f{}_a \nabla_f v^c + \Gamma_f{}^c{}_a \nabla_b v^f$$
$$- \partial_b(\nabla_a v^c) - \Gamma_a{}^f{}_b \nabla_f v^c + \Gamma_f{}^c{}_b \nabla_a v^f$$
$$= \partial_a \partial_b v^c + \partial_a \Gamma_f{}^c{}_b v^f - \Gamma_b{}^f{}_a \partial_f v^c - \Gamma_b{}^f{}_a \Gamma_g{}^c{}_f v^g$$
$$+ \Gamma_f{}^c{}_a \partial_b v^f + \Gamma_f{}^c{}_a \Gamma_g{}^f{}_b v^g - (a \leftrightarrow b)$$
$$= D_a{}^d{}_b \partial_d v^c + (\partial_a \Gamma_f{}^c{}_b - \partial_b \Gamma_f{}^c{}_a + \Gamma_d{}^c{}_a \Gamma_f{}^d{}_b - \Gamma_d{}^c{}_b \Gamma_f{}^d{}_a) v^f$$
$$- 2\Gamma_{[b}{}^d{}_{a]} \nabla_d v^c + D_a{}^d{}_b \nabla_d v^c - D_a{}^d{}_b \Gamma_f{}^c{}_d v^f$$
$$= R^c{}_{fab} v^f - T_a{}^d{}_b \nabla_d v^c$$

∎

Note p214 ⊙ **Vielbeins and $GL(n, \mathbb{R})$ Lie algebra:**

→ Consider an n dimensional spacetime with a moving frame

$$e_a = e_a{}^\mu \partial_\mu \quad \text{and} \quad e^a = e^a{}_\mu dx^\mu$$

where $e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu$ and $e_a{}^\mu e^b{}_\mu = \delta^b_a$. We write $\partial_a f = e_a(f)$. Under a combined frame and coord. transformation, we have $e'_a{}^\mu(x') = \Lambda_a{}^b(x)\, e_b{}^\nu(x)\, \Lambda^\mu{}_\nu(x)$, where $\Lambda^a{}_b \in GL(n, \mathbb{R})$ with $\Lambda_a{}^b = (\Lambda^{-1})^b{}_a$ while

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \quad \text{and} \quad \Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}$$

→ The generators for the Lie algebra of $GL(n,\mathbb{R})$ are denoted by $\Delta_a{}^b$, with $(\Delta_a{}^b)^c{}_d = \delta_a^c \delta_d^b$. They satisfy
$$[\Delta_a{}^b, \Delta_c{}^d] = \delta_c^b \Delta_a{}^d - \delta_a^d \Delta_c{}^b$$

→ In the vector representation, the generators are denoted t_a. We have $\Delta_a{}^b t_c = \delta_c^b t_a$

→ The structure functions are defined by
$$[e_a, e_b] = D_a{}^c{}_b\, e_c \Leftrightarrow de^a = -\frac{1}{2} D_b{}^a{}_c\, e^b e^c$$

→ We assume that there is an affine connection
$$\nabla_a e_b = \Gamma_b{}^c{}_a\, e_c \Leftrightarrow \nabla_b v^a = \partial_b v^a + \Gamma_c{}^a{}_b\, v^c$$
Writing $\Gamma \equiv \Gamma^a{}_b \Delta_a{}^b = \Gamma_b{}^a{}_c\, e^c \Delta_a{}^b$

and $\overset{*}{e} \equiv \overset{*}{e}{}^a t_a$, the torsion tensor and curvature tensors are defined by:
$$\Upsilon = T^a t_a = d\overset{*}{e} + \Gamma \overset{*}{e} \text{ and } R \equiv R^a{}_b \Delta_a{}^b = d\Gamma + \frac{1}{2}[\Gamma, \Gamma]$$
where $[\,,\,]$ is the graded commutator.

↳ Explicitly, $T^a = \frac{1}{2} T_b{}^a{}_c\, e^b e^c = de^a + \Gamma^a{}_b\, e^b$, so that
$$T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \Gamma_b{}^a{}_\mu\, e^b{}_\nu - \Gamma_b{}^a{}_\nu\, e^b{}_\mu$$
$$T_a{}^c{}_b = 2\Gamma_{[b}{}^c{}_{a]} + D_b{}^c{}_a = 2(\Gamma_{[b}{}^c{}_{a]} + D_{[b}{}^c{}_{a]})$$
and $R^a{}_b = \frac{1}{2} R^a{}_{bcd}\, e^c e^d = d\Gamma^a{}_b + \Gamma^a{}_c \Gamma^c{}_b$

→ Under a local frame transformation, we have:
$$e' = \Lambda e \text{ and } \Gamma' = \Lambda \Gamma \Lambda^{-1} + \Lambda d \Lambda^{-1} \text{ so that}$$
$$\Upsilon' = \Lambda \Upsilon \text{ and } R' = \Lambda R \Lambda^{-1}$$

↳ Defining $\Lambda = \mathbb{1} + \omega + \mathcal{O}(\omega^2)$ with $\omega^a{}_b \Delta^a{}_b$ and $\omega_b{}^a = -\omega^a{}_b$, we have $\delta_\omega \Gamma = -(d\omega + [\Gamma, \omega])$
$$\Leftrightarrow \delta_\omega \Gamma^a{}_b = d\omega_b{}^a + \Gamma^a{}_c\, \omega_b{}^c - \Gamma^c{}_b\, \omega_c{}^a$$
and also $\delta_\omega e = [\omega, e] \Leftrightarrow \delta_\omega e^a = \omega^a{}_b e^b$

⊙ Bianchi identities:

→ We have: $R = d\Gamma + \Gamma^2$ and $dR = -[d\Gamma, \Gamma]$
$= -[R - \Gamma^2, \Gamma] = -[R, \Gamma]$

Thm (2nd Bianchi identity)
$$DR = 0 = \underbrace{\{d + [\Gamma, \ ]\}}^{\text{exterior covariant derivative}} R$$

$\boxed{\text{DEMO}}$ $DR = dR + [\Gamma, R]$
$= dd\Gamma + d(\Gamma^2) + [\Gamma, d\Gamma] + [\Gamma, \Gamma^2]$
$= d\Gamma.\Gamma - \Gamma d\Gamma + \Gamma d\Gamma - d\Gamma.\Gamma = 0$ ∎

→ The tension 2-form is $T = d\overset{*}{e} + \Gamma\overset{*}{e}$. Thus,
$$dT = d\Gamma.\overset{*}{e} - \Gamma d\overset{*}{e}$$
$$= d\Gamma.\overset{*}{e} - \Gamma(T - \Gamma\overset{*}{e})$$
$$= (d\Gamma - \Gamma^2)\overset{*}{e} - \Gamma T$$
$$= R\overset{*}{e} - \Gamma T$$

Thm (1st Bianchi identity)
$$DT = R\overset{*}{e} = dT - \Gamma T$$

→ In components, one has:

Action of d on T

$$d\left(\tfrac{1}{2} T_b{}^a{}_c \overset{*}{e}{}^b \overset{*}{e}{}^c\right) + \Gamma^a{}_b \tfrac{1}{2} T_c{}^b{}_d \overset{*}{e}{}^c \overset{*}{e}{}^d = \tfrac{1}{2} R^a{}_{bcd} \overset{*}{e}{}^c \overset{*}{e}{}^d \overset{*}{e}{}^b$$

$$\Leftrightarrow T_b{}^a{}_{c,d}\, e^d e^b e^c - T_b{}^a{}_c\, e^b\left(\tfrac{1}{2} T_f{}^c{}_g\, e^f e^g - 2\Gamma_f{}^c{}_d\, e^d e^f\right)$$
$$+ \tfrac{1}{2} \Gamma_b{}^a{}_d\, e^d\, T_c{}^b{}_f\, e^c e^f = \tfrac{1}{2} R^a{}_{bcd}\, e^c e^d e^b$$

Thm (1BI in components)
$$T^a{}_{[bc;d]} + T_f{}^a{}_{[b} T_c{}^f{}_{d]} = R^a{}_{[bcd]}$$

→ The 2nd BI becomes: $dR^a{}_b + \Gamma^a{}_c R^c{}_b + R^a{}_c \Gamma^c{}_b = 0$
$$\Leftrightarrow \tfrac{1}{2} R^a{}_{bcd,f}\, e^f e^c e^d - R^a{}_{bcd}\, e^c\left(\tfrac{1}{2} T_f{}^d{}_g\, e^f e^g - \Gamma_f{}^d{}_g\, e^g e^f\right)$$
$$+ \Gamma^a{}_c R^c{}_b - R^a{}_c \Gamma^c{}_b = 0$$

Thm (2BI in components)

$$R^a{}_{b[cd;f]} = -R^a{}_{bg[f} T_c{}^g{}_{d]}$$

⊙ Lorentz group and Poincaré algebra:

→ the Poincaré algebra is the Lie algebra of the Poincaré group, wich is the group of isometry of Minkowsky spacetime.
It's a 10-dim noncompact Lie group.

DEF The generators $\mathcal{J}_{ab}$, P_c of the Lorentz group follow the following algebra:

$$[\mathcal{J}_{ab}, \mathcal{J}_{cd}] = \eta_{bc}\mathcal{J}_{ad} - \eta_{ac}\mathcal{J}_{bd} - \eta_{bd}\mathcal{J}_{ac} + \eta_{ad}\mathcal{J}_{bc}$$

$$[\mathcal{J}_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b$$

$$[P_a, P_b] = 0$$

→ We take $\Gamma \equiv \frac{1}{2}\Gamma^{ab}\mathcal{J}_{ab}$ $\quad \overset{*}{e} \equiv \overset{*}{e}{}^a P_a$ $\quad R = \frac{1}{2} R^{ab}\mathcal{J}_{ab}$ and $T = T^a P_a$

We still have $T = d\overset{*}{e} + [\Gamma, \overset{*}{e}]$ with $[,]$ the graded commutator (anticommutator for forms)

Indeed, $[\Gamma, \overset{*}{e}] = \frac{1}{2}\Gamma^{ab}\,{}^*e_c\,[\mathcal{J}_{ab}, P_c]$
$= \frac{1}{2}\left(\Gamma^a{}_b\,\overset{*}{e}{}^b P_a - \Gamma_c{}^b\,\overset{*}{e}{}^c P_b\right) = \Gamma^a{}_b\,\overset{*}{e}{}^b P_a$

↳ $R = d\Gamma + \frac{1}{2}[\Gamma, \Gamma]$

Indeed, $\frac{1}{2}[\Gamma,\Gamma] = \frac{1}{8}\Gamma^{ab}\Gamma^{cd}[\mathcal{J}_{ab}, \mathcal{J}_{cd}]$
$= \frac{1}{8}\left(\Gamma^a{}_b\Gamma^{bd}\mathcal{J}_{ad} - \Gamma_c{}^b\Gamma^{cd}\mathcal{J}_{bd} - \Gamma^a{}_b\Gamma^{cb}\mathcal{J}_{ac} + \Gamma^{ab}\Gamma^c{}_a\mathcal{J}_{bc}\right)$
$= \frac{1}{2}\Gamma^a{}_c\Gamma^{cb}\mathcal{J}_{ab}$

→ Requiring metricity: $0 = \nabla_a g^{bc} = \partial_a g^{bc} + \Gamma_d{}^b{}_a g^{dc} + \Gamma_d{}^c{}_a g^{bd}$

We find $\Gamma^{bc}{}_a = -\Gamma^{cb}{}_a$

4.2 Cartan's action

sec 3.1 → 3.3
Gilkey

⊙ Euler-Lagrange equations:

→ In the standard Cartan formulation, the variables of the variation principle are the components of the vielbein $e_a{}^\mu$ and a Lorentz connection 1-form in the coord. basis, $\Gamma_b{}^a{}_\mu$.

DEF The Cartan's action S^c is given by

$$S^c[e_a{}^\mu, \Gamma^b{}_{c\nu}] \equiv \int d^n x\, |e| \{ R^{ab}{}_{\mu\nu} e_a{}^\mu e_b{}^\nu - 2\Lambda \}$$

where $R^{ab}{}_{\mu\nu} \equiv \partial_\mu \Gamma^{ab}{}_\nu + \Gamma^a{}_{c\mu} \Gamma^{cb}{}_\nu - \partial_\nu \Gamma^{ab}{}_\mu - \Gamma^a{}_{c\nu} \Gamma^{cb}{}_\nu$

→ In the Einstein-Hilbert action, we considered $g_{\mu\nu}$, wich contains 10 fields. Here, we consider $e_a{}^\mu$ as n^2 fields, and $\Gamma^{ab}{}_\mu$ wich contains $\frac{n(n-1)}{2} n$ fields, so for $d=4$, we have 40 fields.

Notice that S^c is 1st order in $e_a{}^\mu$ and $\Gamma^b{}_{c\nu}$.

→ To compute the EOM, let's first vary $R^{ab}{}_{\mu\nu}$:

$$\delta R^{ab}{}_{\mu\nu} = \partial_\mu \delta\Gamma^{ab}{}_\nu + \Gamma^a{}_{c\mu} \delta\Gamma^{cb}{}_\nu + \delta\Gamma^a{}_{c\mu} \cdot \Gamma^{cb}{}_\nu - \partial_\nu \delta\Gamma^{ab}{}_\mu - \Gamma^a{}_{c\nu} \delta\Gamma^{cb}{}_\mu - \delta\Gamma^a{}_{c\nu} \cdot \Gamma^{cb}{}_\mu$$

Now, $\nabla_\mu \delta\Gamma^{ab}{}_\nu = \partial_\mu \delta\Gamma^{ab}{}_\nu + \Gamma^a{}_{c\mu} \delta\Gamma^{cb}{}_\nu + \Gamma^b{}_{c\mu} \delta\Gamma^{ac}{}_\nu$

So we can write $\delta R^{ab}{}_{\mu\nu} = \nabla_\mu \delta\Gamma^{ab}{}_\nu - \nabla_\nu \delta\Gamma^{ab}{}_\mu$

→ The variation of the action reads:

$$\delta S^c = \int d^n x\, |e| \{ (\nabla_\mu \delta\Gamma^{ab}{}_\nu - \nabla_\nu \delta\Gamma^{ab}{}_\mu) e_a{}^\mu e_b{}^\nu + (-e^a{}_\mu (R - 2\Lambda) + R^{ab}{}_{\mu\nu} e_b{}^\nu + R^{ba}{}_{\mu\nu} e_b{}^\nu) \delta e_a{}^\mu \}$$

DEF We define the Einstein tensor $G^a{}_\mu$ by

$$G^a{}_\mu \equiv R^a{}_\mu - \frac{1}{2} e^a{}_\mu R$$

and the Ricci tensor $R^a{}_\mu$ by $R^{ab}{}_{\mu\nu} e_b{}^\nu \equiv R^a{}_\mu$

↳ the variation of the action can be written as

$$\delta S^c = \int d^n x \, |e| \{ (\nabla_\mu \delta \Gamma^{ab}{}_\nu - \nabla_\nu \delta \Gamma^{ab}{}_\mu) e_a{}^\mu e_b{}^\nu + \underbrace{(-e^a{}_\mu (R - 2\Lambda) + 2R^a{}_\mu)}_{= 2(G^a{}_\mu - \Lambda e^a{}_\mu)} \delta e_a{}^\mu \}$$

DEF We define the covariant derivative of a tensor density as

$$\nabla_\mu |e| = |e| (e_b{}^\nu \partial_\mu e^b{}_\nu)$$

↳ Then, $\partial_\mu (|e| v^\mu) = |e| (D_\mu + e_b{}^\nu \partial_\mu e^b{}_\nu) v^\mu = \nabla_\mu (|e| v^\mu)$

→ Integrating by part and dropping the boundary term, we find

$$\delta S^c = \int d^n x \{ 2|e| (G^a{}_\mu + \Lambda e^a{}_\mu) \delta e_a{}^\mu + 2 \nabla_\nu (|e| e_{[a}{}^\mu e_{b]}{}^\nu) \delta \Gamma^{ab}{}_\mu \}$$

So that:

$$\begin{cases} \dfrac{\delta \mathcal{L}^c}{\delta e_a{}^\mu} = 2|e| (G^a{}_\mu + \Lambda e^a{}_\mu) \\[2ex] \dfrac{\delta \mathcal{L}^c}{\delta \Gamma^{ab}{}_\mu} = 2 \nabla_\nu (|e| e_{[a}{}^\mu e_{b]}{}^\nu) \end{cases}$$

→ Let's study $\delta \mathcal{L}^c / \delta \Gamma$:

$$\begin{aligned}
2 \nabla_\nu (|e| e_{[a}{}^\mu e_{b]}{}^\nu) &= |e| (e_c{}^\lambda \partial_\nu e^c{}_\lambda . (e_a{}^\mu e_b{}^\nu - e_b{}^\mu e_a{}^\nu)) \\
&\quad + |e| (\nabla_\nu e_a{}^\mu . e_b{}^\nu + e_a{}^\mu \nabla_\nu e_b{}^\nu - \nabla_\nu e_b{}^\mu . e_a{}^\nu - e_b{}^\mu \nabla_\nu e_a{}^\nu) \\
&= |e| (\partial_b e_a{}^\mu - \Gamma_a{}^d{}_\nu e_d{}^\mu . e_b{}^\nu - \partial_a e_b{}^\mu + \Gamma_b{}^d{}_\nu e_d{}^\mu . e_a{}^\nu) \quad ① \\
&\quad + |e| (e_a{}^\mu (e_c{}^\lambda \partial_\nu e^c{}_\lambda . e_b{}^\nu + \nabla_\nu e_b{}^\nu)) \quad ② \\
&\quad - |e| (e_b{}^\mu (e_c{}^\lambda \partial_\nu e^c{}_\lambda . e_a{}^\nu + \nabla_\nu e_a{}^\nu))
\end{aligned}$$

$$① = D_b{}^\mu{}_a - e_d{}^\mu \Gamma_a{}^d{}_b + e_d{}^\mu \Gamma_b{}^d{}_a = 2 T_{[b}{}^\mu{}_{a]} + D_b{}^\mu{}_a = T_a{}^\mu{}_b$$

$$\begin{aligned}
② &= e_a{}^\mu (e_c{}^\lambda \partial_\nu e^c{}_\lambda + \partial_\lambda e_b{}^\lambda - \Gamma_b{}^c{}_\lambda e_c{}^\lambda) \\
&= e_a{}^\mu (e_c{}^\lambda \partial_\nu e^c{}_\lambda - e_b{}^\nu \partial_\lambda e^c{}_\nu . e_c{}^\lambda - \Gamma_b{}^c{}_c) \\
&= e_a{}^\mu (e_c{}^\lambda e_b{}^\nu (\partial_\nu e^c{}_\lambda - \partial_\lambda e^c{}_\nu) - \Gamma_b{}^c{}_c) \\
&= e_a{}^\mu (e_c{}^\lambda e_b{}^\nu (-D_\nu{}^c{}_\lambda) - \Gamma_b{}^c{}_c + \Gamma_c{}^c{}_b) \rightarrow = 0 \qquad \text{as } \Gamma_c{}^c{}_b = 0 \\
&= e_a{}^\mu (-D_b{}^c{}_c - 2\Gamma_{[b}{}^c{}_{c]}) = e_a{}^\mu T_b{}^c{}_c \\
&= - e_b{}^\mu T_a{}^c{}_c
\end{aligned}$$

We can now write

$$\frac{\delta \mathcal{L}^c}{\delta \Gamma^{ab}{}_\mu} = |e| (T_a{}^\mu{}_b + 2 e_{[a}{}^\mu T_{b]}{}^c{}_c)$$

→ Setting $\delta\mathcal{L}'/\delta\Gamma \overset{!}{=} 0$, we have

$T_a{}^\mu{}_b + 2 e_{[a}{}^\mu T_{b]}{}^c{}_c = 0 \quad \| \, e_\mu{}^b$

$\Leftrightarrow \; T_a{}^b{}_b + T_a{}^c{}_c - n T_a{}^c{}_c = 0$

$\Leftrightarrow \; (n-2) T_a{}^b{}_b = 0$

For $n \neq 2$, we have $T_a{}^b{}_b \overset{!}{=} 0$

↳ Reinjecting, we have

$|e|(T_a{}^\mu{}_b + 2 e_{[a}{}^\mu T_{b]}{}^c{}_c) = 0 \Leftrightarrow T_a{}^b{}_c = 0$

It follows that when the equations of motion of $\Gamma^{ab}{}_\mu$ hold, the connection is torsionless and thus given by

$\{a b c\} = \{a (bc)\}$ → $\Gamma = \{\} + K + M + r \; : \; \Gamma_{abc} = r_{abc}$

PROP The fields $\Gamma^{ab}{}_\mu$ being entirely determined by $e_a{}^\mu$, they are auxiliary fields: $\Gamma^{ab}{}_\mu = \Gamma^{ab}{}_\mu(e_c{}^\nu, \partial_\rho e_d{}^\lambda)$

⊙ Noether identities:

→ Let's recall that the infinitesimal gauge transformation are obtained as follows:

→ Defining $\Lambda = \mathbb{1} + \omega + \mathcal{O}((\omega^2))$ such that $\omega = \frac{1}{2}\omega^{ab} J_{ab}$ with $\omega^{ab} = -\omega^{ba}$

p37 // → We have $\delta_\omega \Gamma = -(d\omega + [\Gamma, \omega])$

$\Leftrightarrow \delta_\omega \Gamma^{ab} = -(d\omega^{ab} + \Gamma^a{}_c \omega^{cb} + \Gamma^b{}_c \omega^{ac})$

→ We also have $\delta_\omega e = [\omega, e] \Leftrightarrow \delta_\omega e^a = \omega^a{}_b e^b$

→ Under a coordinate transformation,

$$\begin{cases} e'_a{}^\mu = \dfrac{\partial x^\mu}{\partial x^\nu} e_a{}^\nu = \Lambda^\mu{}_\nu e_a{}^\nu \\ \Gamma'^{ab}{}_\mu = \dfrac{\partial x^\nu}{\partial x^\mu} \Gamma^{ab}{}_\nu = \Lambda_\mu{}^\nu \Gamma^{ab}{}_\nu \end{cases}$$

→ Under a change of frame,

check

$$\begin{cases} e'_a{}^\mu = \Lambda_a{}^b e_b{}^\mu \\ \Gamma'^{ab}{}_\mu = \Lambda^a{}_c \Gamma^{cd}{}_\mu (\Lambda^{-1})_d{}^b + \Lambda^a{}_c d(\Lambda^{-1})_c{}^b \end{cases}$$

We need $\Lambda^a{}_b \in O(1, n-1)$ since $g_{\mu\nu} = e^a{}_\mu \eta_{ab} e^b{}_\nu$

→ Taking $x'^\mu = x^\mu - \xi^\mu + O(\xi^2)$ so that $\omega_\nu{}^\mu = \partial_\nu \xi^\mu$

$\Lambda^\mu{}_\nu = \delta^\mu_\nu - \partial_\nu \xi^\mu + O(\xi^2)$

We find:

$$\begin{cases} \delta_{\xi,\omega} e_a{}^\mu = \xi^\rho \partial_\rho e_a{}^\mu - \xi^\mu{}_{,\rho} e_a{}^\rho + \omega_a{}^b e_b = \mathcal{L}_\xi e_a{}^\mu + \omega_a{}^b e_b{}^\mu \\ \delta_{\xi,\omega} \Gamma^{ab}{}_\mu = \xi^\rho \partial_\rho \Gamma^{ab}{}_\mu + \xi^\rho{}_{,\mu} \Gamma^{ab}{}_\rho - \nabla_\mu \omega^{ab} = \mathcal{L}_\xi \Gamma^{ab}{}_\mu - \nabla_\mu \omega^{ab} \end{cases}$$

with our previous notation, $f^\alpha = (\xi^\mu, \omega^{ab})$

→ A gauge symmetry had been defined as $\delta_f \phi^i = R^i{}_\alpha [f^\alpha]$

$\Rightarrow \delta_f \mathcal{L} = \partial_\mu k_f^\mu$, and the Noether identity as:

$R^{+i}_\alpha \left[\frac{\delta \mathcal{L}}{\delta \phi^i}\right] = 0$ with $R^i{}_\alpha [f^\alpha] \frac{\delta \mathcal{L}}{\delta \phi^i} = f^\alpha R^{+i}_\alpha \left[\frac{\delta \mathcal{L}}{\delta \phi^i}\right] + \partial_\mu S_f^\mu$

→ For a combined infinitesimal gauge transformation we have

$$\delta_f S^C = \delta_{\xi,\omega} S^C = \int d^n x \left\{ \frac{\delta \mathcal{L}^C}{\delta e_a{}^\mu} \delta_{\xi,\omega} e_a{}^\mu + \frac{\delta \mathcal{L}^C}{\delta \Gamma^{ab}{}_\mu} \delta_{\xi,\omega} \Gamma^{ab}{}_\mu \right\} \stackrel{!}{=} 0$$

→ For ω^{ab}, we have:

$$\frac{\delta \mathcal{L}^C}{\delta e^{a\mu}} \cdot \omega^{ab} e_b{}^\mu + \frac{\delta \mathcal{L}^C}{\delta \Gamma^{ab}{}_\mu} \cdot (-\nabla_\mu \omega^{ab}) = 0$$

$$\Leftrightarrow \frac{\delta \mathcal{L}^C}{\delta e^{[a|\mu|}} e_{b]}{}^\mu + \nabla_\mu \frac{\delta \mathcal{L}^C}{\delta \Gamma^{ab}{}_\mu} = 0 \quad (*)$$

→ For ξ^ρ, we have:

$$\frac{\delta \mathcal{L}^C}{\delta e_a{}^\mu} \cdot (\xi^\rho \partial_\rho e_a{}^\mu - \xi^\mu{}_{,\rho} e_a{}^\rho) + \frac{\delta \mathcal{L}^C}{\delta \Gamma^{ab}{}_\mu} \cdot (\xi^\rho \partial_\rho \Gamma^{ab}{}_\mu + \xi^\rho{}_{,\mu} \Gamma^{ab}{}_\rho) \stackrel{!}{=} 0$$

$$\Leftrightarrow \frac{\delta \mathcal{L}^C}{\delta e_a{}^\mu} \partial_\rho e_a{}^\mu + \frac{\delta \mathcal{L}^C}{\delta \Gamma^{ab}{}_\mu} \partial_\rho \Gamma^{ab}{}_\mu + \partial_\mu \left[\frac{\delta \mathcal{L}^C}{\delta e_a{}^\rho} e_a{}^\mu - \frac{\delta \mathcal{L}^C}{\delta \Gamma^{ab}{}_\mu} \Gamma^{ab}{}_\rho \right] = 0 \quad (\Delta)$$

$(*) \Leftrightarrow$ to $R_{ab} = R_{ba}$ up to torsion term and (Δ) is equivalent to the contracted Bianchi identity with torsion.

4.3 Einstein-Dirac theory

→ Cartan calculus is needed because Dirac fermions form a representation of the Lorentz group in flat space.

$$\Psi : D(\tfrac{1}{2},0) \oplus D(0,\tfrac{1}{2})$$

↳ We use the vielbein to transform 'world' indices into 'flat' tangent space indices

→ Using Weinberg convention,

$$\delta_\omega \Psi = -\frac{1}{2}\omega_{ab} S^{ab} \Psi \quad \text{with } S^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$$

⇒ representation of Lorentz algebra.

↳ We're gonna use the following representation:

$$\gamma^0 = -i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \text{ and } \gamma^k = -i\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \text{ with } \{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{1}_4$$

and $\mathrm{Sgn}(\eta^{ab}) = (-+++)$

↳ We also define $\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ and $\beta = i\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

↳ Then, $\bar{\Psi} \equiv \Psi^\dagger \beta$ since $(\beta\gamma^a)^\dagger = -\beta\gamma^a$ ↳ $\beta^\dagger = \beta$

DEF The Einstein-Dirac Lagrangian S^{ED} is defined as

$$S^{ED}[e,\Psi,\bar{\Psi}] \equiv \frac{1}{16\pi G}\int d^4x\,|e|\left(R - 2\Lambda - \bar{\Psi}(\gamma^\mu \nabla_\mu - m)\Psi\right)$$

$$= S^G + S^F$$

where $\nabla_\mu \Psi \equiv \partial_\mu \Psi - \frac{1}{2}\Gamma_{ab,\mu} S^{ab}\Psi$ and $\gamma^\mu \equiv e_a{}^\mu \gamma^a$ and where Γ is the torsion free Lorentz connection.

→ Equivalently, we can write

$$S^{ED} = \frac{1}{16\pi G}\int d^4x\,|e|\left\{R - 2\Lambda - \bar{\Psi}(\gamma^a \nabla_a - m)\Psi\right\}$$

with $\nabla_a \Psi = e_a{}^\mu \nabla_\mu \Psi$

→ Notice that $v^a(x)$ transforms as a scalar under coord. transformation since $v^a(x) = v^\mu(x)\, e^a{}_\mu(x)$

→ Using the result about auxiliary field, we have:

$$\frac{\delta \mathcal{L}^G}{\delta e_a{}^\mu} = \frac{\delta \mathcal{L}^C}{\delta e_a{}^\mu}\bigg|_{\Gamma=\Gamma[e]} = \frac{|e|}{8\pi G}\left(G^a{}_\mu + \Lambda e^a{}_\mu\right)\bigg|_{\Gamma=\Gamma[e]}$$

$$\frac{\delta \mathcal{L}^{ED}}{\delta e_a{}^\mu} = \frac{\delta \mathcal{L}^C}{\delta e_a{}^\mu}\bigg|_{\Gamma=\Gamma[e]} + \frac{\delta \mathcal{L}^F}{\delta e_a{}^\mu}$$

↳ In the end, $\displaystyle\frac{\delta \mathcal{L}^{ED}}{\delta e_a{}^\mu} = \frac{|e|}{8\pi G}\left(G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2}T_{\mu\nu}\right)$

with $T_{\mu\nu} = \frac{1}{2}\left(-\bar{\Psi}\gamma_{(\mu}\nabla_{\nu)}\Psi + \nabla_{(\mu}\bar{\Psi}\gamma_{\nu)}\Psi\right)$

4.4 Gravity as a Chern-Simons theory

→ We consider Cartan's action S^C in dimension $d=3$ with a negative cosmological constant $\Lambda \equiv \frac{-1}{\ell^2}$ (anti-de Sitter):

$$S^C[e_a{}^\mu, \Gamma^{bc}{}_\nu] = \frac{1}{16\pi G}\int d^3x\, |e|\,(R - 2\Lambda)$$

with convention $\eta_{ab} = (-1, 1, 1)$

→ Defining $\epsilon_{012} \equiv 1$ and $\epsilon^{a_1 a_2 a_3} = \eta^{a_1 b_1}\eta^{a_2 b_2}\eta^{a_3 b_3}\epsilon_{b_1 b_2 b_3}$

↳ We have $\epsilon^{012} = \eta^{0b_1}\eta^{1b_2}\eta^{2b_2} = (-1).1.1 = -1$

p39 → Reminder: Lorentz generators J_{ab} have the following algebra:
$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}$
and we define $\Gamma = \frac{1}{2}\Gamma^{ab}{}_\mu J_{ab}\, dx^\mu$

→ We define $J^a = \frac{-1}{2}\epsilon^{abc}J_{bc} \Leftrightarrow J_{ab} = \epsilon_{abc}J^c$ and
$\Gamma_c = \frac{1}{2}\epsilon_{cab}J^{ab} \Leftrightarrow \Gamma^{ab} = -\epsilon^{abc}\Gamma_c$

Lorentz algebra then reads: $[J_a, J_b] = \epsilon_{abc}J^c$

It's the algebra of $so(1,2)$

→ The action reads $16\pi G S^c = \int d^3x\, |e| \left(-\epsilon^{abc} R_{c\mu\nu} e_a{}^\mu e_b{}^\nu - 2\Lambda\right)$
where we defined $R^{ab} = -\epsilon^{abc} R_c$

→ Now, let's notice that
$\epsilon_{abc}\, e^a \wedge e^b \wedge e^c = \epsilon_{abc}\, e^a{}_\mu e^b{}_\nu e^c{}_\rho\, dx^\mu \wedge dx^\nu \wedge dx^\rho$
$= 3!\, |e|\, d^3x$ so $3! \int d^3x\, |e| = \int d^3x\, \epsilon_{abc}\, e^a e^b e^c$
Furthermore,
$\frac{1}{3!} \epsilon_{abc}\, e^a e^b e^c \cdot (-1) \epsilon^{def} R_{f\mu\nu}\, e_d{}^\mu e_e{}^\nu$

$= \frac{1}{3!} 3!\, \delta^{def}_{[abc]} R_{f\mu\nu}\, e_d{}^\mu e_e{}^\nu \cdot e^a \wedge e^b \wedge e^c$

$= \frac{1}{3!} \cdot 2 \cdot R_{f\mu\nu}\, e_d{}^\mu e_e{}^\nu \left(e^a e^b e^c + e^b e^c e^a + e^c e^a e^b\right)$

$= R_{f\mu\nu}\, dx^\mu \wedge dx^\nu \wedge e^f = 2 R_f\, e^g$

↳ The action now reads:
$16\pi G S^c = \int -\frac{\Lambda}{3} \epsilon_{abc}\, e^a e^b e^c + 2 R_c \wedge e^c)$

$\Leftrightarrow S^c = \frac{1}{8\pi G} \int \left(R_c \wedge e^c + \frac{1}{6\ell^2} \epsilon_{abc}\, e^a \wedge e^b \wedge e^c\right)$

⊙ Adjoint representation:

→ We have $[T_a, T_b] = f_a{}^c{}_b T_c$
→ $(\delta_a)^b{}_c = f_a{}^b{}_c \Rightarrow [\delta_a, \delta_b] = f_a{}^c{}_b\, \delta_c \Leftrightarrow$ Jacobi id.

→ $\delta_a v^b = -f_a{}^b{}_c v^c$ and
$[\delta_a, \delta_b] v^d = \delta_a(-f_b{}^d{}_g v^g) - \delta_b(-f_a{}^d{}_g v^g)$
$= (f_b{}^d{}_g f_a{}^g{}_h - f_a{}^d{}_g f_b{}^g{}_h) v^h + f_a{}^d{}_g f_h{}^g{}_b v^h$
$= -f_h{}^d{}_g f_b{}^g{}_a v^h = -f_a{}^g{}_b f_g{}^d{}_h v^h = f_a{}^g{}_b \delta_g v^d$

→ The coadjoint representation is $(\delta^T_a)^b{}_c = -f_a{}^b{}_c$

→ An invariant inner product is $\delta^T_c g_{ab} = 0$
$\Leftrightarrow f_a{}^d{}_c g_{db} + f_b{}^d{}_c g_{ad} = 0 \Leftrightarrow f_{bac} + f_{abc} = 0$

③ Extend the algebra:

→ We write $A = e^a{}_\mu dx^\mu P_a + \Gamma^a{}_\mu dx^\mu J_a$ where

$$[J_a, J_b] = \epsilon_{abc} J^c$$
$$[J_a, P_b] = \epsilon_{abc} P^c$$
$$[P_a, P_b] = \frac{1}{\ell^2} \epsilon_{abc} J^c$$

PROP $\langle J_a, P_b \rangle = \eta_{ab}$ $\langle J_a, J_b \rangle = 0 = \langle P_a, P_b \rangle$ is invariant

DEMO

→ $\delta_{J_c} \langle J_a, P_b \rangle = \langle [J_c, J_a], P_b \rangle + \langle J_a, [J_c, P_b] \rangle$
$= \epsilon_{cad} \langle J^d, P_b \rangle + \epsilon_{cbd} \langle J_a, P^d \rangle = \epsilon_{ca}{}^d \eta_{db} + \epsilon_{cb}{}^d \eta_{ad}$
$= \epsilon_{cab} + \epsilon_{cba} = \epsilon_{cab} - \epsilon_{cab} = 0$

→ $\delta_J \langle J, J \rangle = \delta_J \langle P, P \rangle = 0$

→ $\delta_{P_c} \langle J_a, J_b \rangle = \langle [P_c, J_a], J_b \rangle + \langle J_a, [P_c, J_b] \rangle$
$= -\epsilon_{cad} \langle P^d, J_b \rangle - \epsilon_{cbd} \langle J_a, P^d \rangle$
$= -\epsilon_{cab} - \epsilon_{cba} = 0$

→ $\delta_{P_c} \langle P_a, P_b \rangle = \langle [P_c, P_a], P_b \rangle + \langle P_a, [P_c, P_b] \rangle$
$= \ell^{-2} \epsilon_{cad} \langle J^d, P_b \rangle + \ell^{-2} \epsilon_{cbd} \langle P_a, J^d \rangle$
$= \ell^{-2} (\epsilon_{cab} + \epsilon_{cba}) = 0$ ∎

DEF The Chern-Simons action S^{CS} is given by

$$S^{CS}[A] = \frac{-k}{4\pi} \int_{\mathcal{M}} \langle A, dA + \frac{2}{3} A^2 \rangle$$

→ Let's develop this action and show that $S^{CS} \leftrightarrow S^C$:

$S^{CS} = -k/4\pi \,.\, \int \langle A, dA + 2/3 \,.\, A^2 \rangle$

$= -k/4\pi \,.\, \int \langle e^a P_a + \Gamma^a J_a, d(e^a P_a + \Gamma^a J_a) + \frac{1}{3} [A, A] \rangle$

$= \frac{-k}{4\pi} \int \langle e^a P_a, d\Gamma^b J_b + \frac{1}{3} [A, A]^b J_b \rangle$
$\qquad + \langle \Gamma^a J_a, de^b P_b + \frac{1}{3} [A, A]^b P_b \rangle$

$= \frac{-k}{4\pi} \int \langle$