5 ADM FORMALISM

→ The Arnowitt-Deser-Misner formalism is a hamiltonian formulation of general relativity that plays an important role in canonical quantum gravity and numerical relativity. It was published in 1959

5.1 Non-null hypersurfaces

DEF In a 4-D spacetime manifold, a hypersurface is a 3-D submanifold that can be either timelike, spacelike or null.

→ A hypersurface can be defined with an explicit form:

$$f(x^\mu) = 0 \quad \text{with } \mu = 0, 1, \dots, n-1$$

or with a parametric form:

$$x^\mu = x^\mu(y^A) \quad \text{with } A = 1, \dots, n-1$$

→ Example: ${}^2S$ in $\mathbb{R}^3$.

The 2-sphere in 3-D flat space is described either by $f = x^2+y^2+z^2-R^2$ (explicit form) or by $\begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}$ with (θ, ϕ) the intrinsic coordinates.

→ The relation $x^\mu(y^A)$ describe curves contained entirely in Σ.

⊙ Normal vector:

→ The vector $f_{,\mu}\, dx^\mu$ is conormal to the hypersurface because the value of f changes only in the direction orthogonal to Σ:

Let $v^\mu \equiv \dfrac{\partial f}{\partial x^\mu}$ be the tangent vector, and $c(t)$ a curve that lies inside of Σ. Then: $\langle df, v \rangle = 0 \Rightarrow df = \dfrac{\partial f}{\partial x^\mu} \cdot dx^\mu$ is conormal to Σ, of coord. $n_\mu \equiv \dfrac{\partial f}{\partial x^\mu}$

DEF We introduce a unit normal n_μ defined by:

$$n_\mu n^\mu \equiv \epsilon \begin{cases} \equiv 1 & \text{if } \Sigma \text{ is timelike} \\ \equiv -1 & \text{if } \Sigma \text{ is spacelike} \\ \equiv 0 & \text{if } \Sigma \text{ is null} \end{cases}$$

and requiring that $n_\mu \partial^\mu f > 0$

↳ always in the direction of $f \nearrow$

PROP For a non-null hypersurface, the unit normal is given by

$$n_\mu = \frac{\epsilon \, \partial_\mu f}{\sqrt{|g^{\mu\nu} \partial_\mu f \, \partial_\nu f|}}$$

DEMO $n_\mu n^\mu = \dfrac{\epsilon^2 \partial_\mu f \, \partial^\mu f}{|\partial_\mu f \, \partial^\mu f|} = \epsilon^3 = \epsilon$ ■

⊙ Induced metric:

→ the metric intrinsic to the hypersurface Σ is obtained by restricting the line element to displacements confined to the hypersurface.

→ Recalling the parametric equations $x^\mu = x^\mu(y^A)$, we have that the vectors $e_A{}^\mu \equiv \dfrac{\partial x^\mu}{\partial y^A}$ are tangent to curves contained in Σ.

PROP $n_\mu e_A{}^\mu = 0$

Check!

DEMO Indeed, $n_\mu e_A{}^\mu \propto \dfrac{\partial f}{\partial x^\mu} \cdot \dfrac{\partial x^\mu}{\partial y^A} = \dfrac{df}{dy^A} = 0$ ■

DEF We define the induced metric or first fundamental form h_{AB} by

$$h_{AB} \equiv g_{\mu\nu}(x(y)) \frac{\partial x^\mu}{\partial y^A} \frac{\partial x^\nu}{\partial y^B}$$ such that,

for displacements within Σ, we have:

$$ds_\Sigma^2 = g_{\mu\nu}(x(y)) \frac{\partial x^\mu}{\partial y^A} \frac{\partial x^\nu}{\partial y^B} dy^A \, dy^B$$

$$= h_{AB} \, dy^A \, dy^B$$

→ Under a spacetime coordinate transformation, $x^\mu \mapsto x^{\mu'}$, h_{AB} is a scalar. Under a surface coord. transfo $y^A \mapsto y^{A'}$, h_{AB} is a tensor

Not We write $y^a = y^A$ for one hypersurface, and

$y^a = (\ell, y^A)$ for a foliation

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PROP The inverse metric can be written as

$$g^{\mu\nu} = \epsilon n^\mu n^\nu + h^{AB} e_A{}^\mu e_B{}^\nu$$

DEMO Indeed, this is equivalent to $g_{\mu\nu} = \epsilon n_\mu n_\nu + h^{AB} e_A{}^\sigma e_B{}^\rho g_{\sigma\mu} g_{\rho\nu}$

We need to check $\langle n, n\rangle = \epsilon$, $\langle n, e_A\rangle = 0$ and $\langle e_A, e_B\rangle = h_{AB}$

→ $\langle n,n\rangle = n_\mu g^{\mu\nu} n_\nu = n_\mu \epsilon n^\mu n^\nu n_\nu = \epsilon^3 = \epsilon$

→ $\langle n, e_A\rangle = n^\mu g_{\mu\nu} e_A{}^\nu = n^\mu (\epsilon n_\mu n_\nu + h^{AB} e_A{}^\sigma e_B{}^\rho g_{\sigma\mu} g_{\rho\nu}) e_A{}^\nu = 0$

→ $\langle e_A, e_B\rangle = e_A{}^\mu (\epsilon n_\mu n_\nu + h^{CD} e_C{}^\sigma e_D{}^\rho g_{\sigma\mu} g_{\rho\nu}) e_B{}^\nu$

$= h^{CD} h_{CA} h_{DB} = h_{AB}$ ∎

?(

→ Another useful way of writing this is the following decomposition of the identity: $\delta^\mu{}_\nu = \epsilon n^\mu n_\nu + h^{AB} e_A{}^\mu e_{B\nu}$

→ This means that if we go from the coord. basis $\partial/\partial x^\mu$ to the moving frame on Σ, we have the following decomposition:

→ $e_a{}^\mu = (n^\mu, e_A{}^\mu)$ with $g_{ab} = e_a{}^\mu g_{\mu\nu} e_b{}^\nu = \left(\begin{array}{c|c} \epsilon & 0 \cdots 0 \\ \hline 0 & \\ \vdots & h_{AB} \\ 0 & \end{array}\right)$

↑ tetrad

→ The cotetrad $^*e^a{}_\mu = (\epsilon n_\mu, h^{AB} e_B{}^\rho g_{\rho\nu})$

where $g^{ab} = \left(\begin{array}{c|c} \epsilon & 0 \cdots 0 \\ \hline 0 & \\ \vdots & h^{AB} \\ 0 & \end{array}\right)$

↳ The components of tensors are:

DEF

$$v^a = e^a{}_\mu v^\mu = \begin{cases} \epsilon n_\mu v^\mu & \equiv v^\perp \\ h^{AB} e_{B\mu} v^\mu & \equiv v^A \end{cases}$$

$$\omega_a = e_a{}^\mu \omega_\mu = \begin{cases} n^\mu \omega_\mu & \equiv \omega_\perp \\ e_A{}^\mu \omega_\mu & \equiv \omega_A \end{cases}$$

→ One suppose that f itself can be taken locally as a coord. of Σ, so that $x^\mu = x^\mu(f, y^A) = x^\mu(x^a)$ is an invertible change of coord. and restricting to Σ means putting $f = 0$.

PROP The rotation coefficients vanishes: $D_A{}^\mu{}_B = 0$

DEMO Indeed,

$$D_A{}^\mu{}_B = \partial_A e_B{}^\mu - \partial_B e_A{}^\mu$$
$$= \partial_A \frac{\partial x^\mu}{\partial y^B} - \partial_B \frac{\partial x^\mu}{\partial y^A} = e_A{}^\nu \frac{\partial}{\partial x^\nu}\frac{\partial x^\mu}{\partial y^B} - e_B{}^\nu \frac{\partial}{\partial x^\nu}\frac{\partial x^\mu}{\partial y^A}$$
$$= \left\{\frac{\partial}{\partial y^A}\frac{\partial}{\partial y^B} - \frac{\partial}{\partial y^B}\frac{\partial}{\partial y^A}\right\} x^\mu = 0$$

■

⊙ Integration over hypersurface Σ:

DEF We define the surface element $d\Sigma$ as
$$d\Sigma \equiv \sqrt{|h|}\, d^{n-1}y$$
where $h \equiv \det[h_{AB}]$

→ The combination $n_\mu d\Sigma$ is a directed surface element that points in the direction of increasing f.

DEF We define the following infinitesimal vector field $d\Sigma_\mu$ as
$$d\Sigma_\mu \equiv \sqrt{|g|}\, \epsilon_{\mu\alpha_1\dots\alpha_{n-1}} e_1{}^{\alpha_1} \dots e_{n-1}{}^{\alpha_{n-1}} d^{n-1}y$$

PROP We have $d\Sigma_\mu = \epsilon\, n_\mu d\Sigma$

DEMO Indeed,

$$n^\mu d\Sigma_\mu = \sqrt{|g|}\, \epsilon_{\mu\alpha_1\dots\alpha_{n-1}} n^\mu e_1{}^{\alpha_1}\dots e_{n-1}{}^{\alpha_{n-1}} d^{n-1}y$$
$$= \sqrt{|g|}\, \det(e_a{}^\mu)\, d^{n-1}y$$

But $\det g_{ab} = \epsilon h = (\det e_a{}^\mu)^2 g \Rightarrow (\det e_a{}^\mu)^2 = \epsilon h / g$

So $n^\mu d\Sigma_\mu = \sqrt{|g|} \cdot \frac{\sqrt{|h|}}{\sqrt{|g|}} \cdot d^{n-1}y = d\Sigma \Leftrightarrow d\Sigma_\mu = \epsilon\, n_\mu d\Sigma$

■

⊙ Tangent tensor fields:

DEF A tensor $T^{\mu\nu\dots}$ defined on Σ and tangent to Σ can be written as
$$T^{\mu\nu\dots}(x(y)) = T^{AB\dots}(y)\, e_A{}^\mu e_B{}^\nu \dots$$

↳ Indeed tangent: $T^{\mu\nu\dots} n_\nu = 0 = n_\mu T^{\mu\nu\dots}$

→ We can also take an arbitrary tensor and project it down to the hypersurface, so that only its tangential components survive.

DEF We define the projector $h^{\mu\nu}$ as
$$h^{\mu\nu} \equiv h^{AB} e_A{}^{\mu} e_B{}^{\nu} = g^{\mu\nu} - \epsilon n^{\mu} n^{\nu}$$

→ For any arbitrary tensor $T^{\mu\nu}$, there is an associated $T^{\alpha\beta} \equiv h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} T^{\mu\nu}$ that is tangent to Σ.

→ Projections $T_{AB} \equiv T_{\mu\nu} e_A{}^{\mu} e_B{}^{\nu} = h_{AC} h_{BD} T^{CD}$ give the associated tensor T^{AB} on Σ wich is a spacetime scalar.

⊙ Intrinsic ∇ and extrinsic curvature:

→ We wish to examine how tangent tensor fields are differentiated. We want to relate the covariant derivative of A^{μ} (with respect to a connection $g_{\mu\nu}$-compatible) to the covariant derivative of A^{A}, defined in terms of a connection h_{AB}-compatible.

→ We recall that for a tangent vector field A^{μ}, we have:
$$A^{\mu} = A^{A} e_A{}^{\mu}, \quad A^{\mu} n_{\mu} = 0 \quad \text{and} \quad A_A = A_{\mu} e_A{}^{\mu}$$

DEF We define the intrinsic covariant derivative of a tangent vector field A_A as the projection of $\nabla_{\nu} A_{\mu}$ onto the hypersurface:
$$A_{A|B} \equiv A_{\mu;\nu} e_A{}^{\mu} e_B{}^{\nu}$$

→ Notice that
$$A_{\mu;\nu} e_A{}^{\mu} e_B{}^{\nu} = (A_{\mu} e_A{}^{\mu})_{;\nu} e_B{}^{\nu} - A_{\mu} e_A{}^{\mu}{}_{;\nu} e_B{}^{\nu}$$

$$= A_{A,\nu} e_B{}^{\nu} - e_A{}^{\mu}{}_{;\nu} e_B{}^{\nu} A_C e^C{}_{\mu}$$

$$= A_{A,B} - e_{A\sigma;\nu} e_B{}^{\nu} e_C{}^{\sigma} A^C = A_{A,B} - \Gamma_{CAB} A^C$$

where $\Gamma_{CAB} \equiv e_C{}^{\sigma} e_{A\sigma;\rho} e_B{}^{\rho}$

PROP The connection Γ_{CAB} is the h_{AB}-compatible, torsion free Levi-Civita connection for the induced metric:

$$\Gamma_{CAB} = \frac{1}{2}\left(h_{CA,B} + h_{CB,A} - h_{AB,C}\right)$$

$\boxed{\text{DEMO}}$ Let's show that $h_{AB|C} = 0 = h_{\mu\nu;\sigma}\, e_A{}^\mu e_B{}^\nu e_C{}^\sigma$

Indeed:

$$h_{\mu\nu;\sigma} e_A{}^\mu e_B{}^\nu e_C{}^\sigma = (g_{\mu\nu} - \epsilon\, n_\mu n_\nu)_{;\sigma}\, e_A{}^\mu e_B{}^\nu e_C{}^\sigma$$
$$= -\epsilon\left(n_{\mu;\sigma} \cancel{n_\nu} + \cancel{n_\mu} n_{\nu;\sigma}\right) e_A{}^\mu e_B{}^\nu e_C{}^\sigma = 0$$

$\rightarrow$ the quantities $A_{A|B} = A_{\mu;\nu}\, e_A{}^\mu e_B{}^\nu$ are the tangential components of the vector $A^\mu{}_{;\nu} e_B{}^\nu$. We would like to explicit its normal components. To do that, we re-express $A^\mu{}_{;\nu} e_B{}^\nu$ as $g^\mu{}_\sigma A^\sigma{}_{;\nu} e_B{}^\nu$ and decompose the metric into its normal and tangential parts: $g^{\mu\nu} = \epsilon n^\mu n^\nu + h^{AB} e_A{}^\mu e_B{}^\nu$. This gives

$$A^\mu{}_{;\nu} e_B{}^\nu = (\epsilon n^\mu n_\sigma + h^{AC} e_A{}^\mu e_{C\sigma})\, A^\sigma{}_{;\nu} e_B{}^\nu$$

$$= \epsilon\,(n_\sigma A^\sigma{}_{;\nu} e_B{}^\nu) n^\mu + h^{AC} \underbrace{(A_{\sigma;\nu}\, e_C{}^\sigma e_B{}^\nu)}_{A_{C|B}} e_A{}^\mu$$

$$= \epsilon\left\{ (\cancel{n_\sigma A^\sigma})_{;\nu} - n_{\sigma;\nu} A^\sigma \right\} e_B{}^\nu n^\mu + h^{AC} A_{C|B}\, e_A{}^\mu$$

$$= A^A{}_{|B}\, e_A{}^\mu - \epsilon A^A (n_{\sigma;\nu}\, e_A{}^\sigma e_B{}^\nu) n^\mu$$

DEF We introduce K_{AB} the extrinsic curvature or second fundamental form of the hypersurface Σ:

$$K_{AB} \equiv -n_{\mu;\nu}\, e_A{}^\mu e_B{}^\nu$$

↳ We have

$$A^\mu{}_{;\nu} e_B{}^\nu = \underbrace{A^A{}_{|B}\, e_A{}^\mu}_{\text{tangential}} + \underbrace{\epsilon\, n^\mu A^A K_{AB}}_{\text{normal}}$$

PROP The Gauss-Weingarten equation is

$$e_B{}^\nu e_{A;\nu} = \nabla_B e_A{}^\mu = \Gamma_A{}^C{}_B\, e_C{}^\mu + \epsilon K_{AB} n^\mu$$

$\boxed{\text{DEMO}}$ Using the above relation:

? $e_A{}^\mu{}_{;B} =$

⊙ Properties of extrinsic curvature:

PROP The extrinsic curvature is symmetric: $K_{AB} = K_{BA}$

DEMO Indeed:

$\rightarrow K_{AB} = -e_A{}^\mu n_{\mu;\nu} e_B{}^\nu = e_A{}^\mu{}_{;\nu} n_\mu e_B{}^\nu = -n_{\mu;\nu} e_A{}^\mu e_B{}^\nu$

$\rightarrow K_{BA} = -e_B{}^\mu n_{\mu;\nu} e_A{}^\nu = e_B{}^\mu{}_{;\nu} n_\mu e_A{}^\nu = -n_{\mu;\nu} e_A{}^\mu e_B{}^\nu$

? using the property that $e_A{}^\mu{}_{;B} = e_B{}^\mu{}_{;A}$. Indeed,

$$K_{AB} - K_{BA} = (\partial_B e_A{}^\mu - \partial_A e_B{}^\mu) n_\mu + (\Gamma^\mu{}_{\sigma\nu} e_A{}^\sigma e_B{}^\nu - \Gamma^\mu{}_{\sigma\nu} e_B{}^\sigma e_A{}^\nu) n_\mu$$

$$= D_A{}^A{}_B + 2\Gamma^\mu_{[AB]} = -T_A{}^\mu{}_B \text{ the torsion.}$$

∎

PROP $K_{AB} = -e_A{}^\mu \left(\frac{1}{2} \mathcal{L}_n g_{\mu\nu}\right) e_B{}^\nu$

DEMO Indeed,

$$-e_A{}^\mu \left(\tfrac{1}{2}\mathcal{L}_n g_{\mu\nu}\right) e_B{}^\nu = -e_A{}^\mu \cdot \tfrac{1}{2}(n_{\mu;\nu} + n_{\nu;\mu}) e_B{}^\nu$$

$$= -e_A{}^\mu n_{(\mu;\nu)} e_B{}^\nu = K_{(AB)} = K_{AB}$$

∎

DEF The trace of the extrinsic curvature is $K \equiv h^{AB} K_{AB}$

$\rightarrow$ We have $K = h^{AB} \cdot (-e_A{}^\mu n_{(\mu;\nu)} e_B{}^\nu)$

$$= -(g^{\alpha\beta} - \epsilon n^\alpha n^\beta) \cdot e^A{}_\alpha e^B{}_\beta \cdot e_A{}^\mu \cdot e_B{}^\nu n_{(\mu;\nu)}$$

$$= -(g^{\mu\nu} - \epsilon n^\mu n^\nu) n_{\mu;\nu}$$

$$= -n^\mu{}_{;\mu}$$

$$\Rightarrow K = -n^\mu{}_{;\mu}$$

PROP A surface is $\begin{cases} \text{convex} & \text{if } K > 0 \\ \text{concave} & \text{if } K < 0 \end{cases}$

$\rightarrow$ While h_{AB} is concerned with the purely intrinsic aspects of a hypersurface's geometry, K_{AB} is concerned with the extrinsic aspects. Taken together, these tensors provide a virtually complete characterization of the hypersurface.

⊙ Gauss-Codazzi equations:

→ We want to know the expression of the projected components of the Riemann tensor: $R_{ABCD} = e_A{}^\mu e_B{}^\nu e_C{}^\rho e_D{}^\sigma R_{\mu\nu\rho\sigma}$ in term of the intrinsic Rieman tensor on the surface ${}^S R_{ABCD}$:

$${}^S R_{ABCD} = h_{AE} {}^S R^E{}_{BCD}$$

→ We have:

$${}^S R_{EDAB} = h_{EC} \left(\partial_A \Gamma_D{}^C{}_B + \Gamma_F{}^C{}_A \Gamma_D{}^F{}_B - \partial_B \Gamma_D{}^C{}_A - \Gamma_F{}^C{}_B \Gamma_D{}^F{}_A - D_A{}^F{}_B \Gamma_D{}^C{}_F \right)$$

→ The question we now examine is whether this 3-D Riemann tensor can be expressed in terms of $R^\gamma{}_{\delta\alpha\beta}$ (the 4-D version) evaluated on Σ.

p77 PROP $R_{ABCD} = {}^S R_{ABCD} + \epsilon (K_{AD} K_{BC} - K_{AC} K_{BD})$

DEMO We start from $e_A{}^\mu{}_{;\nu} e_B{}^\nu = \Gamma_A{}^D{}_B e_D{}^\mu + \epsilon K_{AB} n^\mu \ /{}_{;\lambda} \cdot e_C{}^\lambda$

→ The LHS is $(e_A{}^\mu{}_{;\nu} e_B{}^\nu)_{;\lambda} e_C{}^\lambda$

$= e_A{}^\mu{}_{;\nu\lambda} e_B{}^\nu e_C{}^\lambda + e_A{}^\mu{}_{;\nu} e_B{}^\nu{}_{;\lambda} e_C{}^\lambda$

$= e_A{}^\mu{}_{;\nu\lambda} e_B{}^\nu e_C{}^\lambda + e_A{}^\mu{}_{;\nu} (\Gamma_B{}^D{}_C e_D{}^\nu + \epsilon K_{BC} n^\nu)$

$= e_A{}^\mu{}_{;\nu\lambda} e_B{}^\nu e_C{}^\lambda + \Gamma_B{}^D{}_C (\Gamma_A{}^E{}_D e_E{}^\mu + \epsilon K_{AD} n^\mu) + \epsilon K_{BC} e_A{}^\mu{}_{;\nu} n^\nu$

→ The RHS is $(\Gamma_A{}^D{}_B e_D{}^\mu + \epsilon K_{AB} n^\mu)_{;\lambda} e_C{}^\lambda$

$= (\Gamma_A{}^E{}_{B,C} + \Gamma_D{}^E{}_C \Gamma_A{}^D{}_B - \Gamma_A{}^E{}_D \Gamma_B{}^D{}_C) e_E{}^\lambda$

$+ \epsilon (K_{AB,C} + \Gamma_A{}^D{}_B K_{DC} - \Gamma_B{}^D{}_C K_{AD}) n^\mu + \epsilon (K_{AB} n^\mu{}_{;\lambda} e_C{}^\lambda - K_{BC} e_A{}^\mu{}_{;\nu} n^\nu)$

↳ $R^\mu{}_{\sigma\lambda\nu} e_A{}^\sigma e_B{}^\nu e_C{}^\lambda = (\partial_C \Gamma_A{}^E{}_B + \Gamma_D{}^E{}_C \Gamma_A{}^D{}_B - (B \leftrightarrow C)) e_E{}^\mu$

$- \Gamma_A{}^E{}_B \, 2 \Gamma_{[B}{}^D{}_{C]} e_E{}^\mu + \epsilon \left(\partial_C K_{AB} - \partial_B K_{AC} + \Gamma_A{}^D{}_B K_{DC} - \Gamma_A{}^D{}_C K_{DB} - \Gamma_B{}^D{}_C K_{AD} + \Gamma_C{}^D{}_B K_{AD} \right) n^\mu$

$+ \epsilon (K_{AB} n^\mu{}_{;\lambda} e_C{}^\lambda - K_{AC} n^\mu{}_{;\lambda} e_B{}^\lambda) - \epsilon K_{BC} e_A{}^\mu{}_{;\nu} n^\nu + \epsilon K_{CB} e_A{}^\mu{}_{;\nu} n^\nu$

$\Rightarrow R^\mu{}_{ACB} = {}^S R^E{}_{ACB} e_E{}^\mu + \epsilon (K_{AB|C} - K_{AC|B}) n^\mu$

$+ \epsilon (K_{AB} n^\mu{}_{;\lambda} e_C{}^\lambda - K_{AC} n^\mu{}_{;\lambda} e_B{}^\lambda) \quad /\!/ \cdot e_{D\mu}$

$\Rightarrow R_{DACB} = {}^S R_{DACB} + \epsilon (K_{AB} e_D{}^\sigma g_{\sigma\mu} n^\mu{}_{;\lambda} e_C{}^\lambda - K_{AB} e_D{}^\sigma g_{\sigma\mu} n^\mu{}_{;\lambda} e_B{}^\lambda)$

$+ \epsilon (-K_{AB} K_{DC} + K_{AC} K_{DB})$

▨

PROP $R^\perp{}_{ABC} = K_{AC|B} - K_{AB|C}$ ↺ It's the Gauss-Codazzi eqs.

① Ricci scalar and contracted form

→ The Gauss-Codazzi equations can also be written in contracted form, in terms of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$. The spacetime Ricci tensor is given by

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = (\epsilon n^{\alpha} n_{\rho} + h^{AB} e_A{}^{\alpha} e_{B\rho}) R^{\rho}{}_{\mu\alpha\nu}$$

check

$$= R^{\perp}{}_{\mu\perp\nu} + R_{B\mu A\nu} h^{AB}$$

The Ricci scalar is given by:

$$R = (\epsilon n^{\mu} n^{\nu} + h^{CD} e_C{}^{\mu} e_D{}^{\nu}) \{ R^{\perp}{}_{\mu\perp\nu} + R_{B\mu A\nu} h^{AB} \}$$

$$= R^{\perp\perp}{}_{\perp\perp} + R^{\perp}{}_{C\perp D} h^{CD} + R_B{}^{\perp}{}_{A\perp} h^{AB} + h^{CD} R^{A}{}_{CAD}$$

$$= R^{\perp A}{}_{\perp A} + R^{A\perp}{}_{A\perp} + h^{CD} ({}^{S}R^{A}{}_{CAD} + \epsilon K^{A}{}_{D} K_{CA})$$

$$= R^{\perp A}{}_{\perp A} + R^{A\perp}{}_{A\perp} + {}^{S}R + \epsilon (\mathrm{tr}\{K^2\} - \mathrm{tr}\{K\}^2)$$

$= ($complicated, go in Poisson p. 79 or notes p 205$)$

PROP The 4-D Ricci scalar evaluated on the hypersurface Σ can be written as:

$$R = {}^{S}R + \epsilon \left(\mathrm{tr}\{K^2\} + \mathrm{tr}\{K\}^2 \right) + 2\epsilon \left(n^{\mu}{}_{;\nu} n^{\nu} - n^{\mu} n^{\nu}{}_{;\nu} \right)_{;\mu}$$

5.2 ADM parametrisation of the metric

→ In a 3+1 or $(n-1,1)$ decomposition, one considers the foliation of spacetime by a family of spacelike hypersurface Σ_a defined by

$$f(x^{\mu}) - a = 0$$

↳ Locally, there exist a change of coord. $x^{\mu} = x^{\mu}(f, y^{A})$ with, for each surface, $\epsilon = -1$

↳ The normal is then given by:

$$n_{\mu} = \frac{-\partial_{\mu} f}{\sqrt{|g^{\mu\nu} \partial_{\mu} f \partial_{\nu} f|}}$$

DEF We introduce the Lapse function of an interval N by

$$N = \frac{1}{\sqrt{|g^{\mu\nu} \partial_{\mu} f \partial^{\nu} f|}}$$

→ Consider the frame $\left(n^\mu, \frac{\partial x^\mu}{\partial y^A}\right) = (n^\mu, e_A{}^\mu) = e_a{}^\mu$

The tangent vector along the f coordinate lines is

$$\frac{\partial}{\partial f} = \frac{\partial x^\mu}{\partial f} \cdot \frac{\partial}{\partial x^\mu} \equiv f^\mu \cdot \frac{\partial}{\partial x^\mu}$$

This vector can be decomposed in a normal and tangential component: $\frac{\partial x^\mu}{\partial f} \equiv f^\mu = N' n^\mu + N^A e_A{}^\mu$

PROP The scalar $N' = N$

DEMO Indeed, contracting with n_μ, we have:

$$n_\mu \cdot \frac{\partial x^\mu}{\partial f} = n_\mu \cdot (N' n^\mu + N^A e_A{}^\mu) = -N'$$

? $\Leftrightarrow -N \frac{\partial f}{\partial x^\mu} \cdot \frac{\partial x^\mu}{\partial f} = -N' \Leftrightarrow N = N'$ ▨

DEF We define the lapse function N and the shift vector N^A as

$N \equiv g_{\mu\nu} f^\mu n^\nu$ and $N^A \equiv h^A{}_B f^B$ à vérifier

The set $\{N, N^A, h^{AB}\}$ are the ADM variables

→ Since $dx^\mu = \frac{\partial x^\mu}{\partial f} df + \frac{\partial x^\mu}{\partial y^A} dy^A$, we get:

$$ds^2 = \left(\frac{\partial x^\mu}{\partial f} df + e_A{}^\mu dy^A\right) g_{\mu\nu} \left(\frac{\partial x^\nu}{\partial f} df + e_B{}^\nu dy^B\right)$$

$$= \left(N n^\mu df + e_A{}^\mu (dy^A + N^A df)\right) g_{\mu\nu} \left(N n^\nu df + e_B{}^\nu (dy^B + N^B df)\right)$$

$$= N^2 n^\mu n^\nu g_{\mu\nu} df^2 + g_{\mu\nu} e_A{}^\mu e_B{}^\nu (dy^A + N^A df)(dy^B + N^B df)$$

$$= -N^2 df^2 + h_{AB} (dy^A + N^A df)(dy^B + N^B df)$$

p210 notes DEF The ADM parametrization of the metric is given by

$$ds^2 = -N^2 df^2 + h_{AB} (dy^A + N^A df)(dy^B + N^B df)$$

→ In (ℓ, y^A) coord., the metric is :

$$g_{\cdot\cdot} = \begin{array}{c} \\ \ell \\ A \end{array}\begin{array}{c}\begin{array}{cc} \ell & B \end{array} \\ \left(\begin{array}{c|c} -N^2+N_AN^A & N_B \\ \hline N_A & h_{AB} \end{array}\right)\end{array}$$

PROP The inverse metric is $(g^{-1})^{\cdot\cdot} = \left(\begin{array}{c|c} -1/N^2 & N^C/N^2 \\ \hline N^B/N^2 & h^{BC}-N^BN^C/N^2 \end{array}\right)$

DEMO Indeed :

$(-N^2+N_AN^A)(-1/N^2) + N_B(N^B/N^2) = 1$

$(-N^2+N_AN^A)(N^C/N^2) + N_B(h^{BC}-N^BN^C/N^2)$

$= -N^C + \frac{N_AN^AN^C}{N^2} + N_Bh^{BC} - \frac{N_BN^BN^C}{N^2} = N_Bh^{BC} - N^C = N^C - N^C = 0$

$(N_A)(-1/N^2) + h_{AB}(N^B/N^2) = (-N_A+N_A)/N^2 = 0$

$N_A(N^C/N^2) + h_{AB}(h^{BC}-N^BN^C/N^2) = h_{AB}h^{BC} = \delta_A{}^C$ ▨

PROP $\sqrt{-g} = N\sqrt{h}$

DEMO Indeed, we can use Laplace expansion :

$(g^{-1})^{\ell\ell} = \det(g^{-1}) \cdot \det(h) \Rightarrow g = -N^2 \det h$ ▨

→ Recall that in the moving frame $(n^\mu, e_A{}^\mu)$, the metric is

$g_{\cdot\cdot} = \begin{pmatrix} \epsilon & 0 \\ 0 & h_{AB} \end{pmatrix}$ and its inverse $g^{\cdot\cdot} = \begin{pmatrix} \epsilon & 0 \\ 0 & h^{BC} \end{pmatrix}$

? ↳ In the (ℓ, y^A) coord. system, $n_\ell = -N$ and $n_A = 0$

⊙ Lie derivative of h_{AB}:

→ For later purpose, we also need to compute $\mathcal{L}_\ell h_{AB}$

PROP $$\mathcal{L}_\ell h_{AB} = \mathcal{L}_\ell (g_{\mu\nu}) e_A{}^\mu e_B{}^\nu$$

DEMO Indeed:

$\mathcal{L}_\ell h_{AB} = \mathcal{L}_\ell (g_{\mu\nu} e_A{}^\mu e_B{}^\nu)$ and

$\mathcal{L}_\ell e_A{}^\mu = \ell^\alpha \partial_\alpha e_A{}^\mu - \partial_\alpha \ell^\mu e_A{}^\alpha$

$= \frac{\partial x^\alpha}{\partial \ell} \cdot \partial_\alpha \cdot \frac{\partial x^\mu}{\partial y^A} - \partial_\alpha \frac{\partial x^\mu}{\partial \ell} \cdot \frac{\partial x^\alpha}{\partial y^A} = \frac{\partial^2 x^\mu}{\partial \ell \partial y^A} - \frac{\partial^2 x^\mu}{\partial \ell \partial y^A} = 0$ ▨

→ Now, $\mathcal{L}_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}$

$$= \xi^\sigma \partial_\sigma g_{\mu\nu} + \partial_\mu \xi^\sigma . g_{\sigma\nu} + \partial_\nu \xi^\sigma . g_{\mu\sigma}$$

$$= \xi^\sigma \nabla_\sigma g_{\mu\nu} + \nabla_\mu \xi^\sigma . g_{\sigma\nu} + \nabla_\nu \xi^\sigma . g_{\mu\sigma}$$
$$+ \xi^\sigma \Gamma^\alpha_{\mu\sigma} g_{\alpha\nu} + \xi^\sigma \Gamma^\alpha_{\nu\sigma} g_{\mu\alpha} - \Gamma^\sigma_{\alpha\mu} \xi^\alpha . g_{\sigma\nu} - \Gamma^\sigma_{\alpha\nu} \xi^\alpha . g_{\mu\sigma} \quad \text{(ok)}$$

→ Now, recall that $\xi_\mu = N n_\mu + N_\mu$ and that $N^\mu = e_A{}^\mu N^A \perp n^\mu$

Then, $\mathcal{L}_\xi h_{AB} = e_A{}^\mu e_B{}^\nu (\xi_{\mu;\nu} + \xi_{\nu;\mu})$

$$= e_A{}^\mu e_B{}^\nu \left((N n_\mu + N_\mu)_{;\nu} + (N n_\nu + N_\nu)_{;\mu} \right)$$

$$= e_A{}^\mu e_B{}^\nu \left[N_{,\nu} . n_\mu + N_{,\mu} . n_\nu + N(n_{\mu;\nu} + n_{\nu;\mu}) + N_{\mu;\nu} + N_{\nu;\mu} \right]$$

$$= -N(-n_{\mu;\nu} e_A{}^\mu e_B{}^\nu - n_{\nu;\mu} e_A{}^\mu e_B{}^\nu) + N_{A|B} + N_{B|A}$$

$$\Leftrightarrow \mathcal{L}_\xi h_{AB} = -N(K_{AB} + K_{BA}) + N_{A|B} + N_{B|A}$$

PROP $K_{AB} = -N \Gamma_A{}^t{}_B$

[DEMO] Indeed,

$K_{AB} = -n_{\mu;\nu} e_A{}^\mu e_B{}^\nu = -n_{A;B} = n_{A,B} + n_x \Gamma_A{}^x{}_B$ where $x = \{t, y^A\}$

? $= -N \Gamma_A{}^t{}_B$ using the fact that $n_t = -N$ and $n_A = 0$ ■

5.3 Hamiltonian formulation

→ One consider a $(n-1)+1$ decomposition, in the particular case where $f = x^0$ (foliation by equal-time hypersurfaces).

In our old coord. on the hypersurface,

$f = x^0$, $y^A = x^i$ such that $e_A{}^\mu = \dfrac{\partial x^\mu}{\partial y^A} = \delta^\mu_i$ and $\epsilon = -1$

→ We then have $n_\mu = -N \delta^0_\mu$

↳ To compute it's inverse, one has

$$\frac{\partial}{\partial x^0} = N n^\mu \frac{\partial}{\partial x^\mu} + N^i \frac{\partial}{\partial x^i} \iff n^\mu = \begin{pmatrix} 1/N \\ -N^i/N \end{pmatrix}$$

→ The expression for the metric in the coord. $(f, y^A) \mapsto (x^0, x^i)$ becomes: $ds^2 = -N^2 (dx^0)^2 + g_{ij} (dx^i + N^i dx^0)(dx^j + N^j dx^0)$

↳ It can be interpreted as an invertible change of coord. from ${}^d g_{\mu\nu} \cong ({}^d g_{00}, {}^d g_{0i}, {}^d g_{ij}) \mapsto (N, N^i, g_{ij})$

In matrix form: ${}^d g_{\mu\nu} = \begin{pmatrix} {}^d g_{00} & {}^d g_{0k} \\ {}^d g_{i0} & {}^d g_{ik} \end{pmatrix} = \begin{pmatrix} N_j N^j - N^2 & N_k \\ N_i & g_{ik} \end{pmatrix}$

Not: We use ^{d} suffix to denote the components of a tensor in space-time, while no suffix means the components of a tensor in space alone.

→ We use g_{ij} to raise and lower indices of N^j: $N_i = g_{im} N^m$ and $g_{ik} g^{kj} = \delta_i^j$

PROP The inverse metric is

$${}^d g^{\nu\lambda} = \begin{pmatrix} {}^d g^{00} & {}^d g^{0m} \\ {}^d g^{k0} & {}^d g^{km} \end{pmatrix} = \begin{pmatrix} -1/N^2 & N^m/N^2 \\ N^k/N^2 & g^{km} - N^k N^m/N^2 \end{pmatrix}$$

↳ It gives indeed ${}^d g_{\mu\nu} \, {}^d g^{\nu\lambda} = {}^d \delta_\mu^\lambda$

→ The inverse change of variable is then given by:

$^dg_{00} = -1/N^2 \Leftrightarrow N = \sqrt{\frac{-1}{^dg^{00}}}$ and $N^i = \frac{-1}{^dg} \cdot g^{0i}$

We still have $\sqrt{-^dg} = N\sqrt{g}$ with $g \equiv \det g_{ij}$

⊙ Rewriting Einstein-Hilbert action:

→ We want to write $S^{EH}[^dg_{\mu\nu}] = \int d^dx \sqrt{-^dg}\,(^dR - 2\Lambda)$ in term of the ADM's variables:

Recall $^dR = R - (tr[K]^2 - tr[K^2]) \underbrace{- 2(n^\alpha{}_{;\beta} n^\beta - n^\beta{}_{;\alpha} n^\alpha)_{;\alpha}}_{\text{total derivative}}$

↳ $S[g_{ij}, N^i, N] = \int d^dx\, N\sqrt{g}\,(R - tr[K]^2 + tr[K^2] - 2\Lambda)$

$= \int d^dx\, N\sqrt{g}\{(R - 2\Lambda) - K^i{}_i K^j{}_j + K^i{}_j K^j{}_i\} = \int d^dx\, \mathcal{L}$

DEF We introduce the Wheeler-DeWitt metric G^{ijpq} as

$$G^{ijpq} \equiv \frac{\sqrt{g}}{2}(g^{ip}g^{jq} + g^{iq}g^{jp} - 2g^{ij}g^{pq})$$

↳ A usefull identity is:

$$G^{ijpq} K_{ij} K_{pq} = \frac{\sqrt{g}}{2}(K^{pq}K_{pq} + K^{qp}K_{pq} - 2K^i{}_i K^q{}_q)$$

$$= \sqrt{g}\,(tr[K^2] - tr[K]^2)$$

↳ We can write:

$$S = \int d^dx\, N\sqrt{g}\underbrace{(R - 2\Lambda)}_{\text{pot. term}} + N\underbrace{G^{ijpq} K_{ij} K_{pq}}_{\text{kinetic term}}$$

→ Using previous result that $\mathcal{L}_\ell h_{AB} = -N(K_{AB} + K_{BA}) + N_{A|B} + N_{B|A}$:

? $\mathcal{L}_{\partial/\partial x^0} = \dot{g}_{ij} = 2NK_{ij} + N_{i|j} + N_{j|i}$

$\Leftrightarrow K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - N_{i|j} - N_{j|i})$

↳ Since the spatial curvature only involves spatial derivative of g_{ij}, the only dependence of $\mathcal{L}$ on time derivatives is contained in the dependence of K_{ij} on $\dot{g}_{ij}$

DEF We introduce the canonical momenta associated to g_{ij} as
$$\pi^{ij} \equiv \frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}}$$

↳ We have $\pi^{ij} = N G^{ijpq}\left(\frac{1}{2N} K_{pq} + K_{ij}\, \frac{\delta^i{}_p \delta^j{}_q}{2N}\right) = G^{ijpq} K_{pq}$

We only perform a Legendre transform on variables in wich ∂_t appears.
To invert this relation, we need to compute the inverse of the Wheeler-DeWitt metric:

PROP $G_{pqrs} = \frac{1}{2\sqrt{g}}\left(g_{pr} g_{qs} + g_{ps} g_{qr} - \frac{2}{d-2} g_{pq} g_{rs}\right)$ and $G^{ijpq} G_{pqrs} = \delta^{(i}_r \delta^{j)}_s$

DEMO We have $G^{ijpq} = \sqrt{g}/2\,(g^{ip} g^{jq} + g^{iq} g^{jp} - 2 g^{pq} g^{ij})$. Then, we suppose $G_{pqrs} = \frac{1}{2\sqrt{g}}\left(g_{pr} g_{qs} + g_{ps} g_{qr} - \alpha\, g_{pq} g_{rs}\right)$ and compute

$$G^{ijpq} G_{pqrs} = \frac{1}{4}\Big(\delta^i_r \delta^j_s + \delta^i_s \delta^j_r - 2 g^{ij} g_{rs} + \delta^i_s \delta^j_r + \delta^i_r \delta^j_s - 2 g_{rs} g^{ij}$$
$$- \alpha g^{ij} g_{rs} - \alpha g^{ij} g_{rs} + 2\alpha (d-1) g^{ij} g_{rs}\Big)$$
$$= \frac{1}{2}\left(\delta^i_r \delta^j_s + \delta^i_s \delta^j_r\right) \text{ if } -4 - 2\alpha + 2\alpha(d-1) = 0 \Leftrightarrow \alpha = \frac{2}{d-2}$$

→ Inversing the relation, we get $K_{pq} = G_{pqrs} \pi^{rs}$
$\Rightarrow \dot{g}_{pq} = 2N\, G_{pqrs} \pi^{rs} + N_{p|q} + N_{q|p}$

⊙ Performing the Legendre transform:

→ The Hamiltonian is obtained by doing a Legendre transform with respect to $\dot{g}_{ij}$

DEF The canonical Hamiltonian $\mathcal{H}_c$ is defined as
$$\mathcal{H}_c \equiv \left(\pi^{ij} \dot{g}_{ij} - \mathcal{L}\right)\Big|_{\dot{g}_{ij} = \dot{g}_{ij}(\pi^{kl})}$$

→ $\mathcal{H}_c = \pi^{ij}\left(2N G_{ijrs} \pi^{rs} + N_{i|j} + N_{j|i}\right) - N\sqrt{g}\,(R - 2\Lambda) - N G^{ijpq}\left(K_{ij} K_{pq}\right)\Big|$
and $\left(K_{ij} K_{pq}\right)\Big| = G_{ijrs} \pi^{rs} G_{pqkl} \pi^{kl}$

We get:

$$\mathcal{H}_c = \pi^{ij}\left(2N G_{ijrs}\pi^{rs} + N_{i|j} + N_{j|i}\right) - N\sqrt{g}\,(R-2\Lambda)$$
$$- N G^{ijpq}\left(G_{ijrs}\pi^{rs} G_{pqkl}\pi^{kl}\right)$$
$$= N\left(\pi^{ij} G_{ijrs}\pi^{rs} - \sqrt{g}\,(R-2\Lambda)\right) + \left(N_{i|j} + N_{j|i}\right)\pi^{ij}$$

→ Notice that π^{ij} is a tensor density. One can covariantly integrate by parts and the covariant total derivative is an ordinary one that can be dropped in the first order action, but there is an additional term:

$$\pi^{ij}{}_{;j} = \pi^{ij}{}_{,j} + \Gamma^i_{kj}\pi^{kj} + \Gamma^j_{kj}\pi^{ik} - \Gamma^k_{kj}\pi^{ij}$$

Up to a total derivative, we have:

$$\pi^{ij}(N_{i|j} + N_{j|i}) = \underbrace{2(\pi^{ij}N_j)|_i}_{\text{total derivative}} - 2\pi_{ij}{}^{|j}\, N^i$$

PROP | The canonical Hamiltonian can be written as

$$\mathcal{H}_c = N\mathcal{H}_\perp + N^i\mathcal{H}_i + 2(\pi^{ij}N_j)|_j$$

with $\mathcal{H}_\perp$ is the Hamiltonian constraint:

$$\mathcal{H}_\perp \equiv \pi^{ij} G_{ijrs}\pi^{rs} - \sqrt{g}\,(R-2\Lambda)$$

and $\mathcal{H}_i$ is the diffeomorphism constraint (or momentum constraint):

$$\mathcal{H}_i \equiv -2\pi_{ij}{}^{|j}$$

PROP | The ADM Lagrangian $\mathcal{L}_{ADM}$ is given by "$\mathcal{L} = p_i\dot{q}^i - H$"

$$\mathcal{L}_{ADM}[g_{ij}, \pi^{ij}, N, N^i] = \underbrace{\dot{g}_{ij}\pi^{ij}}_{\text{"}p.\dot{q}\text{"}} - N\mathcal{H}_\perp - N^i\mathcal{H}_i$$

→ The associated EOM are found by varying $\mathcal{L}_{ADM}$ with respect to π^{ij} and g_{ij}, and by imposing $\mathcal{H}_\perp \approx 0$ and $\mathcal{H}_i \approx 0$ ($\Leftrightarrow$ varying $\mathcal{L}$ w.r.t. N and N^i).

$$\hookrightarrow \delta S = \int d^dx \left\{ -\delta N.\mathcal{H}_\perp - \delta N^i.\mathcal{H}_i + \frac{\delta\mathcal{L}^{ADM}}{\delta g_{ij}}.\delta g_{ij} + \frac{\delta\mathcal{L}^{ADM}}{\delta\pi^{ij}}\delta\pi^{ij} \right\}$$

① $\dfrac{\delta\mathcal{L}^{ADM}}{\delta N} = 0 \Leftrightarrow \mathcal{H}_\perp = 0$

② $\dfrac{\delta\mathcal{L}^{ADM}}{\delta N^i} = 0 \Leftrightarrow \mathcal{H}_i = 0$

} Lagrange multiplier. The Hamiltonian of the theory vanishes on the EOM.

③ $$\frac{\delta \mathcal{L}^{ADM}}{\delta \pi^{ij}} = 0 \iff \dot{g}_{ij} = 2N G_{ijrs} \pi^{rs} + N_{i|j} + N_{j|i}$$

④ $$\frac{\delta \mathcal{L}^{ADM}}{\delta g_{ij}} = 0 \iff \dot{\pi}^{ij} = -N\sqrt{g}\left(G^{ij} + \Lambda g^{ij}\right) + \frac{1}{2} N g^{ij} \pi^{pq} G_{pqrs} \pi^{rs}$$
$$- 2N \frac{1}{\sqrt{g}} \left(\pi^{i}{}_{m} \pi^{mj} - \frac{1}{d-2} \pi \pi^{ij}\right) + \sqrt{g}\left(N^{|ij} - g^{ij} N^{|m}{}_{|m}\right)$$
$$+ \left(\pi^{ij} N^{m}\right)_{|m} - N^{i}{}_{|m} \pi^{mj} - N^{j}{}_{|m} \pi^{mi}$$

See notes p. 92 for the computation.