3 APPLICATION TO EFE

3.1 Metric formulation

3.1.1 Variational principle in metric formulation

DEF The Einstein-Hilbert action is defined as

$$S_{EH}[g] \equiv \frac{1}{16\pi G}\int d^n x \sqrt{|g|}\left\{R(g) - 2\Lambda\right\}$$

where $R = R^\alpha{}_\alpha$ is the trace of the Ricci tensor, the Ricci tensor being $R_{\alpha\beta} \equiv R^\mu{}_{\alpha\mu\beta} = g^{\mu\nu} R_{\nu\alpha\mu\beta}$
and where Λ is the cosmological constant.

→ We use the Christoffel connexion $\Gamma \equiv \{\}$ such that

$$R^\mu{}_{\alpha\beta\gamma} \equiv \partial_\beta \left\{{}^{\mu}_{\gamma\alpha}\right\} + \left\{{}^{\mu}_{\sigma\beta}\right\}\left\{{}^{\sigma}_{\gamma\alpha}\right\} - (\beta \leftrightarrow \gamma)$$

$$= \partial_\beta \Gamma_{\gamma\alpha}^{\mu} + \Gamma_{\sigma\beta}^{\mu}\Gamma_{\alpha\gamma}^{\sigma} - \left(\partial_\gamma \Gamma_{\alpha\beta}^{\mu} + \Gamma_{\sigma\gamma}^{\mu}\Gamma_{\beta\alpha}^{\sigma}\right)$$

Prop
$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}\, g_{\alpha\beta}\,\delta g^{\alpha\beta} = +\frac{1}{2}\sqrt{|g|}\, g^{\alpha\beta}\,\delta g_{\alpha\beta}$$

$\boxed{\text{DEMO}}$ ① First, notice that for a matrix $A = S\Lambda S^{-1}$ with Λ diagonal, we have $\det\Lambda = \prod_i \lambda_i$ and

$$\prod_i \lambda_i = \exp\left\{\Sigma_i \ln(\lambda_i)\right\} = \exp\left\{\mathrm{Tr}\left[S.\log\Lambda.S^{-1}\right]\right\} = \exp\left\{\mathrm{Tr}\left[\log A\right]\right\}$$

Hence, $\det A = \exp\{\mathrm{Tr}\log A\}$

② For a variation of the element of A, we have:

$$\delta\ln\left[\det A\right] = \frac{\delta|\det A|}{|\det A|} = \delta\,\mathrm{Tr}\log A = \mathrm{Tr}\,\delta\log A = \mathrm{tr}\left[A^{-1}\delta A\right]$$

Then for a Lorentzian manifold $((1,n-1))$, we have $|\det g_{\mu\nu}| = -g$, and

$$\delta\ln|g| = \frac{\delta(-g)}{-g} = \mathrm{Tr}\left(g^{\mu\nu}\delta g_{\sigma\rho}\right) = g^{\mu\nu}\delta g_{\mu\nu}$$

$$\Leftrightarrow \delta g = g.g^{\mu\nu}\delta g_{\mu\nu} \Leftrightarrow \delta\sqrt{|g|} = \frac{1}{2}\frac{1}{\sqrt{|g|}}\delta|g| = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}$$

③ Since $0 = \delta(\delta^\alpha{}_\gamma) = \delta(g^{\alpha\beta}g_{\beta\gamma}) = \delta g^{\alpha\beta}.g_{\beta\gamma} + \delta g_{\beta\gamma}.g^{\alpha\beta}$, we have

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

$\blacksquare$

→ The variation of the Riemann is given by: (in a FFF)

$$\delta R^\alpha{}_{\beta\gamma\delta} = \delta\{^{\alpha}_{\beta\delta}\}_{;\gamma} - \delta\{^{\alpha}_{\beta\gamma}\}_{;\delta} \quad \text{with}$$

$$\delta\{^{\mu}_{\alpha\beta}\} = \frac{1}{2} g^{\mu\sigma}\left(\delta g_{\sigma\alpha;\beta} + \delta g_{\sigma\beta;\alpha} - \delta g_{\alpha\beta;\sigma}\right)$$

→ We can now compute the variation of S^{EH}:

$$16\pi G \cdot S^{EH} = \int d^n x \sqrt{|g|}\left(R^\sigma{}_{\alpha\sigma\beta} \cdot g^{\alpha\beta} - 2\Lambda\right)$$

$$16\pi G\, \delta S^{EH} = \int d^n x \left\{ \sqrt{|g|}\, \delta g^{\alpha\beta}\left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta}\right) + \sqrt{|g|}\, g^{\alpha\beta}\left(\delta\{^{\mu}_{\alpha\beta}\}_{;\mu} - \delta\{^{\mu}_{\alpha\mu}\}_{;\beta}\right) \right\}$$

Lemma $\sqrt{|g|}\, f^\mu{}_{;\mu} = \partial_\mu\left(\sqrt{|g|}\, f^\mu\right)$

Indeed: $\partial_\mu \sqrt{|g|} = \frac{|g|}{2\sqrt{|g|}} g^{\alpha\beta} g_{\alpha\beta,\mu} = \frac{\sqrt{|g|}}{2} g^{\alpha\beta} g_{\alpha\beta,\mu}$

Now, $\sqrt{|g|}\{^{\nu}_{\mu\nu}\} = \frac{1}{2} g^{\nu\sigma}\left(\cancel{g_{\sigma\mu,\nu}} + g_{\sigma\nu,\mu} - \cancel{g_{\mu\nu,\sigma}}\right)\sqrt{|g|} = \frac{\sqrt{|g|}}{2} g^{\nu\sigma} \partial_\mu g_{\nu\sigma}$

We then get:

$$\partial_\mu\left(\sqrt{|g|}\, f^\mu\right) = \sqrt{|g|}\, f^\mu{}_{,\mu} + f^\mu \sqrt{|g|}\{^{\nu}_{\mu\nu}\} = f^\mu{}_{;\mu} \sqrt{|g|}$$ ∎

↳ $$16\pi G\, \delta S^{EH} = \int d^n x \left\{ \sqrt{|g|}\, \delta g^{\alpha\beta}\left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta}\right) + \left[\partial_\mu\left(\sqrt{|g|}\, g^{\alpha\beta}\, \delta\{^{\mu}_{\alpha\beta}\}\right) - \partial_\beta\left(\sqrt{|g|}\, g^{\alpha\beta}\, \delta\{^{\mu}_{\alpha\mu}\}\right)\right] \right\}$$

→ Neglecting boundary terms and using the metricity of the Christoffel connection, we get

$$16\pi G \frac{\delta \mathcal{L}^{EH}}{\delta g^{\alpha\beta}} = \sqrt{|g|}\left(G_{\alpha\beta} + \Lambda g_{\alpha\beta}\right)$$

3.1.2 Matter couplings:

→ We consider an extended action $S = S^{EH} + S^{M}$ where S^{M} is a bosonic scalar field: $S^{S}[g,\phi] \equiv \frac{-1}{8\pi G}\int d^n x \sqrt{|g|}\left\{\partial_\mu\phi\,\partial^\mu\phi + m^2\phi^2\right\}$

or the e-m field: $S^{M}[g, A_\mu] \equiv \frac{-1}{16\pi G}\int d^n x\sqrt{|g|}\left\{F_{\mu\nu}F^{\mu\nu}\right\}$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ the Maxwell Field

DEF The stress-energy tensor T is defined as

$$T_{\alpha\beta} := \frac{-2}{\sqrt{|g|}}\frac{\delta\mathcal{L}^M}{\delta g^{\alpha\beta}} \Leftrightarrow \frac{\delta\mathcal{L}^M}{\delta g^{\alpha\beta}} = -\frac{\sqrt{|g|}}{2}T_{\alpha\beta}$$

→ The stress-energy tensor is not necessarily the same as the canonical stress-energy tensor: $\Theta^{\mu\nu} \equiv \frac{-\partial\mathcal{L}}{\partial\Phi^i_{\mu}}\partial^\nu\Phi^i + \eta^{\mu\nu}\mathcal{L}$ wich satisfies $\partial_\mu\Theta^{\mu\nu} = 0$

→ The variation of the metric gives

$$\frac{1}{16\pi G}\sqrt{|g|}\left(G_{\alpha\beta} + \Lambda g_{\alpha\beta}\right) + \frac{\delta\mathcal{L}^M}{\delta g^{\alpha\beta}} = 0$$

$$\Leftrightarrow G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G\,.\,T_{\alpha\beta}$$

⊙ Computation of $T_{\alpha\beta}$ for a bosonic scalar field:

→ $$T_{\alpha\beta}^{S} = \frac{-2}{\sqrt{|g|}}\cdot\frac{-1}{8\pi G}\sqrt{|g|}\left\{\partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\left(\partial_\sigma\phi\,\partial^\sigma\phi + m^2\phi^2\right)\right\}$$

$$= \frac{1}{4\pi G}\left(\partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\left(\partial_\sigma\phi\,\partial^\sigma\phi + m^2\phi^2\right)\right)$$

⊙ Computation of $T_{\alpha\beta}$ for Maxwell field:

→ $$T_{\alpha\beta}^{M} = \frac{-2}{\sqrt{|g|}}\cdot\frac{-1}{16\pi G}\sqrt{|g|}\left(2F_\alpha{}^\sigma F_{\beta\sigma} - \frac{1}{2}g_{\alpha\beta}F^{\rho\sigma}F_{\rho\sigma}\right)$$

$$= \frac{1}{4\pi G}\left(F_\alpha{}^\sigma F_{\beta\sigma} - \frac{1}{4}g_{\alpha\beta}F^2\right)$$

⊙ Equations of motion:

Not | We write the covariant box operator as
$\Box \equiv \nabla^\sigma \nabla_\sigma$, not to be confused with $\partial^2 \equiv \partial^\sigma \partial_\sigma$

→ For the bosonic scalar field, we have:

$$\frac{\delta \mathcal{L}^S}{\delta \phi} = \frac{1}{4\pi G} \sqrt{|g|} \left\{ g^{\alpha\beta} \partial_\alpha \partial_\beta \phi - m^2 \phi \right\}$$

$$= \frac{1}{4\pi G} \sqrt{|g|} \left\{ \frac{\partial_\alpha}{\sqrt{|g|}} \left[\sqrt{|g|}\, g^{\alpha\beta} \partial_\beta \phi \right] - m^2 \phi \right\}$$

$$= \frac{1}{4\pi G} \sqrt{|g|} \left\{ \Box \phi - m^2 \phi \right\} = 0$$

→ For the Maxwell field, we have:

$$\frac{\delta \mathcal{L}^M}{\delta A^\mu} = \frac{1}{4\pi G} \sqrt{|g|}\ \partial_\nu F^{\nu\mu}$$

$$= \frac{1}{4\pi G} \sqrt{|g|} \frac{1}{\sqrt{|g|}} \partial_\nu \left(\sqrt{|g|}\, F^{\nu\mu} \right)$$

$$= \frac{1}{4\pi G} \sqrt{|g|}\ \nabla_\nu \left[\nabla^\nu A^\mu - \nabla^\mu A^\nu \right] = 0$$

3.1.3 Gauge invariance:

→ In a flat metric η, we had $-\frac{1}{2} \int d^n x \left(\partial_\mu \phi\, \partial^\mu \phi + m^2 \phi^2 \right)$ invariant under $\Lambda^T \eta \Lambda = \eta$, asking for $\phi'(x') \stackrel{!}{=} \phi(x)$.
For a generic metric g, we have instead
$-\frac{1}{2} \int d^n x \sqrt{|g|} \left(\partial_\mu \phi . g^{\mu\nu} . \partial_\nu \phi + m^2 \phi^2 \right)$ wich is invariant under diffeomorphism: $\partial'_\mu \phi' = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \phi$

→ The gauge invariance requires $d^n x' \sqrt{|g'|} (R' - 2\Lambda) \stackrel{!}{=} d^n x \sqrt{|g|} (R - 2\Lambda)$
where: $x \mapsto x' \equiv x - \xi(x)$

$$g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \cdot g_{\alpha\beta}(x) . \frac{\partial x^\beta}{\partial x'^\nu}$$

→ Notice that:

$$\frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu_\alpha - \frac{\partial \xi^\mu}{\partial x^\alpha} \quad \text{and} \quad \frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu + \frac{\partial \xi^\alpha}{\partial x^\mu}$$

Prop Under an infinitesimal diffeo, the variation of the metric is given by its Lie derivative:

$$\delta_\xi g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} = 2\xi_{(\nu;\mu)}$$

DEMO Indeed,

$$\delta_\xi g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = g'_{\mu\nu}(x'+\xi) - g_{\mu\nu}(x)$$
$$= \xi^\alpha \partial_\alpha g_{\mu\nu}(x) + \frac{\partial x^\alpha}{\partial x'^\mu} g_{\alpha\beta}(x) \frac{\partial x^\beta}{\partial x'^\nu} - g_{\mu\nu}(x)$$
$$= \xi^\alpha \partial_\alpha g_{\mu\nu}(x) + (\delta^\alpha_\mu + \xi^\alpha{}_{,\mu})\, g_{\alpha\beta}(x) (\delta^\beta_\nu + \xi^\beta{}_{,\nu}) - g_{\mu\nu}(x)$$
$$= \xi^\alpha \partial_\alpha g_{\mu\nu}(x) + \xi^\alpha{}_{,\mu} g_{\alpha\nu} + \xi^\beta{}_{,\nu} g_{\mu\beta} = \mathcal{L}_\xi g_{\mu\nu}$$
$$= \xi^\alpha g_{\mu\nu;\alpha} + \xi^\alpha{}_{;\mu} g_{\alpha\nu} + \xi^\beta{}_{;\nu} g_{\mu\beta} = \xi_{\nu;\mu} + \xi_{\mu;\nu}$$

3.1.4 Isometries and Killing vectors:

DEF Under a diffeomorphism, the metric transform as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}(x)\, g_{\alpha\beta}(x)\, \frac{\partial x^\beta}{\partial x'^\nu}(x)$$

A metric $\bar{g}_{\mu\nu}(x)$ is sait to be form-invariant if

$$\bar{g}'_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x') = \partial_\mu x^\alpha \cdot \bar{g}_{\alpha\beta} \cdot \partial_\nu x^\beta$$

Any transformation that satisfies this equality is an isometry

→ An isometry is a special case of a diffeomorphism

→ For an infinitesimal coord. transfo, we have:

$$\delta_\xi g_{\mu\nu}(x) = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$$

→ Those $\xi^\mu(x)$ are Killing vectors.

Prop Let $L^{EH} = \sqrt{g}\,(R - 2\Lambda)$. The invariance of the action under finite transformations gives $\delta_\xi L^{EH} = \partial_\mu(\xi^\mu L^{EH})$

(invariance: $d^4x' \sqrt{|g'|}(R' - 2\Lambda) \overset{!}{=} d^4x \sqrt{|g|}(R - 2\Lambda)$)

Indeed, let's show that

$$d^4x' L^{EH}[g'_{\alpha\beta}(x'), \partial'_\mu g'_{\alpha\beta}(x'), \partial'_\mu \partial'_\nu g_{\alpha\beta}(x')]$$
$$= d^4x\, L^{EH}[g_{\alpha\beta}(x), \partial_\mu g_{\alpha\beta}(x), \partial_\mu \partial_\nu g_{\alpha\beta}(x)]$$

① We write $\Delta g_{\alpha\beta} = \xi^{\mu}{}_{,\alpha}\, g_{\mu\beta} + \xi^{\nu}{}_{,\beta}\, g_{\alpha\nu}$

② Notice that:

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \cdot \frac{\partial}{\partial x^{\nu}} = (\delta^{\nu}_{\mu} + \xi^{\nu}{}_{,\mu}) \frac{\partial}{\partial x^{\nu}} \quad \text{and} \quad d^n x' = (\delta^{\mu}{}_{\nu} - \xi^{\mu}{}_{,\nu})\, d^n x$$

③ We need to show that:

$$(1-\xi^{\sigma}{}_{,\sigma}) L^{EH}[g_{\alpha\beta} + \Delta g_{\alpha\beta}, (\delta^{\nu}_{\mu} + \xi^{\nu}{}_{,\mu})\partial_\nu (g_{\alpha\beta} + \Delta g_{\alpha\beta}), (\delta^{\gamma}_{\mu} + \xi^{\gamma}{}_{,\mu})\partial_\gamma (\delta^{\delta}_{\nu} + \xi^{\delta}{}_{,\nu})\partial_\delta (g_{\alpha\beta} + \Delta g_{\alpha\beta})$$
$$\doteq L^{EH}[g_{\alpha\beta}, \partial_\mu g_{\alpha\beta}, \partial_\mu \partial_\nu g_{\alpha\beta}]$$

This implies

$$\xi^{\rho}{}_{,\rho} L^{EH} + \frac{\partial L^{EH}}{\partial g^{\alpha\beta}} \Delta g_{\alpha\beta} + \frac{\partial L^{EH}}{\partial \partial_\mu g_{\alpha\beta}} (\partial_\mu \Delta g_{\alpha\beta} + \xi^{\rho}{}_{,\mu} \partial_\rho g_{\alpha\beta})$$

$$+ \frac{\partial L^{EH}}{\partial[\partial_\mu \partial_\nu g_{\alpha\beta}]} (\partial_\mu \partial_\nu \Delta g_{\alpha\beta} + \partial_\mu [\xi^{\delta}{}_{,\nu} \partial_\delta g_{\alpha\beta}] + \xi^{\gamma}{}_{,\mu} \partial_\gamma \partial_\nu g_{\alpha\beta}) = 0$$

Now, $\xi^{\rho}{}_{,\rho} L^{EH} = -\partial_\rho (\xi^\rho L^{EH}) + \frac{\partial L^{EH}}{\partial g_{\alpha\beta}} \xi^\rho \partial_\rho g_{\alpha\beta} + \frac{\partial L^{EH}}{\partial[\partial_\mu g_{\alpha\beta}]} \xi^\rho \partial_\rho \partial_\mu g_{\alpha\beta} +$

$$+ \frac{\partial L^{EH}}{\partial[\partial_\mu \partial_\nu g_{\alpha\beta}]} \xi^\rho \partial_\rho \partial_\mu \partial_\nu g_{\alpha\beta}$$

$$\Rightarrow \frac{\partial L^{EH}}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} + \frac{\partial L^{EH}}{\partial[\partial_\mu g_{\alpha\beta}]} \partial_\mu \delta g_{\alpha\beta} + \frac{\partial L^{EH}}{\partial[\partial_\mu \partial_\nu g_{\alpha\beta}]} \partial_\mu \partial_\nu \delta g_{\alpha\beta} = \partial_\rho (\xi^\rho L^{EH})$$

$$\Rightarrow \delta_\xi L^{EH} = \partial_\rho (\xi^\rho L^{EH}) \qquad \blacksquare$$

→ Using this property and neglecting boundary terms, we can write:

$$0 = 16\pi G\, \delta_\xi S^{EH} = -\int d^n x \sqrt{|g|} (G^{\mu\nu} + \Lambda g^{\mu\nu})\, \delta_\xi g$$
$$= -2\int d^n x \sqrt{|g|} (G^{\mu\nu} + \Lambda g^{\mu\nu})\, \xi_{\mu;\nu}$$
$$= 2\int d^n x \sqrt{|g|} (G^{\mu\nu} + \Lambda g^{\nu\mu})_{;\mu}\, \xi_\nu \quad \forall \xi_\nu$$
$$\Rightarrow \quad G^{\mu\nu}{}_{;\nu} = 0 \quad \text{Noether id.}$$