ADVANCED GENERAL RELATIVITY

PHYS-F-418 ~ Glenn Barnich

→ The major aim of this course is to provide technical background material needed for standard computations in GR and its extension.

CH1 AUXILIARY FIELDS

1.1 Generalized auxiliary fields and symmetries

→ Let the action S depends on 2 fields y^i, z^α:

$$S[y^i, z^\alpha] = \int d^n x\ \mathcal{L}[y^i, y^i_\mu, y^i_{\mu\nu}, z^\alpha, z^\alpha_{\mu\nu}, \dots]$$

where $f_\mu \equiv \partial_\mu f$

↳ Varying the action we get:

$$\delta S = \int d^n x \left\{ \frac{\delta L}{\delta y^i} \delta y^i + \frac{\delta L}{\delta z^\alpha} \cdot \delta z^\alpha \right\}$$

→ Reminder: for $L = L(q, \dot{q}, \ddot{q})$, one gets:

$$\delta L = \delta q\, \partial_q L + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} + \delta \ddot{q} \frac{\partial L}{\partial \ddot{q}}$$

Furthermore, $\delta \dot{q} = \frac{d}{dt} \delta q$. Integrating by part, we get:

$$\delta L = \delta q \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} \right\} + (\text{boundary terms})$$

↳ The Euler-Lagrange derivatives are given by:

$$\frac{\delta L}{\delta y^i} = \frac{\partial L}{\partial y^i} - \partial_\mu \frac{\partial L}{\partial y^i_\mu} + \partial_\mu \partial_\nu \frac{\partial L}{\partial y^i_{\mu\nu}} - \dots$$

→ If we assume $\frac{\delta L}{\delta z^\alpha} = 0 \Leftrightarrow z^\alpha = Z^\alpha[y]$ can be solved algebraically.

DEF A field ξ^α is an auxiliary field is

$$\frac{\delta L}{\delta \xi^\alpha} = 0 \iff \xi^\alpha = Z^\alpha(y^i, y^i_\mu, y^i_{\mu\nu}, \dots) \text{ algebraically.}$$

Thm Let $S[y^i, \xi^\alpha]$ such that

$$\begin{cases} \dfrac{\delta L}{\delta y^i} = 0 \\ \dfrac{\delta L}{\delta \xi^\alpha} = 0 \end{cases}$$

. It is equivalent to $\frac{\delta \bar{L}}{\delta y^i} = 0$ where

$\bar{L} \equiv L[y, \xi = Z] := L|_{\xi = Z}$ is the reduced Lagrangian.

DEMO We have to show that $\left.\frac{\delta L}{\delta y^i}\right|_{\substack{\xi = Z \\ \xi_\mu = Z_\mu}} = \frac{\delta \bar{L}}{\delta y^i}$. We consider:

$$\delta \bar{S} = \delta S[y, \xi = Z]$$
$$= \int d^4x \left\{ \left.\frac{\partial L}{\partial y^i}\right| \delta y^i + \left.\frac{\partial L}{\partial y^i_\mu}\right| \delta y^i_\mu + \dots + \left.\frac{\partial L}{\partial \xi^\alpha}\right| \left(\frac{\partial Z^\alpha}{\partial y^i} \delta y^i + \frac{\partial Z^\alpha}{\partial y^i_\mu} \delta y^i_\mu + \dots \right) \right.$$
$$\left. + \left.\frac{\partial L}{\partial \xi^\alpha_\mu}\right| \partial_\mu \left[\frac{\partial Z^\alpha_\mu}{\partial y^i} \delta y^i + \frac{\partial Z^\alpha_\mu}{\partial y^i_\nu} \delta y^i_\nu + \dots \right] \right\}$$

?

$$= \int d^4x \left\{ \left.\frac{\delta L}{\delta y^i}\right| \delta y^i + \left.\frac{\delta L}{\delta \xi^\alpha}\right| \delta Z^\alpha \right\} = \int d^4x \; \delta \bar{L} \qquad \blacksquare$$

① Symmetries:

DEF Let $S = [\varphi^i, \dots] = \int d^4x \, \mathcal{L}[\varphi^i, \varphi^i_\mu, \dots]$. There is a symetry when

$$\delta_Q \varphi^i = Q^i[\varphi^j, \varphi^j_\mu, \dots] \Rightarrow \delta_Q L = \partial_\mu k^\mu_Q$$

and $\delta_Q \varphi^i_\mu = \partial_\mu \delta_Q \varphi^i = \partial_\mu Q^i$

→ In this case, we have:

$$\delta_Q L = Q^i \frac{\partial L}{\partial \varphi^i} + \partial_\mu Q^i \frac{\partial L}{\partial \varphi^i_\mu} + \partial_\mu \partial_\nu \frac{\partial L}{\partial \varphi^i_{\mu\nu}} + \dots = \partial_\mu k^\mu_Q$$

$$\iff Q^i \left(\frac{\partial L}{\partial \varphi^i} - \partial_\mu \frac{\partial L}{\partial \varphi^i_\mu} + \dots \right) = Q^i \frac{\delta L}{\delta \varphi^i} = \partial_\mu \left[k^\mu_Q - \frac{\partial L}{\partial \varphi^i_\mu} Q^i + \dots \right] \equiv \partial_\mu j^\mu_Q$$

? $Q^i(\dots) = \partial_\mu \left[k - \frac{\partial L}{\partial \varphi^i} Q^i \right]$ boundary term?

We found the Noether current j^μ_Q such that $\partial_\mu j^\mu_Q = Q^i \frac{\delta L}{\delta \varphi^i}$

3.

Lemme Let $\delta_Q y^i = Q^i$, $\delta_Q z^\alpha = Q^\alpha$ be a symmetry of $S[y,z]$.
Then, $\bar{\delta} y^i = Q^i\big|_{z=Z}$ is a symmetry of $\bar{S}[y]$

$\boxed{\text{DEMO}}$ By definition, there exist a Noether current such that

$$Q^i \frac{\delta L}{\delta y^i} + Q^\alpha \frac{\delta L}{\delta z^\alpha} = \partial_\mu j^\mu_Q \Big|_{z=Z}$$

? Why not $z = Z$ in j^μ_Q

$$\Leftrightarrow \bar{Q}^i \frac{\delta L}{\delta y^i}\Big| + Q^\alpha\Big| \underbrace{\frac{\delta L}{\delta z^\alpha}\Big|}_{=0} = \partial_\mu j^\mu_Q\Big|$$

What happened ??

$$\Leftrightarrow \bar{Q}^i \frac{\delta \bar{L}}{\delta y^i} = \partial_\mu j^\mu_Q\Big|$$ by the previous result ■

Thm By suitably choosing δz^α, one can arrange that $\delta y^i = \bar{Q}^i$ is a symmetry of $S[y,z]$: $\delta y^i = \bar{\delta} y^i$

$\boxed{\text{DEMO}}$ We write $S[y,z] = \bar{S}[y] + T[y,z]$ with $T = \int d^n x\, M$. Then,

$$\frac{\delta L}{\delta y^i} = \frac{\delta \bar{L}}{\delta y^i} + \frac{\delta M}{\delta y^i} \quad \text{and} \quad \frac{\delta L}{\delta z^\alpha} = \frac{\delta M}{\delta z^\alpha}$$

Since $\frac{\delta L}{\delta y^i}\Big| = \frac{\delta \bar{L}}{\delta y^i}$, we have $\frac{\delta M}{\delta y^i}\Big| = 0$

$$\Rightarrow \frac{\delta M}{\delta y^i} = \lambda^\alpha_i \frac{\delta L}{\delta z^\alpha} + \lambda^{\alpha\mu}_i \partial_\mu \frac{\delta L}{\delta z^\alpha} + \dots$$

$$\hookrightarrow \bar{Q}^i \frac{\delta L}{\delta y^i} + \delta z^\alpha \frac{\delta L}{\delta z^\alpha} = \bar{Q}^i \left(\frac{\delta \bar{L}}{\delta y^i} + \lambda^\alpha_i \frac{\delta L}{\delta z^\alpha} + \lambda^{\alpha\mu}_i \partial_\mu \frac{\delta L}{\delta z^\alpha} + \dots \right) + \delta z^\alpha \frac{\delta L}{\delta z^\alpha}$$

$$= \partial_\mu \left[j^\mu_Q\Big| + t^\mu \right] + \underbrace{\bar{Q}^i \lambda_i \frac{\delta L}{\delta z^\alpha}}_{1} \underbrace{- \partial_\mu \left[\bar{Q}^i \lambda_i^{\alpha\mu} \right] \frac{\delta L}{\delta z^\alpha}}_{2} + \underbrace{\delta z^\alpha \frac{\delta L}{\delta z^\alpha}}_{3}$$

with $t^\mu = \lambda^{\alpha\mu}_i \bar{Q}^i \frac{\delta L}{\delta z^\alpha}$ $\hookrightarrow 3 \stackrel{!}{=} 1+2$

We need to set $\delta z^\alpha \stackrel{!}{=} -\left[\bar{Q}_i \lambda^\alpha_i - \partial_\mu (\bar{Q}_i \lambda_i^{\alpha\mu}) \right]$ ■

⊙ Examples:

① GF YM theories:

$\rightarrow S = \int d^4x \left\{ -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \partial^\mu \bar{C}_a D_\mu C^a - B_a \partial^\mu A^a_\mu + \frac{\xi}{2} B_a B^a \right\}$

On vérifie que B_a sont des champs auxiliaires en regardant les EOM.

$\hookrightarrow \frac{\delta L}{\delta B^a} = 0 \Leftrightarrow -\partial^\nu A_\nu{}^a = \xi B^a \Leftrightarrow B^a = \frac{-1}{\xi} \partial^\nu A_\nu{}^a$

$\Rightarrow$ Lorsque $\xi \neq 0$, B^a est un champ auxiliaire

② First order Hamiltonian systems.

$\rightarrow$ We consider $S_H = \int dt \{ p_a \dot{q}^a - H(q,p,t) \}$ and we compute $\frac{\delta S_H}{\delta p_a} = 0$. We suppose $\left| \frac{\partial^2 H}{\partial p_a \partial p_b} \right| \neq 0$ (regularity). Then,

$\Leftrightarrow \dot{q}^a - \frac{\partial H}{\partial p_a} = 0 \Rightarrow p_a = p_a(q, \dot{q}, t)$

$\hookrightarrow$ Canonical moments are auxiliary fields

1.2 Invertibility and Legendre transforms

Prop Invertibility is guaranteed if H is constructed from L by Legendre transform: $\left| \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b} \right| \neq 0 \Leftrightarrow \left| \frac{\partial^2 H}{\partial p_a \partial p_b} \right| \neq 0$

Prop Legendre transforms are invertible, they square to one.

$\rightarrow$ Example: $H = \frac{1}{2} g^{ab}(q) p_a p_b + k^a(q) p_a + V(q)$

such that $\det g^{ab} \neq 0$ and $g_{ab} g^{bc} = \delta_a{}^c$

$\hookrightarrow$ The reduced action is the Lagrangian action:

$$\bar{S}_H = \int dt \{ \tfrac{1}{2} (\dot{q}^a - k^a) g_{ab} (\dot{q}^b - k^b) - V(q) \}$$

$\hookrightarrow$ For an harmonic oscillator, $H = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2$

$\Rightarrow \frac{\delta S_H}{\delta q} = 0 \Leftrightarrow -\dot{p} = \omega^2 q \Leftrightarrow q = \frac{-1}{\omega^2} \dot{p}$

$\hookrightarrow \bar{S}_H = \int dt \{ p\dot{q} - H \} \Big|_{q = \frac{-1}{\omega^2}\dot{p}} = \int dt \left\{ \frac{-1}{\omega^2} \ddot{p} p - \frac{1}{2} \left(p^2 + \frac{1}{\omega^2} \dot{p}^2 \right) \right\}$

$$= \int dt \{ \tfrac{1}{2} \tfrac{1}{\omega^2} \dot{p}^2 - \tfrac{1}{2} p^2 \}$$

$\hookrightarrow$ Same dynamics.