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Antoine DIERCKX

Théorie de la Gravitation - Exercices notés

PARTIE A

① On considère $\hat{\Box}\hat{\phi}(x^\mu, z) = 0$ avec $\hat{\phi}(x^\mu, z) = \sum_n \phi_n(x^\mu) e^{inz/L}$

$$\hat{\Box}\hat{\phi} = \partial^M \partial_M \hat{\phi}$$
$$= \sum_n (\Box + \partial_z^2)(\phi_n(x^\mu) e^{inz/L})$$
$$= \sum_n (\Box + i^2 n^2/L^2)(\phi_n(x^\mu) e^{inz/L}) \overset{!}{=} 0$$
$$\Leftrightarrow \left(\Box - \frac{n^2}{L^2}\right)\phi_n = 0$$
$$\Leftrightarrow (\Box - m_n^2)\phi_n = 0 \quad \text{où } m_n^2 \equiv \frac{n^2}{L^2}$$

② On a $d\hat{s}^2 = g_{\mu\nu} dx^\mu dx^\nu + (dz + A_\mu dx^\mu)^2$

$$= g_{\mu\nu} dx^\mu dx^\nu + dz^2 + A_\mu A_\nu dx^\mu dx^\nu + A_\mu dx^\mu dz + A_\mu dz dx^\mu$$
$$= (g_{\mu\nu} + A_\mu A_\nu) dx^\mu dx^\nu + A_\mu(dz dx^\mu + dx^\mu dz) + dz^2$$

Explicitement : $$\hat{g}_{MN} = \begin{pmatrix} g_{00} + A_0^2 & g_{01} + A_0 A_1 & g_{02} + A_0 A_2 & g_{03} + A_0 A_3 & A_0 \\ g_{10} + A_1 A_0 & g_{11} + A_1^2 & g_{12} + A_1 A_2 & g_{13} + A_1 A_3 & A_1 \\ g_{20} + A_2 A_0 & g_{21} + A_2 A_1 & g_{22} + A_2^2 & g_{23} + A_2 A_3 & A_2 \\ g_{30} + A_3 A_0 & g_{31} + A_3 A_1 & g_{32} + A_3 A_2 & g_{33} + A_3^2 & A_3 \\ A_0 & A_1 & A_2 & A_3 & 1 \end{pmatrix}$$

$$\hat{g}^{MN} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} & -A^0 \\ g_{10} & g_{11} & g_{12} & g_{13} & -A^1 \\ g_{20} & g_{21} & g_{22} & g_{23} & -A^2 \\ g_{30} & g_{31} & g_{32} & g_{33} & -A^3 \\ -A^0 & -A^1 & -A^2 & -A^3 & 1 + A^\mu A_\mu \end{pmatrix}$$

On rappelle l'expression des symboles de Christoffel en fonction de la métrique :

$$\hat{\Gamma}^A_{MN} = \frac{1}{2}\hat{g}^{AB}(\partial_M \hat{g}_{BN} + \partial_N \hat{g}_{MB} - \partial_B \hat{g}_{MN})$$

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$\rightarrow \hat{\Gamma}^{\mu}_{\nu\lambda} = \frac{1}{2}\hat{g}^{\mu A}(\partial_\nu \hat{g}_{A\lambda} + \partial_\lambda \hat{g}_{A\nu} - \partial_A \hat{g}_{\nu\lambda})$

$= \frac{1}{2}\hat{g}^{\mu\alpha}(\partial \hat{g}_{\alpha\lambda} + \partial_\lambda \hat{g}_{\alpha\nu} - \partial_\alpha \hat{g}_{\nu\lambda})$

$+ \frac{1}{2}\hat{g}^{\mu z}(\partial_\nu \hat{g}_{z\lambda} + \partial_\lambda \hat{g}_{z\nu} - \cancel{\partial_z \hat{g}_{\nu\lambda}})$

$= \frac{1}{2} g^{\mu\alpha}(\partial_\nu[g_{\alpha\lambda} + A_\alpha A_\lambda] + \partial_\lambda[g_{\alpha\nu} + A_\alpha A_\nu] - \partial_\alpha[g_{\nu\lambda} + A_\nu A_\lambda])$

$+ \frac{1}{2}(-A^\mu)(\partial_\nu A_\lambda + \partial_\lambda A_\nu)$

$= \Gamma^{\mu}_{\nu\lambda} + \frac{1}{2} g^{\mu\alpha}(A_\alpha \partial_\nu A_\lambda + A_\lambda \partial_\nu A_\alpha + A_\alpha \partial_\lambda A_\nu + A_\nu \partial_\lambda A_\alpha$

$- A_\nu \partial_\alpha A_\lambda - A_\lambda \partial_\alpha A_\nu)$

$- \frac{1}{2} A^\mu \partial_\nu A_\lambda - \frac{1}{2} A^\mu \partial_\lambda A_\nu$

$= \Gamma^{\mu}_{\nu\lambda} + \frac{1}{2} g^{\mu\alpha}(A_\alpha B_{\nu\lambda} + A_\lambda F_{\nu\alpha} + A_\nu F_{\lambda\alpha}) - \frac{1}{2} A^\mu B_{\nu\lambda}$

$= \Gamma^{\mu}_{\nu\lambda} + \frac{1}{2}(A_\lambda F_\nu{}^\mu + A_\nu F_\lambda{}^\mu)$

$= \Gamma^{\mu}_{\nu\lambda} - \frac{1}{2}(A_\lambda F^\mu{}_\nu + A_\nu F^\mu{}_\lambda)$

$\rightarrow \hat{\Gamma}^{z}_{\nu\lambda} = \frac{1}{2}\hat{g}^{z\alpha}(\partial_\nu \hat{g}_{\alpha\lambda} + \partial_\lambda \hat{g}_{\alpha\nu} - \partial_\alpha g_{\nu\lambda})$

$+ \frac{1}{2}\hat{g}^{zz}(\partial_\nu \hat{g}_{z\lambda} + \partial_\lambda \hat{g}_{z\nu})$

$= \frac{1}{2}(-A^\alpha)(\partial_\nu[g_{\alpha\lambda} + A_\alpha A_\lambda] + \partial_\lambda[g_{\alpha\nu} + A_\alpha A_\nu] - \partial_\alpha[g_{\nu\lambda} + A_\nu A_\lambda])$

$+ \frac{1}{2}(1 + A^2)(\partial_\nu[A_\lambda] + \partial_\lambda[A_\nu])$

$= -\frac{1}{2} A^\sigma g_{\sigma\beta} g^{\beta\alpha}(\{\partial_\nu g_{\alpha\lambda} + \partial_\lambda g_{\alpha\nu} - \partial_\alpha g_{\nu\lambda}\}$

$+ A_\alpha \partial_\nu A_\lambda + A_\lambda \partial_\nu A_\alpha + A_\alpha \partial_\lambda A_\nu + A_\nu \partial_\lambda A_\alpha - A_\nu \partial_\alpha A_\lambda - A_\lambda \partial_\alpha A_\nu)$

$+ \frac{1}{2} B_{\nu\lambda} + \frac{1}{2} A^2 B_{\nu\lambda}$

$= -A_\sigma \Gamma^{\sigma}_{\nu\lambda} - \frac{1}{2} A^\alpha (A_\alpha B_{\nu\lambda} + A_\nu F_{\lambda\alpha} + A_\lambda F_{\nu\alpha}) + \frac{1}{2} B_{\nu\lambda} + \frac{1}{2} A^2 B_{\nu\lambda}$

$= \frac{1}{2} B_{\nu\lambda} - \frac{1}{2} A^\mu (A_\nu F_{\lambda\mu} + A_\lambda F_{\nu\mu}) - A_\mu \Gamma^{\mu}_{\nu\lambda}$

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$$\rightarrow \hat{\Gamma}^{\mu}_{z\lambda} = \frac{1}{2}\hat{g}^{\mu\alpha}\left(\partial_z \hat{g}_{\alpha\lambda} + \partial_\lambda \hat{g}_{\alpha z} - \partial_\alpha \hat{g}_{z\lambda}\right) + 0$$

$$= \frac{1}{2} g^{\mu\alpha}\left(\partial_\lambda A_\alpha - \partial_\alpha A_\lambda\right)$$

$$= \frac{1}{2} F_\lambda{}^{\mu}$$

$$= -\frac{1}{2} F^{\mu}{}_{\lambda}$$

$$\rightarrow \hat{\Gamma}^{z}_{z\mu} = \frac{1}{2}\hat{g}^{z\alpha}\left(\partial_\mu \hat{g}_{z\alpha} - \partial_\alpha \hat{g}_{\mu z}\right)$$

$$= -\frac{1}{2} A^{\alpha} F_{\mu\alpha}$$

$$= -\frac{1}{2} A^{\nu} F_{\mu\nu}$$

$$\rightarrow \hat{\Gamma}^{\mu}_{zz} = \frac{1}{2}\hat{g}^{\mu\alpha}\left(-\partial_\alpha \hat{g}_{zz}\right) + 0$$

$$= 0$$

$$\rightarrow \hat{\Gamma}^{z}_{zz} = \frac{1}{2}\hat{g}^{z\alpha}\left(-\partial_\alpha \hat{g}_{zz}\right) + 0$$

$$= 0$$

On trouve ainsi les résultats escomptés :

$$\hat{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} - \frac{1}{2}\left(A_\lambda F^{\mu}{}_{\nu} + A_\nu F^{\mu}{}_{\nu}\right)$$

$$\hat{\Gamma}^{z}_{\nu\lambda} = \frac{1}{2} B_{\nu\lambda} - \frac{1}{2} A^{\mu}\left(A_\nu F_{\lambda\mu} + A_\lambda F_{\nu\mu}\right) - A_\mu \Gamma^{\mu}_{\nu\lambda}$$

$$\hat{\Gamma}^{\mu}_{z\lambda} = -\frac{1}{2} F^{\mu}{}_{\lambda}$$

$$\hat{\Gamma}^{z}_{z\mu} = -\frac{1}{2} A^{\nu} F_{\mu\nu}$$

$$\hat{\Gamma}^{\mu}_{zz} = \hat{\Gamma}^{z}_{zz} = 0$$

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PARTIE A

(2b) On rappelle l'expression du tenseur de Riemann en terme de ses composantes :

$$R^\alpha_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\beta\sigma} - \Gamma^\sigma_{\mu\beta}\Gamma^\alpha_{\nu\sigma}$$

$$\to \hat R_{\mu\nu} = \hat R^A_{\mu A\nu} = \hat R^\alpha_{\mu\alpha\nu} + \hat R^z_{\mu z\nu}$$

$$= \partial_\alpha \hat\Gamma^\alpha_{\mu\nu} - \partial_\nu \hat\Gamma^\alpha_{\mu\alpha} + \hat\Gamma^\sigma_{\mu\nu}\hat\Gamma^\alpha_{\alpha\sigma} + \hat\Gamma^z_{\mu\nu}\hat\Gamma^\alpha_{\alpha z} - \hat\Gamma^\sigma_{\mu\alpha}\hat\Gamma^\alpha_{\nu\sigma} - \hat\Gamma^z_{\mu\alpha}\hat\Gamma^\alpha_{\nu z}$$

$$+ \cancel{\partial_z \hat\Gamma^z_{\mu\nu}} - \partial_\nu \hat\Gamma^z_{\mu z} + \hat\Gamma^\sigma_{\mu\nu}\hat\Gamma^z_{z\sigma} + \cancel{\hat\Gamma^z_{\mu\nu}\hat\Gamma^z_{zz}} - \hat\Gamma^\sigma_{\mu z}\hat\Gamma^z_{\nu\sigma} - \hat\Gamma^z_{\mu z}\hat\Gamma^z_{\nu z}$$

$$= \partial_\alpha\left[\Gamma^\alpha_{\mu\nu} - \frac{1}{2} F^\alpha{}_\mu A_\nu - \frac{1}{2} F^\alpha{}_\nu A_\mu\right] - \partial_\nu\left[\Gamma^\alpha_{\mu\alpha} - \frac{1}{2} F^\alpha{}_\mu A_\alpha\right]$$

$$+ \left(\cancel{\Gamma^\sigma_{\mu\nu}} - \frac{1}{2} F^\sigma{}_\mu A_\nu - \frac{1}{2} F^\sigma{}_\nu A_\mu\right)\left(\cancel{\hat\Gamma^\alpha_{\alpha\sigma}} - \frac{1}{2} F^\alpha{}_\sigma A_\alpha\right)$$

$$+ \hat\Gamma^z_{\mu\nu}\left(\cancel{-\frac{1}{2} F^\alpha{}_\alpha}\right) - \left(\cancel{\Gamma^\sigma_{\mu\alpha}} - \frac{1}{2} F^\sigma{}_\mu A_\alpha - \frac{1}{2} F^\sigma{}_\alpha A_\mu\right)\left(\cancel{\Gamma^\alpha_{\nu\sigma}} - \frac{1}{2} F^\alpha{}_\nu A_\sigma - \frac{1}{2} F^\alpha{}_\sigma A_\nu\right)$$

$$- \left(\frac{1}{2} B_{\mu\alpha} + \frac{1}{2} A^\beta (A_\mu F_{\beta\alpha} + A_\alpha F_{\beta\mu}) - A_\beta \cancel{\Gamma^\beta_{\mu\alpha}}\right)\left(-\frac{1}{2} F^\alpha{}_\nu\right)$$

$$- \partial_\nu\left(-\frac{1}{2} A^\beta F_{\mu\beta}\right) + \left(\cancel{\Gamma^\sigma_{\mu\nu}} - \frac{1}{2} F^\sigma{}_\mu A_\nu - \frac{1}{2} F^\sigma{}_\nu A_\mu\right)\left(-\frac{1}{2} A^\beta F_{\sigma\beta}\right)$$

$$- \left(-\frac{1}{2} A^\beta F_{\mu\beta}\right)\left(-\frac{1}{2} A^\gamma F_{\nu\gamma}\right)$$

On se place dans un SCLI : $\Gamma = 0$ et $\partial \mapsto \nabla$

$$\hat R_{\mu\nu} = \nabla_\alpha\left[\Gamma^\alpha_{\mu\nu} - \frac{1}{2} F^\alpha{}_\mu A_\nu - \frac{1}{2} F^\alpha{}_\nu A_\mu\right] + \nabla_\nu\left[\Gamma^\alpha_{\mu\alpha} + \frac{1}{2} F^\alpha{}_\mu A_\alpha\right]$$

$$+ \left(\frac{1}{2} F^\sigma{}_\mu A_\nu + \frac{1}{2} F^\sigma{}_\nu A_\mu\right)\left(\frac{1}{2} F^\alpha{}_\sigma A_\alpha\right)$$

$$- \left(\frac{1}{2} F^\sigma{}_\mu A_\alpha + \frac{1}{2} F^\sigma{}_\alpha A_\mu\right)\left(\frac{1}{2} F^\alpha{}_\nu A_\sigma + \frac{1}{2} F^\alpha{}_\sigma A_\nu\right)$$

$$+ \left(\frac{1}{2} B_{\mu\alpha} + \frac{1}{2} A^\beta (A_\mu F_{\beta\alpha} + A_\alpha F_{\beta\mu})\right)\left(\frac{1}{2} F^\alpha{}_\nu\right) + \frac{1}{2}\nabla_\nu \left(A^\beta F_{\mu\beta}\right)$$

$$+ \left(\frac{1}{2} F^\sigma{}_\mu A_\nu + \frac{1}{2} F^\sigma{}_\nu A_\mu\right)\left(\frac{1}{2} A^\beta F_{\sigma\beta}\right) - \left(\frac{1}{2} A^\beta F_{\mu\beta}\right)\left(\frac{1}{2} A^\gamma F_{\nu\gamma}\right)$$

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$$\hat{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} F^{\alpha}{}_{\mu} \nabla_\alpha A_\nu - \frac{1}{2} A_\nu \nabla_\alpha F^{\alpha}{}_{\mu} - \frac{1}{2} F^{\alpha}{}_{\nu} \nabla_\alpha A_\mu - \frac{1}{2} A_\mu \nabla_\alpha F^{\alpha}{}_{\nu}$$

$$+ \frac{1}{2} F^{\alpha}{}_{\mu} \nabla_\nu A_\alpha + \frac{1}{2} A_\alpha \nabla_\nu F^{\alpha}{}_{\mu}$$

$$\cancel{+\frac{1}{4} A_\nu A_\alpha F^{\sigma}{}_{\mu} F^{\alpha}{}_{\sigma}} \cancel{+ \frac{1}{4} A_\mu A_\alpha F^{\sigma}{}_{\nu} F^{\alpha}{}_{\sigma}}$$

$$- \frac{1}{4} A_\alpha A_\sigma F^{\sigma}{}_{\mu} F^{\alpha}{}_{\nu} \cancel{- \frac{1}{4} A_\alpha A_\nu F^{\sigma}{}_{\mu} F^{\alpha}{}_{\sigma}} \cancel{- \frac{1}{4} A_\mu A_\sigma F^{\sigma}{}_{\alpha} F^{\alpha}{}_{\nu}}$$

$$- \frac{1}{4} A_\mu A_\nu F^{\sigma}{}_{\alpha} F^{\alpha}{}_{\sigma}$$

$$+ \frac{1}{4} B_{\mu\alpha} F^{\alpha}{}_{\nu} + \frac{1}{4} A_\mu A_\beta F^{\beta}{}_{\alpha} F^{\alpha}{}_{\nu} \cancel{+ \frac{1}{4} A_\alpha A_\beta F^{\beta}{}_{\mu} F^{\alpha}{}_{\nu}}$$

$$+ \frac{1}{2} A_\beta \nabla_\nu F_{\mu}{}^{\beta} + \frac{1}{2} F_{\mu}{}^{\beta} \nabla_\nu A_\beta$$

$$+ \frac{1}{4} A_\nu A_\beta F^{\sigma}{}_{\mu} F_{\sigma}{}^{\beta} + \frac{1}{4} A_\mu A_\beta F^{\sigma}{}_{\nu} F_{\sigma}{}^{\beta} \cancel{- \frac{1}{4} A_\beta A_\gamma F_{\mu}{}^{\beta} F_{\nu}{}^{\gamma}}$$

$$= R_{\mu\nu} + \frac{1}{2} \left(A_\nu \nabla_\alpha F_{\mu}{}^{\alpha} + A_\mu \nabla_\alpha F_{\nu}{}^{\alpha} \right) + \frac{1}{4} A_\mu A_\nu F^{\sigma\alpha} F_{\sigma\alpha}$$

$$+ \frac{1}{2} F^{\alpha}{}_{\mu} F_{\nu\alpha} - \frac{1}{2} F^{\alpha}{}_{\nu} \nabla_\alpha A_\mu + \frac{1}{2} A_\alpha \nabla_\nu F^{\alpha}{}_{\mu} + \frac{1}{2} A_\alpha \nabla_\nu F_{\mu}{}^{\alpha} + \frac{1}{2} F_{\mu}{}^{\alpha} \nabla_\nu A_\alpha$$

$$- \frac{1}{4} A_\nu A_\beta F^{\sigma}{}_{\mu} F^{\beta}{}_{\sigma} + \frac{1}{4} F^{\alpha}{}_{\nu} \nabla_\mu A_\alpha + \frac{1}{4} F^{\alpha}{}_{\nu} \nabla_\alpha A_\mu$$

$$= R_{\mu\nu} + \frac{1}{2} \left(A_\nu \nabla_\alpha F_{\mu}{}^{\alpha} + A_\mu \nabla_\alpha F_{\nu}{}^{\alpha} \right) + \frac{1}{4} A_\mu A_\nu F^{\sigma\alpha} F_{\sigma\alpha} + \frac{1}{2} F_{\mu}{}^{\alpha} F_{\alpha\nu}$$

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$\longrightarrow \hat{R}_{z\mu} = \hat{R}^{A}{}_{zA\mu} = \hat{R}^{\alpha}{}_{z\alpha\mu} + \hat{R}^{z}{}_{zz\mu}$

$$= \partial_\alpha \hat{\Gamma}^\alpha_{z\mu} - \partial_\mu \hat{\Gamma}^\alpha_{z\alpha} + \hat{\Gamma}^\sigma_{z\mu}\hat{\Gamma}^\alpha_{\alpha\sigma} + \hat{\Gamma}^z_{z\mu}\hat{\Gamma}^\alpha_{\alpha z} - \hat{\Gamma}^\sigma_{z\alpha}\hat{\Gamma}^\alpha_{\mu\sigma} - \hat{\Gamma}^z_{z\alpha}\hat{\Gamma}^\alpha_{\mu z}$$

$$+ \partial_z \hat{\Gamma}^z_{z\mu} - \partial_\mu \hat{\Gamma}^z_{zz} + \hat{\Gamma}^\sigma_{z\mu}\hat{\Gamma}^z_{z\sigma} + \hat{\Gamma}^z_{z\mu}\hat{\Gamma}^z_{zz} - \hat{\Gamma}^\sigma_{zz}\hat{\Gamma}^z_{\mu\sigma} - \hat{\Gamma}^z_{zz}\hat{\Gamma}^z_{\mu z}$$

$$= \partial_\alpha \hat{\Gamma}^\alpha_{z\mu} - \partial_\mu \hat{\Gamma}^\alpha_{z\alpha} + \hat{\Gamma}^\sigma_{z\mu}\hat{\Gamma}^\alpha_{\alpha\sigma} - \hat{\Gamma}^\sigma_{z\alpha}\hat{\Gamma}^\alpha_{\mu\sigma} - \hat{\Gamma}^z_{z\alpha}\hat{\Gamma}^\alpha_{\mu z}$$

$$+ \hat{\Gamma}^\sigma_{z\mu}\hat{\Gamma}^z_{z\sigma}$$

$$= \partial_\alpha\left[-\frac{1}{2}F^\alpha{}_\mu\right] + \left[-\frac{1}{2}F^\sigma{}_\mu\right]\left[\Gamma^\alpha_{\alpha\sigma} - \frac{1}{2}F^\alpha{}_\sigma A_\alpha\right]$$

$$+ \left(+\frac{1}{2}F^\sigma{}_\alpha\right)\left(\Gamma^\alpha_{\mu\sigma} - \frac{1}{2}F^\alpha{}_\mu A_\sigma - \frac{1}{2}F^\alpha{}_\sigma A_\mu\right)$$

$$+ \left(+\frac{1}{2}A^\sigma F_{\alpha\sigma}\right)\left(-\frac{1}{2}F^\alpha{}_\mu\right) + \left(+\frac{1}{2}F^\sigma{}_\mu\right)\left(+\frac{1}{2}A^\alpha F_{\sigma\alpha}\right)$$

On passe en SCLI.

$$\hat{R}_{z\mu} = -\frac{1}{2}\nabla_\alpha F^\alpha{}_\mu + \frac{1}{4}A_\alpha F^\sigma{}_\mu F^\alpha{}_\sigma - \frac{1}{4}A_\sigma F^\sigma{}_\alpha F^\alpha{}_\mu - \frac{1}{4}A_\mu F^\sigma{}_\alpha F^\alpha{}_\sigma$$

$$-\frac{1}{4}A_\sigma F_\alpha{}^\sigma F^\alpha{}_\mu + \frac{1}{4}A_\alpha F^\sigma{}_\mu F_\sigma{}^\alpha$$

$$= \frac{1}{2}\nabla_\alpha F_\mu{}^\alpha + \frac{1}{4}A_\mu F^{\sigma\alpha}F_{\sigma\alpha}$$

$\longrightarrow \hat{R}_{zz} = \hat{R}^{A}{}_{zAz} = \hat{R}^{\alpha}{}_{z\alpha z} + \hat{R}^{z}{}_{zzz}$

$$= \partial_\alpha \hat{\Gamma}^\alpha_{zz} - \partial_z \hat{\Gamma}^\alpha_{z\alpha} + \hat{\Gamma}^\sigma_{zz}\hat{\Gamma}^\alpha_{\alpha\sigma} + \hat{\Gamma}^z_{zz}\hat{\Gamma}^\alpha_{\alpha z} - \hat{\Gamma}^\sigma_{z\alpha}\hat{\Gamma}^\alpha_{z\sigma} - \hat{\Gamma}^z_{z\alpha}\hat{\Gamma}^\alpha_{zz}$$

$$+ \partial_z \hat{\Gamma}^z_{zz} - \partial_z \hat{\Gamma}^z_{zz} + \hat{\Gamma}^\sigma_{zz}\hat{\Gamma}^z_{z\sigma} + \hat{\Gamma}^z_{zz}\hat{\Gamma}^z_{zz} - \hat{\Gamma}^\sigma_{zz}\hat{\Gamma}^z_{z\sigma} - \hat{\Gamma}^z_{zz}\hat{\Gamma}^z_{zz}$$

$$= +\left(+\frac{1}{2}F^\sigma{}_\alpha\right)\left(-\frac{1}{2}F^\alpha{}_\sigma\right) = -\frac{1}{4}F^\sigma{}_\alpha F^\alpha{}_\sigma$$

$$= \frac{1}{4}F^{\sigma\alpha}F_{\sigma\alpha}$$

[redacted]

PART A

$\boxed{2c}$ Montrer que $\hat{R} = R - \frac{1}{2} F^{\lambda\rho} F_{\lambda\rho}$

$\rightarrow$ $\hat{R} = \hat{R}^M{}_M = \hat{g}^{MN} \hat{R}_{MN} = \hat{g}^{\mu\nu} \hat{R}_{\mu\nu} + \hat{g}^{\mu z} \hat{R}_{\mu z} + \hat{g}^{z\nu} \hat{R}_{z\nu} + \hat{g}^{zz} \hat{R}_{zz}$

$$= \hat{g}^{\mu\nu} \hat{R}_{\mu\nu} + 2 \hat{g}^{\mu z} \hat{R}_{\mu z} + \hat{g}^{zz} \hat{R}_{zz}$$

$$= g^{\mu\nu} \left(R_{\mu\nu} + \frac{1}{2} F_\mu{}^\alpha F_{\alpha\nu} + \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} A_\mu A_\nu + \frac{1}{2} \left(A_\nu \nabla_\alpha F_\mu{}^\alpha + A_\mu \nabla_\alpha F_\nu{}^\alpha \right) \right)$$

$$+ 2 (-A^\mu) \left(\frac{1}{2} \nabla_\alpha F_\mu{}^\alpha + \frac{1}{4} A_\mu F^{\alpha\beta} F_{\alpha\beta} \right) + (1 + A_\sigma A^\sigma) \left(\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \right)$$

$$= R + \frac{1}{2} F_\mu{}^\alpha F_\alpha{}^\mu + \frac{1}{4} F^2 A^2 + \frac{1}{2} \left(A_\nu \nabla_\alpha F^{\nu\alpha} + A_\mu \nabla_\alpha F^{\mu\alpha} \right)$$

$$- A^\mu \nabla_\alpha F_\mu{}^\alpha - \frac{1}{2} A^2 F^2 + \frac{1}{4} F^2 + \frac{1}{4} A^2 F^2$$

$$= R - \frac{1}{2} F^{\mu\alpha} F_{\mu\alpha} + \frac{1}{4} F^2$$

$$= R - \frac{1}{4} F^2 = R - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}$$

On a utilisé les notations suivantes: $A_\mu A^\mu = A^2$ et $F^{\mu\nu} F_{\mu\nu} = F^2$

$\boxed{2d}$ Montrer que sous $A_\mu \mapsto \lambda A_\mu$, $\hat{R} \mapsto R - \frac{\lambda^2}{4} F^{\lambda\rho} F_{\lambda\rho}$

$\rightarrow$ On peut écrire :

$$\hat{R} = R - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} = R - \frac{1}{4} F_{\alpha\beta} F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta}$$

Or, sous $A_\mu \mapsto \lambda A_\mu$, on a :

$$\hat{g}^{\mu\nu} \mapsto \hat{g}^{\mu\nu} \, ; \, \hat{g}^{\mu z} \mapsto -\lambda A^\mu \text{ et } \hat{g}^{zz} \mapsto 1 + \lambda^2 A^2$$

Ainsi, sous cette même transformation, on aura :

$$A^2 \mapsto \lambda^2 A^2 \text{ et } F_{\mu\nu} \mapsto \lambda F_{\mu\nu}$$

On aura ainsi :

$$R - \frac{1}{4} F^2 \mapsto R - \frac{1}{4} \lambda^2 F^2$$

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PARTIE A

$\boxed{2.e}$ Montrer que $\hat{S} = \frac{1}{8\pi\hat{G}} \int d^5x \sqrt{-\hat{g}}\, \hat{R}$ peut se réécrire comme

$$\hat{S} = \frac{1}{8\pi G} \int \sqrt{-g}\, d^4x\, R - \frac{1}{4} \int \sqrt{-g}\, d^4x\, F^2 \quad \text{avec } G = \frac{\hat{G}}{2\pi L} \text{ et } \lambda^2 = 8\pi G.$$

On a :

$$\hat{S} = \frac{1}{8\pi\hat{G}} \int d^4x \int_{S^1} dz \sqrt{-\hat{g}} \left(R - \frac{\lambda^2}{4} F^2\right)$$

$$= \frac{1}{8\pi\hat{G}} \int d^4x \sqrt{-\hat{g}} \left(R - \frac{\lambda^2}{4} F^2\right) \underbrace{\int_{S^1} dz}_{= 2\pi L}$$

$$= \frac{1}{8\pi G} \int d^4x \sqrt{-\hat{g}} \left(R - \frac{\lambda^2}{4} F^2\right)$$

Or $\hat{g} = \det \hat{g}_{MN} = \det \begin{pmatrix} g_{\mu\nu} + \lambda^2 A_\mu A_\nu & \lambda A_\mu \\ \lambda A_\nu & 1 \end{pmatrix}$

ligne $\mu \leftrightarrow$ ligne μ $- \lambda A_\nu \cdot$ ligne 5 $\quad = \det \begin{pmatrix} g_{\mu\nu} & 0 \\ \lambda A_\nu & 1 \end{pmatrix}$

$= \det g$

Ainsi, $\sqrt{-\hat{g}} = \sqrt{-g}$. On trouve alors :

$$\hat{S} = \frac{1}{8\pi G} \int d^4x \sqrt{-g}\, R - \frac{\lambda^2}{4} \cdot \frac{1}{8\pi G} \int d^4x \sqrt{-g}\, F^2$$

$$= \frac{1}{8\pi G} \int d^4x \sqrt{-g}\, R - \frac{1}{4} \int d^4x \sqrt{-g}\, F^{\alpha\beta} F_{\alpha\beta}$$

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PARTIE A

$\boxed{3}$ Considérons l'action suivante :

$$\hat{S} = \frac{1}{8\pi G}\int \sqrt{-g}\, d^4x\, R - \frac{1}{4}\int \sqrt{-g}\, d^4x\, F^2$$

$$= 2\, S_{EH} + S_M$$

avec $S_{EH} = \frac{1}{16\pi G}\int \sqrt{-g}\, d^4x\, R$ et $S_M = \frac{-1}{4}\int \sqrt{-g}\, d^4x\, F^2$

→ Considérons une variation de l'action sous $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$

→ On utilise le résultat suivant :

$$\delta S_{EH} = \frac{1}{16\pi G}\int \sqrt{-g}\, d^4x\, (-G^{\mu\nu})\, \delta g_{\mu\nu}$$

avec $G^{\mu\nu} = R^{\mu\nu} - \frac{R}{2} g^{\mu\nu}$

$$\rightarrow \delta S_M = \frac{-1}{4}\int d^4x \Big\{ (\delta\sqrt{-g})\, F^2 + \sqrt{-g}\,.\,\delta(g^{\alpha\mu})\, g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu} + \sqrt{-g}\, g^{\alpha\mu}\, \delta(g^{\beta\nu})\, F_{\alpha\beta} F_{\mu\nu} \Big\}$$

On utilise les résultats suivants :

$$\rightarrow \delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\, \delta g_{\mu\nu}$$

$$\rightarrow \delta g^{\mu\nu} = -g^{\alpha\mu} g^{\beta\nu}\, \delta g_{\alpha\beta}$$

En effet, $\delta(g^{\mu\alpha} g_{\alpha\nu}) = 0 = g^{\mu\alpha}\, \delta g_{\alpha\nu} + \delta g^{\mu\alpha}.\, g_{\alpha\nu} \quad \| \cdot g^{\nu\sigma}$

$\Leftrightarrow \delta g^{\mu\alpha}.\, \delta^\sigma_\alpha = -g^{\mu\alpha} g^{\nu\sigma}\, \delta g_{\alpha\nu}$

$\Leftrightarrow \delta g^{\mu\sigma} = -g^{\mu\alpha} g^{\nu\sigma}\, \delta g_{\alpha\nu}$

On trouve alors :

$$\delta S_M = -\frac{1}{4}\int d^4x \Big\{ \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\, \delta g_{\mu\nu}\, F^2 - \sqrt{-g}\, g^{\beta\nu} g^{\alpha\sigma} g^{\mu\lambda}\, \delta g_{\sigma\mu}\, F_{\alpha\beta} F_{\lambda\nu} - \sqrt{-g}\, g^{\alpha\mu} g^{\beta\sigma} g^{\nu\lambda}\, \delta g_{\sigma\nu}\, F_{\alpha\sigma} F_{\mu\lambda} \Big\}$$

$$= -\frac{1}{4}\int d^4x \sqrt{-g} \Big\{ \frac{1}{2} g^{\mu\nu} F^2 - F^{\mu\alpha} F^{\nu}{}_{\alpha} - F^{\alpha\mu} F_{\alpha}{}^{\nu} \Big\}\, \delta g_{\mu\nu}$$

$$= -\frac{1}{4}\int d^4x \sqrt{-g} \Big\{ \frac{1}{2} g^{\mu\nu} F^2 - 2 F^{\mu\alpha} F^{\nu}{}_{\alpha} \Big\}\, \delta g_{\mu\nu}$$

10) Antoine DIERCKX

→ On trouve alors :

$$\delta\hat{S} = \frac{1}{8\pi G}\int\sqrt{-g}\,d^4x\,(-G^{\mu\nu})\,\delta g_{\mu\nu} - \frac{1}{4}\int d^4x\sqrt{-g}\left\{\frac{1}{2}g^{\mu\nu}F^2 - 2F^{\mu\alpha}F^{\nu}{}_{\alpha}\right\}\delta g_{\mu\nu}$$

$$= \int d^4x\sqrt{-g}\left\{\frac{-G^{\mu\nu}}{8\pi G} - \frac{1}{8}g^{\mu\nu}F^2 + \frac{1}{2}F^{\mu\alpha}F^{\nu}{}_{\alpha}\right\}\delta g_{\mu\nu}$$

$$\stackrel{!}{=} 0 \quad \forall\,\delta g_{\mu\nu}$$

$$\Rightarrow \quad G^{\mu\nu} = 4\pi G\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F^2\right)$$

→ Considérons une variation de l'action sous $A_\mu \mapsto A_\mu + \delta A_\mu$

→ On a $\delta S_{EH} = 0$

→ On a $\delta S_M = -\frac{1}{4}\int\sqrt{-g}\,d^4x\;\delta(F^{\alpha\beta}F_{\alpha\beta})$

$$\rightarrow \delta(F^{\alpha\beta}F_{\alpha\beta}) = \delta F^{\alpha\beta}.F_{\alpha\beta} + F^{\alpha\beta}\,\delta F_{\alpha\beta}$$

$$= 2F^{\alpha\beta}\,\delta F_{\alpha\beta}$$

$$= 2F^{\alpha\beta}(\partial_\alpha\delta A_\beta - \partial_\beta\delta A_\alpha)$$

$$= 4F^{\alpha\beta}\,\partial_\alpha\delta A_\beta$$

Ainsi, $\delta S_M = -\int\sqrt{-g}\,d^4x\;F^{\alpha\beta}\partial_\alpha\,\delta A_\beta$

$$= +\int\sqrt{-g}\,d^4x\;\partial_\alpha F^{\alpha\beta}.\,\delta A_\beta$$

$$\stackrel{!}{=} 0 \quad \forall\,\delta A_\beta$$

$$\Rightarrow \partial_\alpha F^{\alpha\beta} = 0$$

Si on se place dans un SCLI, on rend notre EOM tensorielle :

$$\nabla_\alpha F^{\alpha\beta} = 0$$

11) Antoine DIERCKX

Considérons l'action suivante :

$$\hat{S} = \frac{1}{8\pi\hat{G}} \int d^5x \sqrt{-\hat{g}}\, \hat{R}$$

On veut la faire varier sous $\hat{g}_{MN} \mapsto \hat{g}_{MN} + \delta\hat{g}_{MN}$

Pour cela, de manière analogue au cas en 4-D, on veut se placer dans un SCLI. Puisque la métrique est de la forme

$d\hat{s}^2 = g_{\mu\nu} dx^\mu dx^\nu + (dz + A_\mu dx^\mu)^2$, avoir

$d\hat{s}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dz^2$ implique que $A^\mu = 0$ dans ce référentiel. On suppose cela possible, pour un bon choix de jauge.

→ Les Γ s'annulent

→ Le tenseur de Ricci prend la forme suivante :

$\hat{R}_{MN} = \hat{R}^A{}_{MAN} = \nabla_A \hat{\Gamma}^A_{MN} - \nabla_N \hat{\Gamma}^A_{MA}$. Sa variation se donnée par :

$$\delta\hat{R}_{MN} = \nabla_A \delta\hat{\Gamma}^A_{MN} - \nabla_N \delta\hat{\Gamma}^A_{MA}$$

→ Le scalaire de Ricci et sa variation sont donnés par :

$$\hat{R} = \hat{g}^{AB} \hat{R}_{AB}$$

$$\begin{aligned}\delta\hat{R} &= \hat{R}_{AB}\, \delta\hat{g}^{AB} + \hat{g}^{AB} \delta\hat{R}_{AB} \\ &= \hat{R}_{AB}\left(-\hat{g}^{AM}\hat{g}^{BN}\delta\hat{g}_{MN}\right) + \hat{g}^{MN}\left(\nabla_A \delta\hat{\Gamma}^A_{MN} - \nabla_N \delta\hat{\Gamma}^A_{MA}\right)\end{aligned}$$

Ainsi, $\delta(\sqrt{-\hat{g}}\,\hat{R})$ s'écrit :

$$\begin{aligned}&\delta(\sqrt{-\hat{g}})\,\hat{R} + \sqrt{-\hat{g}}\,\delta\hat{R} \\ &= \frac{1}{2}\sqrt{-\hat{g}}\,\hat{g}^{AB}\delta\hat{g}_{AB}.\hat{R} + \sqrt{-\hat{g}}\left(-\hat{R}^{MN}\delta\hat{g}_{MN} + \hat{g}^{MN}\left(\nabla_A\delta\hat{\Gamma}^A_{MN} - \nabla_N\delta\hat{\Gamma}^A_{MA}\right)\right) \\ &= \sqrt{-\hat{g}}\left(\frac{\hat{R}}{2}\hat{g}^{AB} - \hat{R}^{AB}\right)\delta\hat{g}_{AB} \\ &\quad + \sqrt{-\hat{g}}\,\nabla_P\left(\hat{g}^{MN}\delta\hat{\Gamma}^P_{MN} - \hat{g}^{MP}\delta\hat{\Gamma}^A_{MA}\right)\end{aligned}$$

On utilise le résultat du cours suivant :

$$\sqrt{-\hat{g}}\,\nabla_P V^P = \partial_P(\sqrt{-\hat{g}}\, V^P)$$

Ainsi, le 2e terme est une dérivée totale.

12) Antoine DIERCKX

La variation de l'action prend alors la forme :

$$\delta \hat{S} = \frac{1}{8\pi \hat{G}} \int d^5x \Big(\sqrt{-\hat{g}} \left\{ \frac{\hat{R}}{2} \hat{g}^{AB} - \hat{R}^{AB} \right\} \delta \hat{g}_{AB}$$

$$+ \partial_P \left[\sqrt{-\hat{g}} \left\{ \hat{g}^{MN} \delta \hat{\Gamma}^P_{MN} - \hat{g}^{MP} \delta \hat{\Gamma}^A_{MA} \right\} \right] \Big)$$

$$\overset{!}{=} 0 \quad \forall \, \delta \hat{g}_{AB}$$

$$\Rightarrow \quad \hat{R}^{AB} - \frac{1}{2} \hat{R} \, \hat{g}^{AB} = 0$$

→ Pour comparer les différentes expressions, il faut les développer.

→ $\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2} R \, \hat{g}_{\mu\nu}$ $\qquad \hat{g}_{\mu\nu} = g_{\mu\nu} + \lambda^2 A_\mu A_\nu$

$$= R_{\mu\nu} + \frac{1}{2} \lambda^2 F_\mu{}^\alpha F_{\alpha\nu} + \frac{1}{4} \lambda^4 F^2 A_\mu A_\nu$$
$$+ \frac{1}{2} \lambda^2 (A_\nu \nabla_\alpha F_\mu{}^\alpha + A_\mu \nabla_\alpha F_\nu{}^\alpha) - \frac{R}{2} g_{\mu\nu} - \frac{\lambda^2 R}{2} A_\mu A_\nu$$
$$+ \frac{\lambda^2 F^2}{8} g_{\mu\nu} + \frac{\lambda^4 F^2}{8} A_\mu A_\nu$$

$$= G_{\mu\nu} + \frac{\lambda^2}{2} F_\mu{}^\alpha F_{\alpha\nu} + \frac{1}{2} \lambda^2 (A_\nu \nabla_\alpha F_\mu{}^\alpha + A_\mu \nabla_\alpha F_\nu{}^\alpha)$$
$$+ \frac{3}{8} \lambda^4 F^2 A_\mu A_\nu - \frac{\lambda^2}{2} R A_\mu A_\nu + \frac{\lambda^2 F^2}{8} g_{\mu\nu} = 0$$

→ $\hat{G}_{\mu z} = \hat{R}_{\mu z} - \frac{1}{2} R \, \hat{g}_{\mu z}$ $\qquad \hat{g}_{\mu z} = \lambda A_\mu$

$$= \frac{\lambda}{2} \nabla_\alpha F_\mu{}^\alpha + \frac{\lambda^3}{4} A_\mu F^2 - \frac{R}{2} \lambda A_\mu + \frac{\lambda^3 F^2}{8} A_\mu$$

$$= \frac{3}{8} \lambda^3 F^2 A_\mu + \frac{\lambda}{2} \nabla_\alpha F_\mu{}^\alpha - \frac{R\lambda}{2} A_\mu = 0$$

→ $\hat{G}_{zz} = \hat{R}_{zz} - \frac{1}{2} R \, \hat{g}_{zz}$ $\qquad \hat{g}_{zz} = 1$

$$= \frac{\lambda^2 F^2}{4} - \frac{R}{2} + \frac{1}{8} \lambda^2 F^2$$

$$= \frac{3}{8} \lambda^2 F^2 - \frac{R}{2} = 0$$

13] Antoine DIERCKX

Formons plusieurs combinaisons pour retomber sur les résultats précédents :

→ $0 = \hat{G}_{\mu\nu} - \lambda A_\nu \hat{G}_{\mu z}$

$$= G_{\mu\nu} + \frac{\lambda^2}{2} F_\mu{}^\alpha F_{\alpha\nu} + \frac{1}{2}\lambda^2 (A_\nu \nabla_\alpha F_\mu{}^\alpha + A_\mu \nabla_\alpha F_\nu{}^\alpha)$$

$$+ \frac{3}{8}\lambda^4 F^2 A_\mu A_\nu - \frac{\lambda^2}{2} R A_\mu A_\nu + \frac{\lambda^2 F^2}{8} g_{\mu\nu}$$

$$- \lambda^4 A_\nu \cdot \frac{3}{8} F^2 A_\mu - \frac{\lambda^2}{2} \nabla_\alpha F_\mu{}^\alpha \cdot A_\nu + \frac{\lambda^2 R}{2} A_\mu A_\nu$$

$$= G_{\mu\nu} + \frac{\lambda^2}{2}\left(F_\mu{}^\alpha F_{\alpha\nu} + A_\mu \nabla_\alpha F_\nu{}^\alpha - \frac{1}{4} F^2 g_{\mu\nu}\right) = 0$$

Si on suppose que $\nabla_\alpha F_\nu{}^\alpha = 0$, on retrouve que

$$G_{\mu\nu} = \frac{\lambda^2}{2}\left(F_\mu{}^\alpha F_{\alpha\nu} - \frac{1}{4} F^2 g_{\mu\nu}\right)$$

ce qui est exactement l'équation du mouvement précédemment obtenue.

→ Justifions $\nabla_\alpha F_\nu{}^\alpha = 0$. Pour cela, on considère la combinaison suivante :

$0 = \hat{G}_{\mu z} - \lambda A_\mu \hat{G}_{zz}$

$$= \frac{3}{8}\lambda^3 F^2 A_\mu + \frac{\lambda}{2} \nabla_\alpha F_\mu{}^\alpha - \frac{R\lambda}{2} A_\mu - \lambda^3 \cdot \frac{3}{8} F^2 A_\mu + \frac{\lambda R}{2} A_\mu$$

$$= \frac{\lambda}{2} \nabla_\alpha F_\mu{}^\alpha = 0 \Leftrightarrow \nabla_\alpha F_\mu{}^\alpha = 0$$

En conclusion, 2 combinaisons des EOM issues de l'action (1) nous donnent des équations équivalentes à celles obtenues via la variation de l'action (17). Cependant, il reste une contrainte supplémentaire issue de l'action (1) (qui n'est, a priori, pas respectée par (17)) puisque nous avons effectué seulement 2 combinaisons à partir de 3 relations. Par exemple, l'équation $\hat{G}_{zz} = 0 \Leftrightarrow F^2 = \frac{R}{6\pi G}$ n'est pas vérifiée par les EOM issues de (1).

14) Dubois DIERCKX

PARTIE B

$\boxed{1}$ On considère l'action $S = \int \sqrt{-g}\, d^3x \left(R - \frac{1}{12} H^2\right)$

Pour trouver les équations du mouvement, on fait varier S sous

$g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$ et $B_{\mu\nu} \mapsto B_{\mu\nu} + \delta B_{\mu\nu}$

→ Variation de la métrique :

$$\delta S = \int d^3x \left\{ \delta\sqrt{-g}\,(R - H^2/12) - \sqrt{-g}\,(\delta R - \delta(H^2)/12) \right\}$$

$$= \int d^3x \left\{ \delta(\sqrt{-g}\,R) - \frac{1}{12}\left(\delta\sqrt{-g}\,H^2 + \sqrt{-g}\,.\,\delta(H^2)\right)\right\}$$

$$\rightarrow \delta H^2 = \delta\left(H^{\alpha\beta\gamma} H_{\alpha\beta\gamma}\right)$$

$$= \delta\left(g^{\mu\alpha} g^{\nu\beta} g^{\sigma\gamma} H_{\mu\nu\sigma} H_{\alpha\beta\gamma}\right)$$

$$= \delta g^{\mu\alpha}.\, g^{\nu\beta} g^{\sigma\gamma} H_{\mu\nu\sigma} H_{\alpha\beta\gamma}$$
$$+ g^{\mu\alpha} \delta g^{\nu\beta}.\, g^{\sigma\gamma} H_{\mu\nu\sigma} H_{\alpha\beta\gamma}$$
$$+ g^{\mu\alpha} g^{\nu\beta}.\, \delta g^{\sigma\gamma} H_{\mu\nu\sigma} H_{\alpha\beta\gamma}$$

Or, $\delta g^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$

$$= \left(-g^{\lambda\mu} g^{\eta\alpha} g^{\nu\beta} g^{\sigma\gamma} - g^{\lambda\nu} g^{\eta\beta} g^{\mu\alpha} g^{\sigma\gamma} - g^{\lambda\sigma} g^{\eta\gamma} g^{\mu\alpha} g^{\nu\beta}\right)$$
$$.\, \delta g_{\lambda\eta} H_{\mu\nu\sigma} H_{\alpha\beta\gamma}$$

$$= \left(-H^{\lambda\beta\gamma} H^{\eta}{}_{\beta\gamma} - H^{\alpha\lambda\gamma} H_{\alpha}{}^{\eta}{}_{\gamma} - H^{\alpha\beta\lambda} H_{\alpha\beta}{}^{\eta}\right) \delta g_{\lambda\eta}$$

$$= -3\, H^{\lambda\alpha\beta} H^{\eta}{}_{\alpha\beta}\, \delta g_{\lambda\eta}$$

$$\rightarrow \delta(\sqrt{-g}\,R) = -\sqrt{-g}\, G^{\alpha\beta} \delta g_{\alpha\beta} + \cancel{\partial_\alpha V^\alpha}$$

dérivée totale

$$\rightarrow \delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}\, g^{\alpha\beta} \delta g_{\alpha\beta}$$

On trouve :

$$\delta S = \int d^3x \left(-\sqrt{-g}\, G^{\alpha\beta} - \frac{1}{12}\left(\frac{1}{2}\sqrt{-g}\, g^{\alpha\beta} H^2 - \sqrt{-g}\, 3 H^{\alpha\sigma\eta} H^{\beta}{}_{\sigma\eta}\right)\right).\, \delta g_{\alpha\beta}$$

$$= \int d^3x \sqrt{-g}\left(-G^{\alpha\beta} - \frac{1}{24} H^2 g^{\alpha\beta} + \frac{1}{4} H^{\alpha\sigma\eta} H^{\beta}{}_{\sigma\eta}\right).\, \delta g_{\alpha\beta}$$

$$\rightarrow G^{\alpha\beta} = \frac{1}{4} H^{\alpha\sigma\eta} H^{\beta}{}_{\sigma\eta} - \frac{1}{24} H^2 g^{\alpha\beta}$$

D'EKCKX

→ Variation du champ B :

$$\delta S = \int d^3x \sqrt{-g} \left(-\frac{1}{12} \delta(H^2) \right)$$

$$\rightarrow \delta H^2 = \delta \left(g^{\mu\alpha} g^{\nu\beta} g^{\sigma\gamma} H_{\mu\nu\sigma} H_{\alpha\beta\gamma} \right)$$
$$= 2 H^{\alpha\beta\gamma} \delta H_{\alpha\beta\gamma}$$

Or, $H = \frac{1}{3!} H_{\sigma\alpha\beta} dx^\sigma \wedge dx^\alpha \wedge dx^\beta$ et

$$dB = \frac{1}{2!} \partial_\sigma B_{\alpha\beta} dx^\sigma \wedge dx^\alpha \wedge dx^\beta$$

et $H_{\sigma\mu\nu} = (dB)_{\sigma\mu\nu} = 3 \partial_{[\nu} B_{\alpha\beta]} = 3 \cdot \frac{2}{3!} \left(\partial_\sigma B_{\alpha\beta} + \partial_\alpha B_{\beta\sigma} + \partial_\beta B_{\sigma\alpha} \right)$

$$\hookrightarrow \delta H^2 = 6 H^{\alpha\beta\gamma} \delta \left(\partial_{[\alpha} \delta B_{\beta\gamma]} \right)$$
$$= 2 H^{\alpha\beta\gamma} \left(\partial_\alpha \delta B_{\beta\gamma} + \partial_\beta \delta B_{\gamma\alpha} + \partial_\gamma \delta B_{\alpha\beta} \right)$$

$$\rightarrow \delta S = \int dx^3 \sqrt{-g} \left(\frac{-1}{12} \right) \cdot 2 H^{\alpha\beta\gamma} \left(\partial_\alpha \delta B_{\beta\gamma} + \partial_\beta \delta B_{\gamma\alpha} + \partial_\gamma \delta B_{\alpha\beta} \right)$$

$$= \int d^3x \sqrt{-g} \frac{1}{6} \left(\partial_\alpha H^{\alpha\beta\gamma} \delta B_{\beta\gamma} + \partial_\beta H^{\alpha\beta\gamma} \delta B_{\gamma\alpha} + \partial_\gamma H^{\alpha\beta\gamma} B_{\alpha\beta} \right)$$

$$= \int d^3x \sqrt{-g} \frac{1}{6} \left(\partial_\alpha H^{\alpha\beta\gamma} \delta B_{\beta\gamma} + \partial_\alpha H^{\alpha\beta\gamma} \delta B_{\beta\gamma} + \partial_\alpha H^{\alpha\beta\gamma} B_{\beta\gamma} \right)$$

$$= \int d^3x \sqrt{-g} \frac{1}{2} \partial_\alpha H^{\alpha\beta\gamma} \cdot \delta B_{\beta\gamma}$$

$$\overset{!}{=} 0 \quad \forall \delta B_{\beta\gamma}$$

$\rightarrow \partial_\alpha H^{\alpha\beta\gamma} = 0 \rightarrow$ En SCLI, on a $\nabla_\alpha H^{\alpha\beta\gamma} = 0$

(6) Antoine DIERCKX

PARTIE B

$\boxed{2}$ On considère la métrique $ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2$
et $B = h(r)\, dt \wedge d\varphi$

→ La métrique est diagonale et de la forme :

$$g_{\mu\nu} = \mathrm{diag}(g_{tt}, g_{rr}, g_{\phi\phi}) = \mathrm{diag}\left(-f(r), \frac{1}{f(r)}, r^2\right)$$

La métrique inverse est alors simplement donnée par :

$$g^{\mu\nu} = \mathrm{diag}(g^{tt}, g^{rr}, g^{\phi\phi}) = \mathrm{diag}\left(\frac{-1}{f(r)}, f(r), \frac{1}{r^2}\right)$$

→ Les composantes du champ B sont :

$$B = h(r)\, dt \wedge d\varphi = \frac{1}{2} B_{\mu\nu}\, dx^\mu \wedge dx^\nu$$
$$= \frac{1}{2} B_{t\varphi}\, dt \wedge d\varphi + \frac{1}{2} B_{\varphi t}\, d\varphi \wedge dt$$
$$= B_{t\varphi}\, dt \wedge d\varphi$$
$$\rightarrow \quad B_{t\varphi} = -B_{\varphi t} = h(r)$$

→ Première équation du mouvement : on avait

$$\nabla_\alpha H^{\alpha\beta\gamma} = 0 \iff \nabla_\alpha H^\alpha{}_{\beta\gamma} = 0$$
$$\iff g^{\alpha\sigma} \nabla_\sigma H_{\alpha\beta\gamma} = 0$$
$$\iff g^{tt} \nabla_t H_{t\mu\nu} + g^{rr} \nabla_r H_{r\mu\nu} + g^{\varphi\varphi} \nabla_\varphi H_{\varphi\mu\nu} = 0$$

Or, $H = dB = dB(r)$. Ainsi, $\nabla_t H_{t\mu\nu} = \nabla_\varphi H_{\varphi\mu\nu} = 0$
On obtient :

$$g^{rr} \nabla_r H_{r\mu\nu} = 0$$

Pour obtenir une équation non triviale, il faut $r \neq \mu$ et $r \neq \nu \neq \mu$.
On choisit d'étudier le cas où $\mu = t$, $\nu = \varphi$:

$$f(r) \nabla_r H_{rt\varphi} = 0$$

→ Pour calculer la dérivée covariante, il faut connaître les symboles de Christoffel.

7) Antoine DIERCKX

→ On rappelle l'expression suivante :

$$\Gamma^{\alpha}_{\rho\gamma} = \frac{1}{2} g^{\alpha\sigma}\left(\partial_\rho g_{\sigma\gamma} + \partial_\gamma g_{\sigma\rho} - \partial_\sigma g_{\rho\gamma}\right)$$

L'expression devient :

$$f(r)\left(\partial_r H_{rt\varphi} - \Gamma^{\alpha}_{rr} H_{\alpha t\varphi} - \Gamma^{\alpha}_{rt} H_{r\alpha\varphi} - \Gamma^{\alpha}_{r\varphi} H_{rt\varphi}\right) = 0$$

↳ $\Gamma^{\alpha}_{rr} = \frac{1}{2} g^{\alpha\sigma}\left(\partial_r g_{\sigma r} + \partial_r g_{\sigma r} - \partial_\sigma g_{rr}\right)$

$\alpha \overset{!}{=} r$ $\quad = \frac{1}{2} g^{rr}\left(2\partial_r g_{rr} - \partial_r g_{rr}\right) = \frac{1}{2} f \partial_r\left(\frac{1}{f}\right) = -\frac{1}{2} f'/f$

↳ $\Gamma^{\sigma}_{rt} = \frac{1}{2} g^{\alpha\sigma}\left(\underbrace{\partial_r g_{\alpha t}}_{\neq 0 \text{ si } \alpha = t} - \cancel{\partial_t g_{\alpha r}} - \cancel{\partial_\alpha g_{rt}}\right)$

$\sigma \overset{!}{=} t$

$\quad = \frac{1}{2} g^{tt}\left(\partial_r g_{tt}\right) = \frac{1}{2}\left(\frac{-1}{f}\right)\partial_r(-f) = \frac{1}{2} f'/f$

↳ $\Gamma^{\alpha}_{r\varphi} = \frac{1}{2} g^{\alpha\sigma}\left(\partial_r g_{\sigma\varphi} + \cancel{\partial_\varphi g_{\sigma r}} - \cancel{\partial_\sigma g_{r\varphi}}\right)$

$\alpha \overset{!}{=} \varphi$

$\quad = \frac{1}{2} g^{\varphi\varphi}\partial_r g_{\varphi\varphi} = \frac{1}{2}\left(\frac{1}{r^2}\right)\partial_r(r^2) = \frac{1}{r}$

→ L'expression devient :

$$f(r)\left(\partial_r H_{rt\varphi} - \Gamma^{r}_{rr} H_{rt\varphi} - \Gamma^{t}_{rt} H_{rt\varphi} - \Gamma^{\varphi}_{r\varphi} H_{rt\varphi}\right) = 0$$

$\Leftrightarrow f(r)\left(\partial_r - \frac{1}{r}\right) H_{rt\varphi} = 0$

$\Leftrightarrow \partial_r H_{rt\varphi} = \frac{1}{r} H_{rt\varphi} \quad \Leftrightarrow \ln H_{rt\varphi} = \ln r + K$

$\Leftrightarrow H_{rt\varphi} = r . C_1$ avec $C_1 \in \mathbb{R}^+_0$

→ Or, $H_{rt\varphi} = 3\partial_{[r} B_{t\varphi]} = \frac{3}{3!}\left(\partial_r B_{t\varphi} + \partial_t B_{\varphi r} + \partial_\varphi B_{rt} - \partial_r B_{\varphi t} - \partial_t B_{r\varphi} - \partial_\varphi B_{tr}\right)$

$\quad = \partial_r B_{t\varphi} = h'(r)$

$\Rightarrow r . C_1 = h'(r) \Leftrightarrow h(r) = \frac{r^2}{2} C_1 + C_2 \ , \ C_2 \in \mathbb{R}$

→ Deuxième équation du mouvement :

On avait $G^{\alpha\beta} = \frac{1}{4} H^{\alpha\mu\nu} H^{\beta}{}_{\mu\nu} - \frac{1}{24} H^2 g^{\alpha\beta}$

$\Leftrightarrow G_{\alpha\beta} = \frac{1}{4} H_{\alpha\mu\nu} H_{\beta}{}^{\mu\nu} - \frac{1}{24} H^2 g_{\alpha\beta}$

→ Calculons H^2 :

$$H^2 = H^{\alpha\beta\gamma} H_{\alpha\beta\gamma}$$
$$= g^{\mu\alpha} g^{\nu\beta} g^{\sigma\gamma} H_{\mu\nu\sigma} H_{\alpha\beta\gamma}$$
$$= g^{tt} g^{rr} g^{\varphi\varphi} H_{tr\varphi} H_{tr\varphi} \times 6$$
$$= 6 \cdot \left(\frac{-1}{f}\right) \cdot f \cdot \frac{1}{r^2} H_{tr\varphi} H_{tr\varphi}$$
$$= -\frac{6}{r^2} \cdot r^2 C_1^2 = -6 C_1^2$$

→ Calculons $H_{\alpha\mu\nu} H_{\beta}{}^{\mu\nu}$:

$$H_{\alpha\mu\nu} H_{\beta\sigma\eta}\, g^{\sigma\mu} g^{\eta\nu}$$
$$= g^{tt} g^{rr} H_{\alpha tr} H_{\beta tr} + g^{tt} g^{\varphi\varphi} H_{\alpha t\varphi} H_{\beta t\varphi}$$
$$+ g^{rr} g^{tt} H_{\alpha rt} H_{\beta rt} + g^{\varphi\varphi} g^{tt} H_{\alpha\varphi t} H_{\beta\varphi t}$$
$$+ g^{rr} g^{\varphi\varphi} H_{\alpha r\varphi} H_{\beta r\varphi} + g^{\varphi\varphi} g^{rr} H_{\alpha\varphi r} H_{\beta\varphi r}$$
$$+ (g^{tt} g^{tt} + g^{rr} g^{rr} + g^{\varphi\varphi} g^{\varphi\varphi}) \cdot 0$$
$$= 2\left(g^{tt} g^{rr} H_{\alpha tr} H_{\beta tr} + g^{tt} g^{\varphi\varphi} H_{\alpha t\varphi} H_{\alpha t\varphi} + g^{rr} g^{\varphi\varphi} H_{\alpha r\varphi} H_{\alpha r\varphi}\right)$$

→ Si $\alpha \neq \beta$; tout les termes sont nuls car α (ou β) serait déjà présent parmis les autres indices. De plus, $g^{\alpha\beta} = 0$ si $\alpha = \beta$.

→ Si $\alpha = \beta$, il y a 3 possibilités :

$\boxed{\alpha = t}$

$$\frac{1}{2} H_{t\mu\nu} H_t{}^{\mu\nu} = g^{rr} g^{\varphi\varphi} H_{tr\varphi} H_{tr\varphi} = f \cdot \frac{1}{r^2} (-r C_1)^2 = f C_1^2$$

$\boxed{\alpha = r}$

$$\frac{1}{2} H_{r\mu\nu} H_r{}^{\mu\nu} = g^{tt} g^{\varphi\varphi} H_{rt\varphi} H_{rt\varphi} = \left(\frac{-1}{f}\right) \frac{1}{r^2} (r C_1)^2 = -\frac{C_1^2}{f}$$

$\boxed{\alpha = \varphi}$

$$\frac{1}{2} H_{\varphi\mu\nu} H_\varphi{}^{\mu\nu} = g^{tt} g^{rr} H_{\varphi tr} H_{\varphi tr} = \left(\frac{-1}{f}\right) f (-r C_1)^2 = -r^2 C_1^2$$

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On trouve alors :

$\boxed{\alpha=\beta=t}$ $G_{tt} = \frac{1}{4} 2 f C_1^2 - \frac{1}{24}(-6C_1)^2 g_{tt}$

$= \frac{1}{2} f C_1^2 + \frac{1}{4} C_1^2 (-f)$

$= \frac{1}{4} f C_1^2$

$\boxed{\alpha=\beta=r}$ $G_{rr} = \frac{1}{2}\left(-\frac{C_1^2}{f}\right) + \frac{1}{4} C_1^2 \left(\frac{1}{f}\right)$

$= -\frac{1}{4} \frac{C_1^2}{f}$

$\boxed{\alpha=\beta=\varphi}$ $G_{\varphi\varphi} = \frac{1}{2}\left(-r^2 C_1^2\right) + \frac{1}{4} C_1^2 (r^2)$

$= -\frac{1}{4} r^2 C_1^2$

→ Afin de trouver une expression avec f, on doit écrire $G_{\alpha\alpha}$ en fonction de la métrique. Avant cela, regardons que symboles de Christoffel sont non nuls.

① $\Gamma^t_{\alpha\beta} = \frac{1}{2} g^{tt} \left(\underbrace{\partial_\alpha g_{\beta t}}_{\neq 0 \text{ si } \alpha=r, \beta=t} + \underbrace{\partial_\beta g_{\alpha t}}_{\neq 0 \text{ si } \beta=r, \alpha=t} - \cancel{\partial_t g_{\alpha\beta}}^{0} \right)$

$\rightarrow \Gamma^t_{rt} = \frac{1}{2}\left(-\frac{1}{f}\right) \partial_r (-f) = f'/2f = \Gamma^t_{tr}$

② $\Gamma^r_{\alpha\beta} = \frac{1}{2} g^{rr} \left(\partial_\alpha g_{\beta r} + \partial_\beta g_{\alpha r} - \partial_r g_{\alpha\beta} \right)$

$\rightarrow \Gamma^r_{rr} = -\frac{1}{2} f'/f$

$\rightarrow \Gamma^r_{\varphi\varphi} = \frac{1}{2} f \left(-\partial_r (r^2)\right) = -rf$

$\rightarrow \Gamma^r_{tt} = -\frac{1}{2} f \, \partial_r (-f) = f' f/2$

③ $\Gamma^\varphi_{\alpha\beta} = \frac{1}{2} g^{\varphi\varphi} \left(\partial_\alpha g_{\beta\varphi} + \partial_\beta g_{\alpha\varphi} - \partial_\varphi g_{\alpha\beta} \right)$

$\rightarrow \Gamma^\varphi_{\varphi r} = \frac{1}{2} \frac{1}{r^2} \left(+\partial_r r^2\right) = 1/r$

↳ Tout les autres Γ sont nuls.

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Les expressions des tenseurs de Ricci se réduisent à présent à :

$\rightarrow R_{tt} = R^{\alpha}{}_{t\alpha t}$ $\quad \left(R_{\mu\nu} = \partial_\alpha \Gamma_{\mu\nu}^{\alpha} - \partial_\nu \Gamma_{\alpha\mu}^{\alpha} + \Gamma_{\mu\nu}^{\sigma}\Gamma_{\alpha\sigma}^{\alpha} - \Gamma_{\mu\alpha}^{\sigma}\Gamma_{\nu\sigma}^{\alpha} \right)$

$$= \partial_\alpha \Gamma_{tt}^{\alpha} - \partial_t \Gamma_{\alpha t}^{\alpha} + \Gamma_{tt}^{\sigma}\Gamma_{\alpha\sigma}^{\alpha} - \Gamma_{t\alpha}^{\sigma}\Gamma_{t\sigma}^{\alpha}$$

$$= \partial_r \Gamma_{tt}^{r} + \Gamma_{tt}^{r}\Gamma_{tr}^{t} + \Gamma_{tt}^{r}\Gamma_{rr}^{r} + \Gamma_{tt}^{r}\Gamma_{\varphi r}^{\varphi} - \Gamma_{tr}^{t}\Gamma_{tt}^{r} - \Gamma_{tr}^{r}\Gamma_{tr}^{r}$$

$$= \partial_r(f'f/2) + (f'f/2)(f'/2f) + (f'f/2)\left(-\tfrac{1}{2}f'/f\right) + (f'f/2)(1/r) - (f'/2f)(f'f/2) - (0)$$

$$= \partial_r\left(\frac{f'f}{2}\right) - \frac{1}{4}f'^2 + \frac{1}{2r}f'f$$

$$= \frac{f''}{2}f + \frac{f'^2}{2} - \frac{1}{4}f'^2 + \frac{1}{2r}f'f$$

$$= \frac{1}{2}f\left(f'' + \frac{f'}{r}\right)$$

$\rightarrow R_{rr} = \partial_\alpha \Gamma_{rr}^{\alpha} - \partial_r \Gamma_{\alpha r}^{\alpha} + \Gamma_{rr}^{\sigma}\Gamma_{\alpha\sigma}^{\alpha} - \Gamma_{r\alpha}^{\sigma}\Gamma_{r\sigma}^{\alpha}$

$$= \partial_r \Gamma_{rr}^{r} - \partial_r \Gamma_{rr}^{r} - \partial_r \Gamma_{\varphi r}^{\varphi} - \partial_r \Gamma_{tr}^{t}$$

$$+ \Gamma_{rr}^{r}\Gamma_{tr}^{t} + \Gamma_{rr}^{r}\Gamma_{rr}^{r} + \Gamma_{rr}^{r}\Gamma_{\varphi r}^{\varphi}$$

$$- \Gamma_{rr}^{r}\Gamma_{rr}^{r} - \Gamma_{tr}^{t}\Gamma_{rt}^{t} - \Gamma_{r\varphi}^{\varphi}\Gamma_{\varphi r}^{\varphi}$$

$$= -\partial_r\left(\frac{1}{r}\right) - \partial_r\left(\frac{f'}{2f}\right) + \left(\frac{-f'}{2f}\right)\left(\frac{f'}{2f}\right) + \left(\frac{-f'}{2f}\right)\left(\frac{1}{r}\right) - \left(\frac{f'}{2f}\right)^2 - \frac{1}{r^2}$$

$$= +\frac{1}{r^2} - \frac{1}{2}\frac{f''}{f} + \frac{1}{2}\frac{f'^2}{f^2} - \frac{1}{4}\frac{f'^2}{f^2} - \frac{f'}{2rf} - \frac{f'^2}{4f^2} - \frac{1}{r^2}$$

$$= -\frac{1}{2}\frac{1}{f}\left(f'' + \frac{f'}{r}\right)$$

$\rightarrow R_{\varphi\varphi} = \partial_\alpha \Gamma_{\varphi\varphi}^{\alpha} - \partial_\varphi \Gamma_{\alpha\varphi}^{\alpha} + \Gamma_{\varphi\varphi}^{\sigma}\Gamma_{\alpha\sigma}^{\alpha} - \Gamma_{\varphi\alpha}^{\sigma}\Gamma_{\varphi\sigma}^{\alpha}$

$$= \partial_r \Gamma_{\varphi\varphi}^{r} + \Gamma_{\varphi\varphi}^{r}\Gamma_{tr}^{t} + \Gamma_{\varphi\varphi}^{r}\Gamma_{rr}^{r} + \Gamma_{\varphi\varphi}^{r}\Gamma_{\varphi r}^{\varphi} - \Gamma_{\varphi\varphi}^{r}\Gamma_{\varphi r}^{\varphi} - \Gamma_{\varphi r}^{\varphi}\Gamma_{\varphi\varphi}^{r}$$

$$= \partial_r(-rf) + (-rf)\left(\frac{f'}{2f}\right) + (-rf)\left(\frac{-f'}{2f}\right) - \left(\frac{1}{r}\right)(-rf)$$

$$= -f - rf' - \frac{rf'}{2} + \frac{rf'}{2} + f$$

$$= -rf'$$

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Maintenant que nous avons nos Ricci, on peut calculer :

$\rightarrow G_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} = R_{\alpha\beta} - \frac{R_{\mu\nu}}{2} g^{\mu\nu} g_{\alpha\beta}$

$\hookrightarrow G_{tt} = R_{tt} - \frac{1}{2}\left(R_{tt} g^{tt} + R_{rr} g^{rr} + R_{\varphi\varphi} g^{\varphi\varphi}\right) g_{tt}$

$= \frac{R_{tt}}{2} - \frac{R_{rr}}{2} g^{rr} g_{tt} - \frac{R_{\varphi\varphi}}{2} g^{\varphi\varphi} g_{tt}$

$= \frac{1}{4} f \left(f'' - \frac{f'}{r}\right) + \frac{1}{4}\frac{1}{f}\left(f'' + \frac{f'}{r}\right)\left(-f^2\right) + \frac{rf'}{2}\left(-\frac{f}{r^2}\right)$

$= -\frac{1}{2}\frac{f'f}{r}$

$\hookrightarrow G_{rr} = R_{rr} - \frac{1}{2}\left(R_{tt} g^{tt} + R_{rr} g^{rr} + R_{\varphi\varphi} g^{\varphi\varphi}\right) g_{rr}$

$= \frac{R_{rr}}{2} - \frac{1}{2} R_{tt} g^{tt} g_{rr} - \frac{1}{2} R_{\varphi\varphi} g^{\varphi\varphi} g_{rr}$

$= \frac{-1}{4f}\left(f'' + \frac{f'}{r}\right) - \frac{1}{4} f \left(f'' - \frac{f'}{r}\right)\left(\frac{-1}{f^2}\right) + \frac{rf'}{2}\frac{1}{r^2 f}$

$= \frac{1}{2}\frac{f'}{rf}$

$\hookrightarrow G_{\varphi\varphi} = R_{\varphi\varphi} - \frac{1}{2}\left(R_{tt} g^{tt} + R_{rr} g^{rr} + R_{\varphi\varphi} g^{\varphi\varphi}\right) g_{\varphi\varphi}$

$= \frac{R_{\varphi\varphi}}{2} - \frac{1}{2} R_{tt} g^{tt} g_{\varphi\varphi} - \frac{1}{2} R_{rr} g^{rr} g_{\varphi\varphi}$

$= -\frac{rf'}{2} - \frac{1}{4} f \left(f'' + \frac{f'}{r}\right)\left(\frac{-r^2}{f}\right) + \frac{1}{4}\frac{1}{f}\left(f'' + \frac{f'}{r}\right)\left(\frac{f}{r^2}\right)$

$= \frac{1}{2} r^2 f''$

On obtient 3 équations :

$\boxed{A}$ $-\frac{1}{2}\frac{ff'}{r} = \frac{1}{4} f C_1^2 \Leftrightarrow -2f' = r C_1^2$

$\boxed{B}$ $\frac{1}{2}\frac{f'}{rf} = -\frac{1}{4}\frac{C_1^2}{f} \Leftrightarrow -2f' = r C_1^2$

$\boxed{C}$ $\frac{1}{2} r^2 f'' = -\frac{1}{4} r^2 C_1^2 \Leftrightarrow -2f'' = C_1^2$

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En résolvant ces équations, on obtient :

$$f(r) = -\frac{C_1^2}{4} r^2 + C_3 \quad \text{avec } C_1 \in \mathbb{R}_0^+ \text{ et } C_3 \in \mathbb{R}.$$

On a trouvé les formes des fonctions $h(r)$ et $f(r)$:

$$\begin{cases} h(r) = \frac{C_1}{2} r^2 + C_2 \\ f(r) = -\frac{C_1^2}{4} r^2 + C_3 \end{cases} \quad \text{avec } C_1 \in \mathbb{R}_0^+ ;\ C_2, C_3 \in \mathbb{R}.$$

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PARTIE B

$\boxed{3}$ Pour connaître la courbure scalaire, prenons la trace du Ricci.

$$R = R^{\alpha}{}_{\alpha} = R_{\alpha\beta}\, g^{\alpha\beta}$$

$$= R_{tt}\, g^{tt} + R_{rr}\, g^{rr} + R_{\varphi\varphi}\, g^{\varphi\varphi}$$

$$= \left(\frac{1}{2} f\left(f'' + \frac{f'}{r}\right)\right)\left(-\frac{1}{f}\right) + \left(-\frac{1}{2}\frac{1}{f}\left(f'' + \frac{f'}{r}\right)\right) f + (-r f')\,\frac{1}{r^2}$$

$$= -\frac{1}{2} f'' - \frac{1}{2}\frac{f'}{r} - \frac{1}{2} f'' - \frac{1}{2}\frac{f'}{r} - \frac{f'}{r}$$

$$= -f'' - 2\frac{f'}{r} = +\frac{C_1^2}{2} - \frac{2}{r}\cdot\left(-\frac{r}{2} C_1^2\right)$$

$$= \frac{3}{2} C_1^2$$

La courbure scalaire est $R = \frac{3}{2} C_1^2$

→ La constante cosmologique Λ apparait selon :

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = K\, T_{\alpha\beta}$$

Nous avons considéré l'équation du mouvement suivante :

$$G_{\alpha\beta} + \frac{1}{24} H^2 g_{\alpha\beta} = \frac{1}{4} H_{\alpha\sigma\eta}\, H_{\beta}{}^{\sigma\eta}$$

On peut alors identifier :

$$K\, T_{\alpha\beta} \equiv \frac{1}{4} H_{\alpha\sigma\eta} H_{\beta}{}^{\sigma\eta} \quad \text{et} \quad \Lambda = \frac{1}{24} H^2$$

Or, $H^2 = -6\, C_1^2 \Rightarrow \Lambda = \frac{-1}{4} C_1 \Leftrightarrow C_1 = -4\Lambda$

Par ailleurs le champ B est donné par :

$$B = h(r)\, dt \wedge d\varphi = \left(\frac{C_1 r^2}{2} + C_2\right) dt \wedge d\varphi$$

$$= (-2\Lambda r^2 + C_2)\, dt \wedge d\varphi$$

Le champ B est donc une fonction de la constante cosmologique.